An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.16, No.1, pp.321-348 (2026)
https://doi.org/10.36922/IJOCTA025330143

RESEARCH ARTICLE

A Hybrid LOPCOW–PROMETHEE framework under linear Diophantine fuzzy sets for sustainable planning

Jeevitha Kannan[1,2], Vimala Jayakumar [3*], Dragan Pamucar[4*], and S. Rajareega[5]

[1]Center for Computational Biology, SRM Institute of Science and Technology, Ramapuram, Chennai, India
[2]Center for Research, Easwari Engineering College, Ramapuram, Chennai, India
[3]Department of Mathematics, Alagappa University, Karaikudi, India
[4]Széchenyi István University, Győr, Hungary
[5]Department of Mathematics, Basic Sciences and Humanities, GMR Institute of Technology, Rajam, Andhra Pradesh, India
jeevithk1@srmist.edu.in, vimaljey@alagappauniversity.ac.in, dpamucar@gmail.com, rajareega.s@gmrit.edu.in

ARTICLE INFO

Article History:
Received: August 13, 2025
Revised: October 6, 2025
Accepted: October 24, 2025
Published Online: January 13, 2026

Keywords:
Linear Diophantine fuzzy set
Distance measure
LOPCOW
PROMETHEE
Sustainable emergency service station
Investment portfolio

AMS Classification 2010:
06D72, 03E72, 08A72, 15B15

ABSTRACT

Imprecision, uncertainty, and conflicting criteria often complicate the process of identifying the optimal conclusions in real-world decision-making scenarios. This paper suggests a novel multi-criteria decision-making (MCDM) framework that combines a hybrid logarithmic precursor chain-driven objective weighting–preference ranking organization method for enrichment of evaluations technique with linear Diophantine fuzzy sets to address uncertain frameworks. The selection of the location of a Sustainable Emergency Service Station and the selection of an investment portfolio are two real-world and socially significant decision-making challenges where traditional MCDM approaches fail to address the uncertainties. Our suggested fuzzy-based paradigm demonstrates the adaptability of both infrastructure design and financial decision-making. The results provide the optimal solutions based on our requirements, even under unpredictable conditions. The outcomes of sensitivity analysis and comparative analysis demonstrate how well the suggested approach handles ambiguous and imprecise data, particularly when expert opinions are presented in a linguistically or incompletely articulated manner. This work provides a solid, scalable, and precise method for resolving MCDM issues in the face of ambiguity, offering improved support to decision-makers in a range of fields.

1. Introduction

Data are frequently ambiguous, partial, or imprecise in many real-world decision-making situations because of human judgment, measurement constraints, or a lack of accurate information. This intrinsic ambiguity makes it more difficult to choose the best option, particularly in situations when there are linguistic or subjective evaluations involved. To overcome these difficulties, Zadeh [1] introduced fuzzy set theory that can handle ambiguity and partial truth in information. To better capture the subtleties of uncertainty, this fundamental concept has evolved to incorporate more sophisticated models. For instance, intuitionistic fuzzy set (IFS) [2] offers an intense modeling structure by including a value of hesitancy in addition to membership(MD) and non-membership degrees (ND). The domain available to decision-makers is later expanded by pythagorean fuzzy sets (PFS), which build upon IFS and permit the squared total of MD and ND to lie between 0 and 1. [3] Furthermore, the q-Rung orthopair fuzzy set

*Corresponding Author

(q-ROFS) was developed, [4] which provides more flexibility in high-uncertainty situations by allowing the total of the q-th power of MD and ND to remain within the interval [0,1].

