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Modelling the Role of Fluctuations in Volume on Self-purification of Natural Water Bodies

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Abstract: We present a simple mathematical model for examining the role of seasonal fluctuations of volume on self-purification of natural water bodies such as a lake ecosystem. The evolution of self-purification process, activated by natural mechanisms of flushing by clean water and degradation of pollutants by aeration, is influenced by a number of parameters depending on lake-specific characteristics. These parameters have been identified in the formulation of the model. The model consists of an initial value problem which has been solved analytically. Numerical results are presented graphically to highlight the effect of these parameters on clean-up time corresponding to all pollution levels. Results for a few simpler situations derived as particular cases from the general solution are also included and discussed.

Key words: Natural water bodies, volume fluctuations, pollution level, self-purification, clean-up time.

Introduction

Since the last few decades, many countries world-wide have been coping with the major crisis of scarcity of safe drinking water on account of a large number of natural water bodies reeling under unprecedented high pollution from unrestricted entry of industrial effluents and domestic wastes. In the long-term, high pollution levels may cause irreversible damage to the ecosystem of the water body and its ability to survive and sustain aquatic life. Apparently, totally preventing or partially curbing the entry of pollutants is an essential step to let the water body self-purify itself by aeration and flushing by clean water. Numerous studies devoted to the problem of pollution management have advocated natural self-purification as an appropriate clean-up strategy to revive natural water bodies into a clean fresh water resource. Using the concepts of acting pollution load and bearing capacity of a water body, Altunin (1996) adopted the general model of development of systems to obtain the maximum allowable critical value of the load for protecting the water body from an ecological catastrophe. Jiang and Shen (2006) in their extensive study on restoration strategies for Lake Donghu estimated the natural purification rate of a eutrophic lake after pollutant removal. In a case study on the system of Selenga River and its Delta, Makushkin and Korsunov (2005) examined the rate of microbiological transformation of organic matter to determine the self-purification elements of the water currents. Ostroumov (2005) gave a systematic account of the concepts concerning the multiple functions of the biota in the self-purification of water bodies and watercourses.

In this paper, we present a simple mathematical model for examining the role of seasonal fluctuations of volume on the evolution of the self-purification process in natural

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water bodies such as a lake ecosystem. It is assumed that the pollutants, largely biological wastes, are removed by flushing and bio-chemical reactions with dissolved oxygen. The main purpose of this study is to investigate the effect of identified parameters on the evolving pollution level and clean-up time for partial or full recovery when volume fluctuations are taken into account. The direct impact of these parameters on self-purification is described by figures showing the pollution profiles for different values of the parameters. Results for a few limiting cases are also included and discussed in this paper.

Model Description and Problem Formulation

The pattern of fluctuations in the volume of water in a lake during seasonal cycles, observed and recorded over an extended span of time can provide estimates of maximum deviation in the volume and the periodic time of volume fluctuations. If it varies about its mean value V_m from minimum value $V_{\min}$ to maximum value $V_{\max}$ with periodic time T, then we may express the lake volume $V(t)$ (km^3) at time t by

$$V(t) = V_m(1 + b\sin\omega t),\ 0 < b < 1 \tag{1}$$

in which the mean volume V_m, frequency ω ($year^{-1}$) and the scalar b quantifying the extent of maximum deviation in the volume, namely, the amplitude bV_m are known from

$$V_m = \frac{1}{2}(V_{\max} + V_{\min}),\ \omega = \frac{2\pi}{T} \text{ and } b = \frac{1}{2}(V_{\max} - V_{\min})$$

If r_i, r_o, r_p and r_e respectively denote the volumetric flow rates ($km^3\ year^{-1}$) of inflow, outflow, precipitation and evaporation, then the water flow balance in the lake gives

$$\dot{V} = r_i + r_p - r_e - r_o \tag{2}$$

Here dot denotes derivative with respect to time variable t.

For a lake system, the flow rates r_i, r_p and r_e may separately vary with time; but if the net rate of water accretion resulting from inflow and the excess (deficit) of precipitation over evaporation is a very slowly varying function of time, then the sum $r_i + (r_p - r_e)$ can be regarded to have a constant value r, say and we write

$$r_i + (r_p - r_e) = r \tag{3}$$

Equation (2) can now be written as

$$\dot{V} = r - r_o \tag{4}$$

It is observed that $b = 0$ corresponds to the case of a lake of constant volume V_m. Also, in this case: (i) $r_i = r_0 = r$ i.e. inflow and outflow are uniform and equal, and (ii) $r_p = r_e$ i.e. precipitation is balanced by evaporation. From equations (3) and (4), the rates of inflow and outflow can be expressed as

$$r_i = r(1 - c) \text{ and } r_o = r\left(1 - \frac{\dot{V}}{r}\right) \tag{5}$$

In which $c = \dfrac{r_p - r_e}{r}$ is introduced as precipitation-evaporation parameter; theoretically it can assume any value in the interval $(-\infty, 1)$.

Let $x(t)$ and $y(t)$ denote the concentration (tons km^{-3}) of pollutants at time t in the inflow and the lake respectively, and k, the rate at which the pollutants in the lake undergo depletion due to bio-chemical reactions with the dissolved oxygen. Assuming that (i) pollutants are removed from the lake by outflow-flushing and bio-chemical-aeration, and (ii) the mixing of pollutants in the lake is instantaneous and uniform in the sense of time scale relevant to lake dynamics, the rate of change of concentration of pollutants in the lake is governed by the following mass balance equation

$$\frac{d}{dt}(yV) = r_i x - r_0 y - k yV \tag{6}$$

which, on using the expressions for r_1 and r_0 from equation (5), takes the form

$$\dot{y} + \left(k + \frac{r}{V}\right)y = (1 - c)r\frac{x}{V}. \tag{7}$$

Equation (7) together with the prescribed initial condition $y(0) = y_0$ constitutes an initial value problem.

Equation (7) can be solved provided the inflow-pollution concentration $x(t)$ is known. We consider an $x(t)$ of the form

$$x(t) = x_0 e^{-a\frac{t}{\tau}},\ x(0) = x_0\ a \geq 0 \tag{8}$$

in which lake parameter τ (years), defined by $\tau = \dfrac{V_m}{r}$ is the residence time of water in the lake, and a is the dimensionless inflow-pollution control parameter which can be varied to regulate $x(t)$ for achieving lake pollution levels satisfying the primary water quality standards for various uses of a fresh water resource.

We note that $x_0 = 0$ or the limit $a \to \infty$ implies pollution-free inflow while $a = 0$ corresponds to an inflow of uniform pollution level x_0 which can be assigned any arbitrary value.

Introducing the following non-dimensional barred quantities

$$\bar{t} = \frac{t}{\tau}, \bar{x} = \frac{x}{x_0}, \bar{y} = \frac{y}{y_0}, \bar{V} = \frac{V}{V_m}, \bar{\omega} = \omega\tau, \bar{k} = k\tau, \quad (9)$$

equation (7) in non-dimensional form becomes

$$\dot{\bar{y}} + \left(\bar{k} + \frac{1}{\bar{V}}\right)\bar{y} = (1-c)\frac{x_0}{y_0}\frac{\bar{x}}{\bar{V}} \quad (10)$$

with $\bar{y}(0) = 1$.

Here the dot denotes the derivative with respect to $\bar{t}$.

In the above equation (10), $\bar{x}(\bar{t}) = e^{-a\bar{t}}$, $\bar{V}(\bar{t}) = 1 + b\sin\bar{\omega}\bar{t}$ and $\bar{\omega}$ is the non-dimensional frequency parameter and $\bar{k}$ is also a non-dimensional parameter depending on lake profile and bio-chemical depletion rate.

Solution

Solution of the above initial value problem (10) is given by

$$\bar{y}(\bar{t}) = \exp\{-A(\bar{t})\}\left[\exp\{A(0)\} + \int_0^{\bar{t}} B(\bar{s})\,d\bar{s}\right] \quad (11)$$

in which

$$A(\bar{t}) = \bar{k}\,\bar{t} + \frac{2}{\bar{\omega}\sqrt{1-b^2}}\tan^{-1}\left(\frac{b + \tan(\bar{\omega}\bar{t}/2)}{\sqrt{1-b^2}}\right)$$

and

$$B(\bar{t}) = (1-c)\frac{x_0}{y_0}(1 + b\sin\bar{\omega}\bar{t})^{-1}\exp(-a\bar{t} + A(\bar{t})).$$

This solution is bounded for $\bar{t} > 0$ provided $\bar{k} \le a$.