The linear Diophantine fuzzy set (LDFS), the precursor of a peculiar fuzzy concept, overcomes all these constraints by incorporating reference parameters. [5] When traditional binary or crisp models are insufficient for MCDM, these fuzzy frameworks have proven widely applicable in medical diagnosis [6] and optimization challenges. [7] Within the linear Diophantine fuzzy (LDF) environment, Kannan et al. [8] proposed and applied the decision-making trial and evaluation laboratory (DEMATEL) technique in the theory of climate change. Jayakumar et al. [9] created the LDF-measurement of alternatives and ranking according to the compromise solution (MARCOS) method in addition to detailing its features and uses. Kannan et al. [10] utilized the LDF correlation coefficient to explain the LDFS clustering approach. Also, the LDF-combinative distance-based assessment (CODAS) approach was developed by Kannan et al. [11] for the logistic selection of specialists. To illustrate their relevance in situations where standard fuzzy logic lacks structural resilience, Zia et al. [12] proposed a complex LDFS for multi-attribute decision making issues. Similarly, to deal with weight uncertainty in multi-criteria decision-making (MCDM) contexts, Gul et al. [13] and Asif et al. [14] introduced aggregation operators for LDF environments and applied them in real-world case scenarios. Furthermore, the work of Sri et al. [15] explains the method for the diagnosis of lung cancer. A complex LDF environment was utilized for decision management in urban transportation, incorporating relationships among LDFS. [16] Aydodu et al. [17] demonstrate encouraging results when employing LDF variations in sustainable applications such as green supply chains and data storage system selection. LDF Dombi aggregation operators were introduced by Anjum et al. [18] in combination with the technique for order preference by similarity to ideal solution (TOPSIS) and VIKOR methods.

The Preference Ranking Organization METHOD for Enrichment of Evaluations (PROMETHEE) approach was developed by Brans et al. in 1986, where the decision-makers can analyze and rank complex options according to several criteria. [19] Experts can obtain prominent insights into the optimal course of action by using this method, which involves a systematic process of assessing alternatives and computing their overall rankings. PROMETHEE provides a versatile and effective framework for addressing decision issues in a variety of fields because of its ability to manage both quantitative and qualitative data. Xu and Li [20] introduced the fuzzy PROMETHEE to fully solve the constraints of the standard PROMETHEE method in tackling decision-making problems in a fuzzy environment. PROMETHEE was used in various aspects by several researchers for its applicability. [21] Recognizing the need for further improvements the framework [22] was later expanded to include a number of fuzzy extensions, which resulted in notable breakthroughs in decision-making analysis. [23] Specialized extensions of q-rung orthopair fuzzy PROMETHEE [24] were developed as a result of these extensions, which contained ideas like q-ROFS. The logarithmic precursor chain-driven objective weighting (LOPCOW) approach [25] addressed the MCDM limitations effectively. The LOPCOW method computes objective weights with logarithmic percentage changes to increase accuracy and objectivity. Furthermore, it can be combined with other approaches like analytic hierarchy process, simple additive weighting, DOmbi Bonferroni, etc. This method shows that the balance between subjective preferences and factual information produces precise and equitable outcomes. [26] Decisions made with LOPCOW are more precise, as it takes into account variations as well as changes in the scenario or environment being evaluated. [27] From resource allocation to alternative selection, this approach is particularly useful for addressing the dynamic aspects of decision-making across a variety of scenarios. The potential of the LOPCOW method to dynamically modify criteria weights in response to changes in objective data is one of its prominent merits. Moreover, to make the choice results more dependable, it LOPCOW can solve the issues of ambiguity and fluctuations that arise during the decision-making process.

There exists some hybrid MCDM approaches that combine PROMETHEE and LOPCOW with other techniques, such as LOPCOW–additive ratio assessment [28] and TOPSIS–PROMETHEE–VIKOR. [29] However, no work has incorporated LOPCOW–PROMETHEE within the LDF framework. Even though earlier research has used various approaches of LDF in other fields, it still lacks reliable weighing and ranking systems. The hybrid LOPCOW–PROMETHEE algorithm was introduced in the LDF domain to fill this methodological gap improves ranking quality and objectivity. The development of a new distance measure for LDFS is another noteworthy methodological advancement in this investigation. The suggested distance measure provides greater discriminative power among alternatives in attribute-rich decision-making. Additionally, it is more consistent with the mathematical framework of LDF, particularly in large-scale applications. Two real-world issues, such as sustainable emergency service station (SESS) placement selection and investment portfolio prioritization, have been used to validate the suggested methodology. Although classical fuzzy PROMETHEE models have been used in similar contexts, these methods have limited coverage of the decision space and domain limitations. By adding LDFS, the decision space is enlarged, allowing for more sophisticated prioritization.