Solution (11) is evaluated numerically for different values of the parameters $a, b, c, \bar{k}, \frac{x_0}{y_0}$ and $\bar{\omega}$. The results, followed by analysis and discussion, are presented in Figures 1-6, to bring out the direct impact of these parameters on the self-purification process.

Special Cases

Lake of Constant Volume

The solution in this case is recovered from the general solution by setting $b = 0$. This leads to $c = 0$, $A(\bar{t}) = (1+\bar{k})\bar{t}$ and $B(\bar{t}) = \frac{x_0}{y_0}\exp\{(1+\bar{k}-a)\bar{t}\}$ and the pollution-concentration profiles are then given by

$$\bar{y}(\bar{t}) = \exp\{-(1+\bar{k})\ \bar{t}\} + \frac{x_0}{y_0}(1+\bar{k}-a)^{-1}\left[\exp(-a\bar{t}) - \exp\{-(1+\bar{k})\ \bar{t}\}\right] \quad (12)$$

Inflow of Uniform Concentration

If, in addition, the inflow has uniform pollution level x_0 i.e., $a = 0$, the solution (12) reduces to

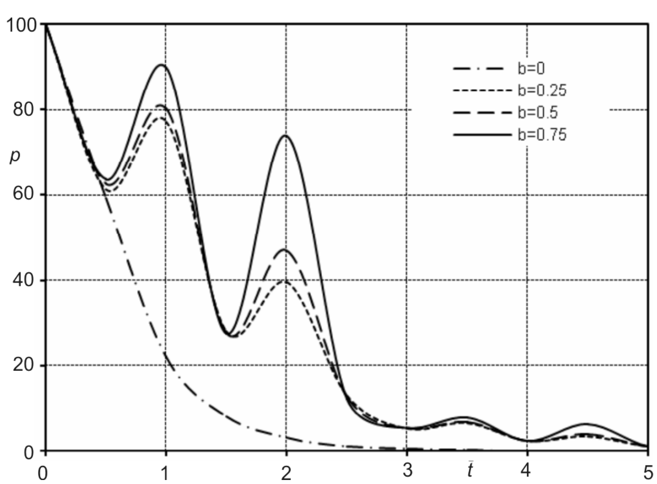


Figure 1: Response of pollution profiles to deviations in volume (*b*).

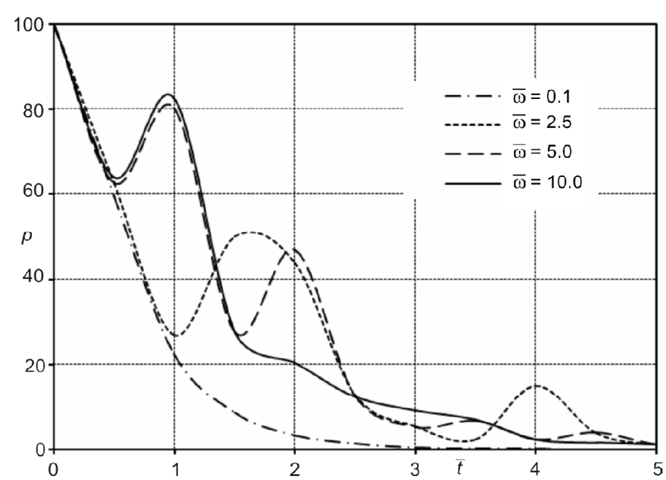
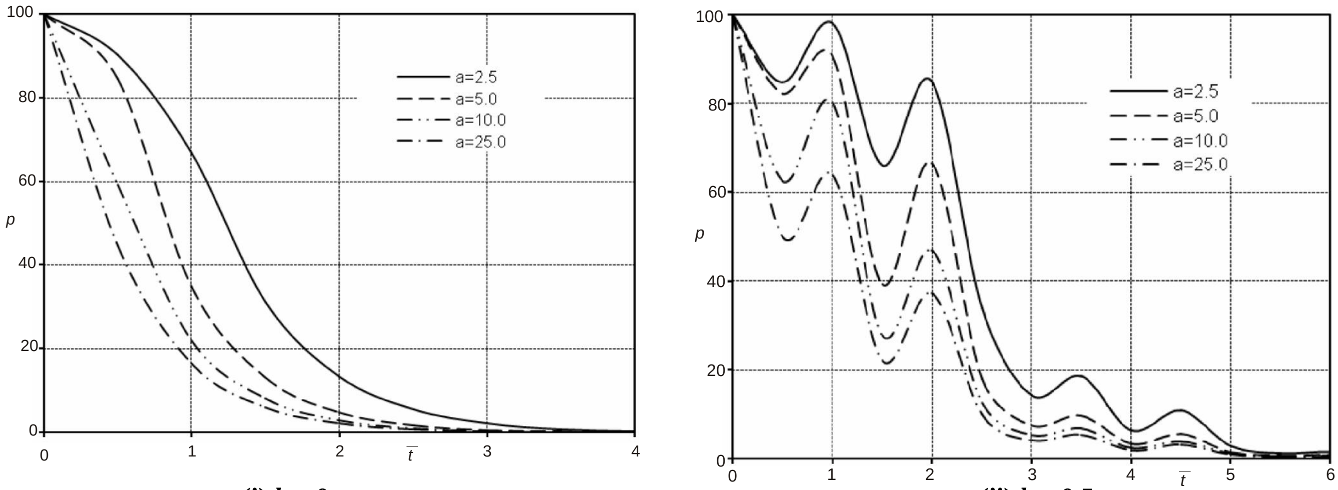
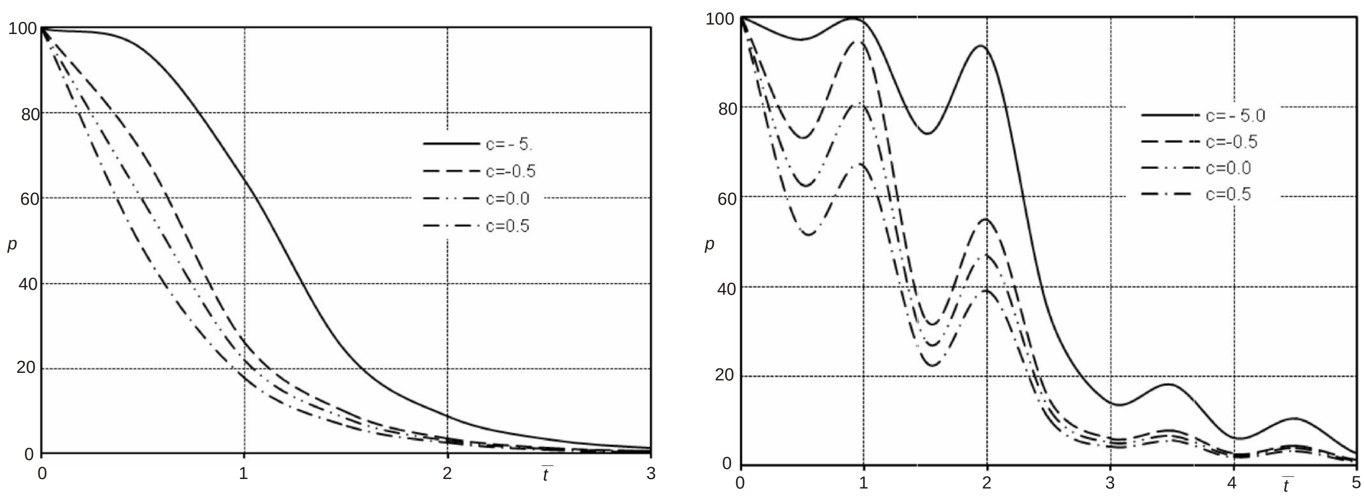


Figure 2: Response of pollution profiles to variation in frequency of volume fluctuations ($\bar{\omega}$).