1.1. Contribution and novelty of this study

This research addresses difficult decision-making situations, characterized by uncertainty, conflicting criteria, and limited data accuracy by introducing a novel hybrid LOPCOW-PROMETHEE strategy that integrates the LDF framework. The following is a summary of this work's inventions and contributions:

(i) The LDF set theory is combined with the LOPCOW and PROMETHEE approaches to create a new MCDM model. This combination uses the mathematical accuracy of LDF sets to address ambiguity in decision matrices, while PROMETHEE offers a trustworthy ranking system and LOPCOW objectively determines criteria weights.

(ii) This study is the first that we are aware of that uses LDF numbers in a ranking framework based on LOPCOW–PROMETHEE, which allows for improved handling of both quantitative and qualitative criteria under uncertainty.

(iii) A novel distance metric is put forth to assess the differences of LDF, tackling the shortcomings of current metrics in encapsulating the arithmetic and structural characteristics of LDF numbers. This new metric improves comparison accuracy in fuzzy situations and strengthens the mathematical underpinnings of LDF-based MCDM.

(iv) The proposed model is validated through two impactful case studies:
- SESS location selection, involving rapid response and infrastructure deployment under constrained resources.
- Sustainable Investment Portfolio Selection (SIPS) aiming at optimized returns under uncertainty in the market values and constraints.

These examples demonstrate the flexibility and relevance of the proposed approach across diverse decision-making domains.

(v) Beyond methodological validation, it reflects actual societal requirements like urban safety, where well-informed decision-making helps reduce hazards and improve public welfare, and climate resilience, where effective resource allocation supports environmental uncertainties.

(vi) The study emphasizes the usefulness of the suggested approach in tackling both theoretical and practical issues by incorporating these viewpoints.

1.2. Research gap and motivation

Although numerous traditional methods that adopt various MCDM approaches, only a few studies have focused on hybrid algorithms, in which one algorithm is used to determine criteria weights and another to rank alternatives. In objective weighting, LOPCOW is widely regarded for its effectiveness and dependability. However, based on the literature review, it is evident that no previous study has applied LOPCOW in extended fuzzy environment's such as the LDFS. Hence, the present study presents a novel framework that takes advantage of LOPCOW and PROMETHEE to bridge this gap. In our suggested MCDM, we employ LDF as earlier LOPCOW and PROMETHEE methods with conventional fuzzy sets fail to handle certain situations due to constraints on range spaces.

The advantages of this integration are that the LDF framework ensures an extended space for degree values, allowing higher precision; LOPCOW provides an objective and impartial weighting mechanism; and PROMETHEE delivers a robust MCDM algorithm for ranking with strong discriminative ability. Altogether, these three features create a more comprehensive and robust decision-making framework. Moreover, the previous LDF algorithms, such as LDF-DEMATEL, LDF-MARCOS, LDF-TOPSIS, and LDF-CODAS, have not explored the hybridization of objective weighting with ranking methods.

They often rely on either subjective weighting or single algorithms. Thus, the absence of such hybrid approaches motivates the development on the hybrid LOPCOW–PROMETHEE–LDF algorithm.

2. Preliminaries

Definition 1. [1] *The fuzzy set $\mathfrak{A}$ is a mapping of $\mathfrak{x} : \Lambda \to [0,1]$ and it is depicted as*

$$\mathfrak{A} = \{(\mathfrak{k}, \mathfrak{x}(\mathfrak{k})) : \mathfrak{k} \in \Lambda\}$$

where $\mathfrak{x}(\mathfrak{k})$ is a membership grade ($\mathbb{MG}$) and it lies between 0 to 1.

Subsequently, the fuzzy set (FS) was expanded to include a non-membership grade (NMG) and it was termed an IFS, which has the following definition:

Definition 2. [2] *The IFS $\mathfrak{B}$ is depicted as*

$$\mathfrak{I} = \{(\mathfrak{k}, \mathfrak{x}(\mathfrak{k}), \mathfrak{y}(\mathfrak{k})) : \mathfrak{k} \in \Lambda\}$$

where $\mathfrak{x}(\mathfrak{k})$, $\mathfrak{y}(\mathfrak{k}) \in [0,1]$ are $\mathbb{MG}$ and $\mathbb{NMG}$, respectively. The restriction on grades is $0 < \mathfrak{x}(\mathfrak{k}) + \mathfrak{y}(\mathfrak{k}) < 1$.

It fails when MG and NMG sum to more than one. To address this, pythagorean fuzzy sets and q-ROFS were implemented. LDFS was presented by Riaz and Hashmi [5] as an expansion of q-ROFS that included reference parameters.

Definition 3. [5] *A linear Diophantine fuzzy set $\mathfrak{L}$ on Λ is depicted as*

$$\mathfrak{L} = \{(\mathfrak{k}, \langle \mathfrak{x}(\mathfrak{k}), \mathfrak{y}(\mathfrak{k})\rangle, \langle \tau(\mathfrak{k}), \upsilon(\mathfrak{k})\rangle) : \mathfrak{k} \in \Lambda\}$$

where $\mathfrak{x}(\mathfrak{k}), \mathfrak{y}(\mathfrak{k}), \tau(\mathfrak{k}), \upsilon(\mathfrak{k}) \in$ [0, 1] are $\mathbb{MG}$, $\mathbb{NMG}$, and reference parameters, respectively. These degrees have a restriction that $0 \leq \tau(\mathfrak{k})\mathfrak{x}(\mathfrak{k}) + \upsilon(\mathfrak{k})\mathfrak{y}(\mathfrak{k}) \leq 1$ for all $\mathfrak{k} \in \Lambda$ with $0 \leq \tau(\mathfrak{k}) + \upsilon(\mathfrak{k}) \leq 1$. The hesitancy degree is $\mathfrak{w} = 1 - (\mathfrak{x}(\mathfrak{k})\tau(\mathfrak{k}) + \mathfrak{y}(\mathfrak{k})\upsilon(\mathfrak{k}))$.

Definition 4. [5] *The term $\mathfrak{E} = \langle \mathfrak{x}, \mathfrak{y}\rangle, \langle \tau, \upsilon\rangle$ is stated to be a linear Diophantine fuzzy number (LDFN) with $0 \leq \tau\mathfrak{x} + \upsilon\mathfrak{y} \leq 1$ and $0 \leq \tau + \upsilon \leq 1$.*

Definition 5. [5] *Let $\mathfrak{E} = \langle \mathfrak{x}, \mathfrak{y}\rangle, \langle \tau, \upsilon\rangle$ be the LDFN over Λ. Then the score function of LDFN is depicted as*

$$\mathfrak{S}(\mathfrak{E}) = \frac{1}{2}\Big[(\mathfrak{x}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{E}}) + (\tau_{\mathfrak{E}} - \upsilon_{\mathfrak{E}})\Big] \tag{1}$$

Definition 6. [5] *Let $\mathfrak{E} = \langle \mathfrak{x}, \mathfrak{y}\rangle, \langle \tau, \upsilon\rangle$ be the LDFN over Λ. Then the accuracy function of LDFN is depicted as*

$$\mathfrak{A}(\mathfrak{E}) = \frac{1}{2}\Big[(\frac{\mathfrak{x}_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{E}}}{2}) + (\tau_{\mathfrak{E}} + \upsilon_{\mathfrak{E}})\Big] \tag{2}$$

Proposition 1. [5] *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two LDFN over Λ. Then the following axioms hold:*