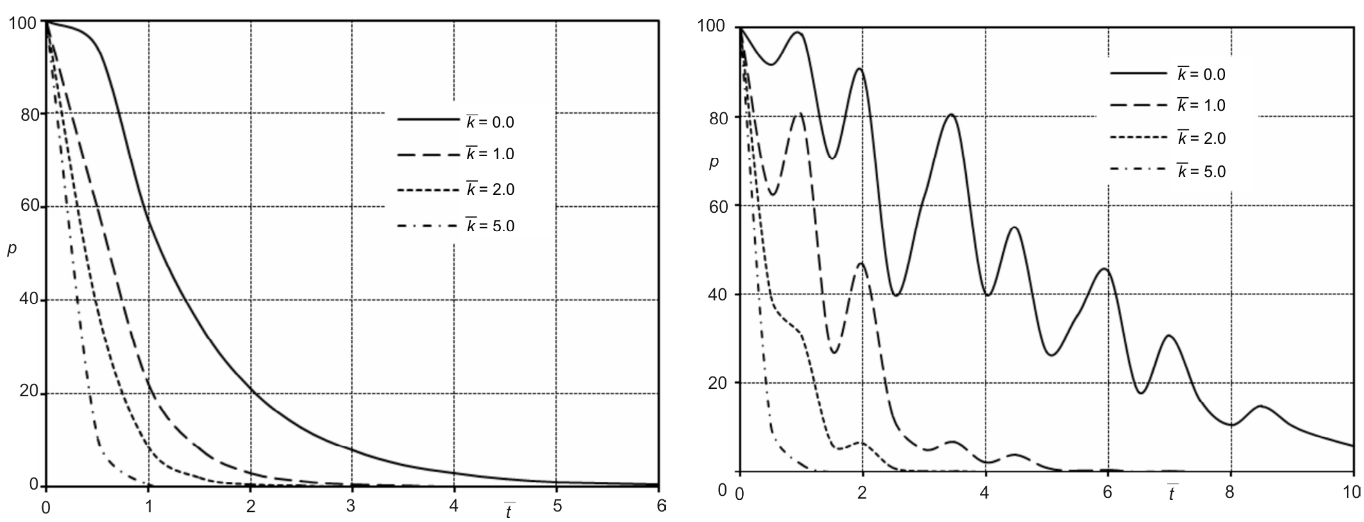
(i) $b = 0$ **(ii) $b = 0.5$**

Figure 3: Response of pollution profiles to the rate of inflowing pollution (*a*).



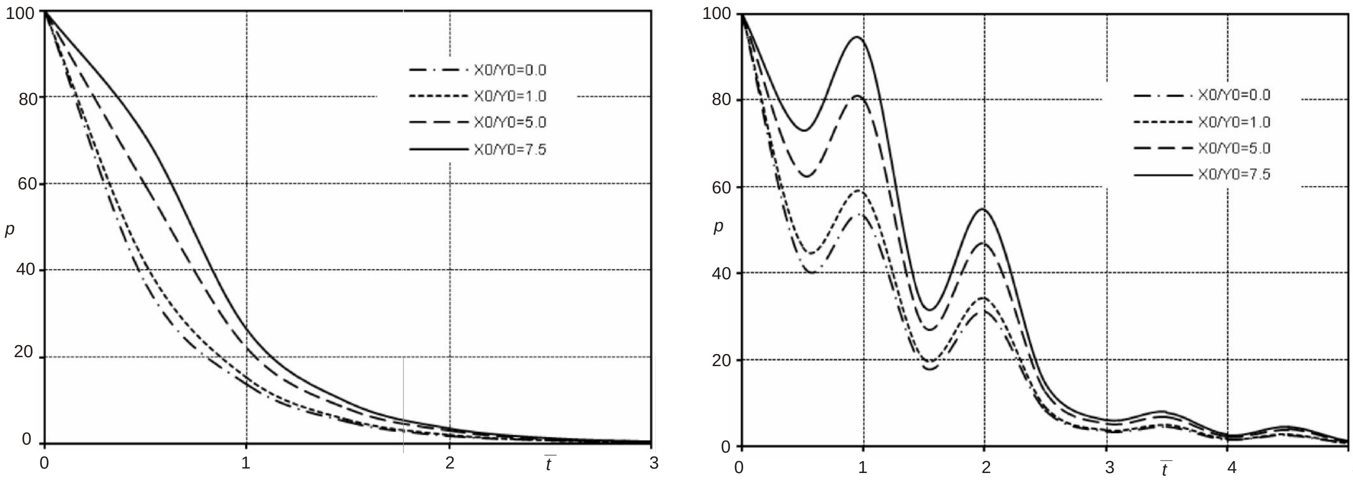
(i) $b = 0$ **(ii) $b = 0.5$**

Figure 4: Response of pollution profiles to excess (deficit) of precipitation over evaporation (*c*).



(i) $b = 0$ **(ii) $b = 0.5$**

Figure 5: Response of pollution profiles to bio-chemical reaction rate ($\bar{k}$).