(1) If $\mathfrak{S}(\mathfrak{E}_1) < \mathfrak{S}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 < \mathfrak{E}_2$
(2) If $\mathfrak{S}(\mathfrak{E}_1) > \mathfrak{S}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 > \mathfrak{E}_2$
(3) If $\mathfrak{S}(\mathfrak{E}_1) = \mathfrak{S}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 = \mathfrak{E}_2$
(4) If $\mathfrak{A}(\mathfrak{E}_1) < \mathfrak{A}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 < \mathfrak{E}_2$
(5) If $\mathfrak{A}(\mathfrak{E}_1) > \mathfrak{A}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 > \mathfrak{E}_2$
(6) If $\mathfrak{A}(\mathfrak{E}_1) = \mathfrak{A}(\mathfrak{E}_2)$, then $\mathfrak{E}_1 = \mathfrak{E}_2$

Gül and Aydoğdu [30] proposed the distance measures between two LDFN $\mathfrak{E}$ and $\mathfrak{P}$ as follows:
Hamming distance:

$$\mathfrak{d}_1 = \frac{1}{4}\big(|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}| + |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}| + |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}| + |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}|\big)$$

Euclidean distance:

$$\mathfrak{d}_2 = \sqrt{\frac{1}{4}\big(|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}|^2 + |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}|^2 + |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}|^2 + |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}|^2\big)} \tag{3}$$

Kamaci et al. [31] introduced the Minkowski and Chebyshev distance measures as follows:
Minkowski distance:

$$\mathfrak{d}_3 = \frac{1}{4}\big(|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}|^L + |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}|^L + |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}|^L + |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}|^L\big)^{\frac{1}{L}} (L \geq 1) \tag{4}$$

Chebyshev distance:

$$\mathfrak{d}_4 = max\,\{|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}|, |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}|, |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}|, |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}|\}$$

Aydoğdu et al. [32] defined a new distance measure for LDFN as follows:

$$\mathfrak{d}_5 = \frac{1}{4}\big(|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}| + |\mathfrak{y}_{\mathfrak{P}}(1 - \mathfrak{y}_{\mathfrak{P}}) - \mathfrak{y}_{\mathfrak{E}}(1 - \mathfrak{y}_{\mathfrak{E}})| + |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}| + |\upsilon_{\mathfrak{P}}(1 - \upsilon_{\mathfrak{P}}) - \upsilon_{\mathfrak{E}}(1 - \upsilon_{\mathfrak{E}})|\big) \tag{5}$$

Based on the distance measures of LDFS introduced by Aydogdu, [32] we derive the distance measure for LDFN as follows:

$$\mathfrak{d}_6 = \frac{1}{8}\big(|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}| + |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}| + |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}| + |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}| + 4max\,\{|\mathfrak{x}_{\mathfrak{E}} - \mathfrak{x}_{\mathfrak{P}}|, |\mathfrak{y}_{\mathfrak{E}} - \mathfrak{y}_{\mathfrak{P}}|, |\tau_{\mathfrak{E}} - \tau_{\mathfrak{P}}|, |\upsilon_{\mathfrak{E}} - \upsilon_{\mathfrak{P}}|\}\big) \tag{6}$$

Assume $J = \langle 0.70, 0.17\rangle, \langle 0.82, 0.09\rangle$, $K_1 = \langle 0.57, 0.27\rangle, \langle 0.70, 0.22\rangle$, and $K_2 = \langle 0.82, 0.04\rangle, \langle 0.72, 0.22\rangle$ be three LDFN.

Table 1. Comparison of existing distance measures of linear Diophantine fuzzy set

DM	(J, K_1)	(J, K_2)
$\mathfrak{d}_1$	0.12	0.12
$\mathfrak{d}_2$	0.1206	0.1206
$\mathfrak{d}_3$	0.1212	0.1212
$\mathfrak{d}_4$	0.13	0.13
$\mathfrak{d}_6$	0.125	0.125