(i) *b* = 0 **(ii) *b* = 0.5**

Figure 6: Response of pollution profiles to the ratio of initial pollution levels of the inflow and the lake.

$$\bar{y}(\bar{t}) = \frac{x_0}{y_0}\left(1+\bar{k}\right)^{-1} + \left\{1 - \frac{x_0}{y_0}\left(1+\bar{k}\right)^{-1}\right\} \exp\left\{-\left(1+\bar{k}\right)\ \bar{t}\right\} \qquad (13)$$

Absence of Bio-chemical Reactions

For this case $\left(\bar{k} = 0\right)$, and solution (13) further simplifies to

$$y(\bar{t}) = \frac{x_0}{y_0} + \left(1 - \frac{x_0}{y_0}\right)\exp\left\{-\bar{t}\right\} \qquad (14)$$

Clean-up Time

If the concentration $y(t)$ of the lake pollutants reduces to p% of its initial value y_0 in t_p years, then t_p, called the clean-up time for p% reduction in pollution level, is obtained by solving the equation

$$\bar{y}\left(\bar{t}_p\right) = \frac{y\left(t_p\right)}{y_0} = \frac{p}{100},\ \bar{t}_p = \frac{t_p}{\tau} \qquad (15)$$

For the lake concentration profile given by solution (13), the expression for t_p comes out to be

$$t_p = \tau\left(1+\bar{k}\right)^{-1} \ln C\left(p\right) \qquad (16)$$

with $C(p) = \left\{1 - \frac{x_0}{y_0}\left(1+\bar{k}\right)^{-1}\right\}\left\{\frac{p}{100} - \frac{x_0}{y_0}\left(1+\bar{k}\right)^{-1}\right\}^{-1}$

where $\frac{x_0}{y_0} < \left(1+\bar{k}\right)\frac{p}{100}$.

In the absence of bio-chemical reactions, expression (16) simplifies to

$$t_p = \tau \ln\left\{\left(1 - \frac{x_0}{y_0}\right)\left(\frac{p}{100} - \frac{x_0}{y_0}\right)^{-1}\right\}, \quad \frac{x_0}{y_0} < \frac{p}{100} \qquad (17)$$

which, for pollution-free inflow, further reduces to the basic result

$$t_p = \tau \ln\left(\frac{100}{p}\right) \qquad (18)$$

used many decades ago for predicting the clean-up time of the Great Lakes of North America-Canada.

Numerical Results and Discussion

Figures 1-6 show the plot of the relation $p = 100\,\bar{y}\left(\bar{t}\right)$, $\bar{y}\left(\bar{t}\right) = y\left(t\right)/y_0$, $t = \tau\bar{t}$ for different values of parameters a, b, c, $\bar{k}$, x_0/y_0 and $\bar{\omega}$.

In the co-ordinates $\left(\bar{t}, p\right)$ of any point on a $p - \bar{t}$ curve, called the pollution profile, the value of p is the non-dimensional measure of the pollution level at real time $t = \tau\bar{t}$ years and the value of $t = \tau\bar{t}$ is the clean-up time in years for p% reduction in the pollution level; one unit of $\bar{t}$ is equivalent to τ years of real time. Also, the values of p close to 0 imply almost total clean-up while those close to 100 mean hardly any clean-up. The concentration of the lake pollutants corresponding to a definite value of p is readily known from $y(t) = \frac{p}{100} y_0$.

Figures 1-6 depict the response of non-dimensional clean-up time $\bar{t}$ corresponding to all pollution levels p, $0<p<100$ (and vice-versa) to variation in one of the parameters while the others are held fixed at their respective characteristic values; $a = 10$, $b = 0.5$, $c = 0$, $\bar{k} = 1$ and $x_0/y_0 = 5$ and $\bar{\omega} = 5$.

Figures 1 and 2 graphically depict the role of parameters b and $\bar{\omega}$ influencing volume fluctuations on the evolution of self-purification process.

Figures 3-6 have two parts each: Figure 3-6(i) and Figure 3-6(ii) refer to lakes of constant volume ($b = 0$) and variable volume ($0 < b < 1$) respectively.

In the discussion that follows, we assume the value of τ to be 20 years for the lake system under study; this is done for the sake of getting a definite sense of time that elapses before the desired pollution level is achieved.

Some main results inferred from these figures are as follows:

In a lake of constant volume, the pollution level p decreases monotonically with increase in $\bar{t}$, and p assumes each value in the interval $0 < b < 100$ only once. On the other hand, in a lake of variable volume ($0 < b < 1$), pollution profiles exhibit fluctuations in values of p induced by changing volume. Also, depending on the value of b, p can assume each value, lying in a definite interval, more than once.