Table 1 shows that the distance measures $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{d}_4$, and $\mathfrak{d}_6$. These measures fail when $\mathfrak{d}_i(J, K_1)$ and $\mathfrak{d}_i(J, K_2)$(i=1,2,3,4,6) produce the same output for different sets K_1 and K_2. Moreover, for the case $L_1 = \langle 0.5, 0.6\rangle, \langle 0.3, 0.7\rangle$ and $L_1 = \langle 0.5, 0.4\rangle, \langle 0.3, 0.3\rangle$, the distance measure $\mathfrak{d}_5$ between L_1 and $\mathfrak{L}_2$ is zero, but $L_1 \neq \mathfrak{L}_2$. The distance measure $\mathfrak{d}_5$ fails in this case. In the following section, we address a unique distance metric based on the Jensen–Shannon divergence measure to overcome such problems.

3. New novel distance measures of linear Diophantine fuzzy set

In this section, we propose two new distance measures for LDFS by utilizing the Jensen–Shannon divergence. We demonstrate their properties and theorems to ensure the dependability.

Definition 7. *Assume that*

$$\mathfrak{E} = \langle \mathfrak{x}_{\mathfrak{E}}, \mathfrak{y}_{\mathfrak{E}}\rangle, \langle \tau_{\mathfrak{E}}, \upsilon_{\mathfrak{E}}\rangle and \mathfrak{P} = \langle \mathfrak{x}_{\mathfrak{P}}, \mathfrak{y}_{\mathfrak{P}}\rangle, \langle \tau_{\mathfrak{P}}, \upsilon_{\mathfrak{P}}\rangle$$

be two LFDN with hesitance $\mathfrak{w}_{\mathfrak{E}} = 1 - (\mathfrak{x}_{\mathfrak{E}}\tau_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{E}}\upsilon_{\mathfrak{E}})$ and $\mathfrak{w}_{\mathfrak{P}} = 1 - (\mathfrak{x}_{\mathfrak{P}}\tau_{\mathfrak{P}} + \mathfrak{y}_{\mathfrak{P}}\upsilon_{\mathfrak{P}})$, respectively. The $\mathfrak{J}(\mathfrak{E}, \mathfrak{P})$ can also be formulated as:

$$\mathfrak{J}(\mathfrak{E}, \mathfrak{P}) = \frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}} + \mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}} log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}} + \mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}} + \tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}} + \tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}} + \upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}} + \upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}} + \mathfrak{w}_{\mathfrak{P}}} + \\ \mathfrak{w}_{\mathfrak{P}} log\dfrac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}} + \mathfrak{w}_{\mathfrak{P}}} \end{array}\right\} \tag{7}$$

Definition 8. *Let $\mathfrak{E}$ and $\mathfrak{P}$ be two LDFN over the universe Λ. Then, the distance measure between $\mathfrak{E}$ and $\mathfrak{P}$ is depicted as*

$$\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) = \sqrt{\mathfrak{J}(\mathfrak{E}, \mathfrak{P})}$$

Theorem 1. *Consider that $\mathfrak{E}$ and $\mathfrak{P}$ be two LDFNs over the universe Λ.*

$$\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) = \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}} + \mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}} log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}} + \mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}} + \tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}} + \tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}} + \upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}} + \upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}} + \mathfrak{w}_{\mathfrak{P}}} + \\ \mathfrak{w}_{\mathfrak{P}} log\dfrac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}} + \mathfrak{w}_{\mathfrak{P}}} \end{array}\right\}\right]^{\frac{1}{2}}$$

be the distance measure between $\mathfrak{E}$ and $\mathfrak{P}$. Then, $\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P})$ hold the following properties:

(P1) Non-degeneracy: $\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) = 0$ iff $\mathfrak{E} = \mathfrak{P}$.
(P2) Symmetry: $\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) = \mathfrak{D}_1(\mathfrak{P}, \mathfrak{E})$
(P3) Boundedness: $0 \leq \mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) \leq 1$.
(P4) Triangular Inequality: $\mathfrak{D}_1(\mathfrak{Q}, \mathfrak{E}) + \mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) \geq \mathfrak{D}_1(\mathfrak{Q}, \mathfrak{P})$