The profile corresponding to $b = 0$ (Figure 1) shows 90% reduction in the value of p in about 30 years and total clean-up in 50 years. In a lake of variable volume, total clean-up ($p = 0$) is not attainable even in 100 years.

The percentage drop of slightly less than 40% in the value of p, attained in 10 years, is the same for all values of frequency $\bar{\omega}$ lying in the range 0.1-10. For small values of $\bar{\omega}(\bar{\omega} = 0.1)$, p, decreases monotonically with $\bar{t}$ and pollution levels $p<5$ are attainable in not less than 40 years. As anticipated from physical considerations, higher values of frequency are observed to induce fluctuations in the pollution level and retard the rate of purification.

Since larger values of the inflowing-pollution control parameter a imply a faster rate of decrease in the concentration of pollutants entering the lake, we expect a faster clean-up to be attained on a profile corresponding to a larger value of a; this fact is confirmed in Figure 3. In a lake of constant volume, the drop in pollution level in 20 years is about 35% for $a = 2.5$ compared to about 85% for $a = 25$ [Figure 3(i)]. Apparently, fastest clean-up will be achieved when the inflow is pollution-free For all values of ($a \to \infty$) in the range 2.5–2.5, pollution levels $p<10$ are achievable in a lake of variable volume in about 80 years [Figure 3(ii)].

It is clear from the definition of the precipitation-evaporation parameter c that $c \geq$ or < 0 implies that the rate of precipitation is greater, equal or less than the rate of evaporation r_e. Figure 4(i) shows that precipitation exceeding evaporation ($c = 0.5$) results in faster clean-up. Also for a value of c close to one but less than one, the pollution level is at its lowest value for all $\bar{t} > 0$. This is an expected result as $c \approx 1$ implies $r_p - r_e >> r_i$, which, in turn, describes a situation where the rate of inflow r_i is subdued by a long spell of very heavy downpours. On the other extreme, the limit $c \to -\infty$ describes a condition of severe drought when the rate of evaporation exceeds the rate of precipitation by the rate of inflow. In this eventuality, the pollution level will assume its highest value as corroborated in Figure 4.

The profiles in Figures 5 and 6 manifest a faster rate of purification for higher values of $\bar{k}$ and lower values of x_0/y_0 respectively. This outcome is expected because a stronger bio-chemical reaction and an inflow of lower pollution level will both result in reduced pollution in the lake.

Conclusions

We have presented a simple mathematical model which describes the natural self-purification process in a lake of variable volume. After having identified the primary dimensionless parameters and assuming their hypothetical values depending on the dimensional lake-specific characteristics, an attempt is made for theoretically analyzing the effect of these parameters on the evolving pollution level and the rate of self-purification. Though the predicted time for achieving the goal of partial or full recovery of an excessively defiled natural water body may run into many decades, yet it is hoped that this model will be applied to real situations after overcoming the hard task of successfully estimating the values of these parameters with reliable accuracy.

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RIVERS OF SOUTH ASIA
To Link or Not to Link

By V. Subramanian

The book addresses key issues related to the current questions that are important in the plans for linking various rivers in South Asia. The book tries to present views from both sides – for and against – and leave the reader to make his/her individual opinion and not swayed by either academic or non-Governmental views. Extrapolations have been made on possible implication of global warming and climate change on the water sector in the sub-continent. As far as possible, efforts have been made to make the reading simple by keeping out high level technical words. The primary question that is being addressed in this book is: "do we want a large number of individual and mighty river basins or a single 'SARC' river basin?"

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About the Author

Prof. V. Subramanian, after completing his PhD from U.S.A with a Fulbright Fellowship and teaching briefly at McGill University, Canada, joined Jawaharlal Nehru University in 1975. Since then he has been working on global rivers for over three decades; incidentally, all the students hostels in JNU are named after rivers of South Asia. More than 35 students did their PhD under him over the years and he has authored or co-authored more than 175 technical international publications as well as books. He set up academic activities on rivers in South Asia at JNU way back in 1975 and has since been associated with several international institutions/organizations either as member or guest faculty from time to time. At present he is working as Emeritus Fellow in Environmental Sciences at JNU.

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