Proof. (P1) $\mathfrak{D}_1(\mathfrak{E}, \mathfrak{P}) = 0$ if $\mathfrak{E} = \mathfrak{P}$.
Assuming that $\mathfrak{E}$ and $\mathfrak{P}$ be two LDFN on Λ, if $\mathfrak{E} = \mathfrak{P}$ it follows:

$$\begin{aligned} \mathfrak{x}_{\mathfrak{E}} &= \mathfrak{x}_{\mathfrak{P}}, \mathfrak{y}_{\mathfrak{E}} = \mathfrak{y}_{\mathfrak{P}}, \tau_{\mathfrak{E}} = \tau_{\mathfrak{P}} \\ \upsilon_{\mathfrak{E}} &= \upsilon_{\mathfrak{P}}, \mathfrak{w}_{\mathfrak{E}} = \mathfrak{w}_{\mathfrak{P}} \end{aligned}$$

Then, $\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})$ becomes:

$$\mathfrak{D}_1(\mathfrak{E},\mathfrak{P}) = \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{E}}} + \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{E}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{E}}} + \\ \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{E}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{E}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{E}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{E}}} + \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{E}}} + \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{E}}} + \\ \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{E}}} \end{array}\right\}\right]^{\frac{1}{2}} = 0$$

In addition, if $\mathfrak{D}_1(\mathfrak{E},\mathfrak{P}) = 0$, it implies

$$\begin{aligned} \mathfrak{x}_{\mathfrak{E}} &= \mathfrak{x}_{\mathfrak{P}}, \mathfrak{y}_{\mathfrak{E}} = \mathfrak{y}_{\mathfrak{P}}, \tau_{\mathfrak{E}} = \tau_{\mathfrak{P}} \\ \upsilon_{\mathfrak{E}} &= \upsilon_{\mathfrak{P}}, \mathfrak{w}_{\mathfrak{E}} = \mathfrak{w}_{\mathfrak{P}} \end{aligned}$$

Proof. (P2). $\mathfrak{D}_1(\mathfrak{E},\mathfrak{P}) = \mathfrak{D}_1(\mathfrak{P},\mathfrak{E})$

$$\mathfrak{D}_1(\mathfrak{E},\mathfrak{P}) = \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}} log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} + \\ \mathfrak{w}_{\mathfrak{P}} log\dfrac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} \end{array}\right\}\right.$$

$$= \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{P}} log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{E}}} + \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{P}} log\dfrac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} + \\ \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} \end{array}\right\}\right]^{\frac{1}{2}} = \mathfrak{D}_1(\mathfrak{P},\mathfrak{E}) \tag{8}$$

Proof. (P3) $0 \leq \mathfrak{D}_1(\mathfrak{E},\mathfrak{P}) \leq 1$.

$$\mathfrak{D}(\mathfrak{E},\mathfrak{P}) = \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}}(\mathfrak{k}) log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\dfrac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} + \\ \mathfrak{w}_{\mathfrak{P}} log\dfrac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} \end{array}\right\}\right]^{\frac{1}{2}}$$

$$\mathfrak{D}(\mathfrak{E},\mathfrak{P})=\left[\frac{1}{2}\begin{Bmatrix}(\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{P})\Big[\frac{\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}+\frac{\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}\Big]+\\(\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{P})\Big[\frac{\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}}+\frac{\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}}\Big]+\\(\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}})\Big[\frac{\tau_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}}+\frac{\tau_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}}\Big]+\\(\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}})\Big[\frac{\upsilon_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}}+\frac{\upsilon_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}}\Big]+\\(\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}\Big[\frac{\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}}+\frac{\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}}log\frac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}}\Big]\end{Bmatrix}\right]^{\frac{1}{2}}$$

$$=\left[\frac{1}{2}(\mathfrak{E}+\mathfrak{P})(1-H(\frac{\mathfrak{E}}{\mathfrak{E}+\mathfrak{P}}-\frac{\mathfrak{P}}{\mathfrak{E}+\mathfrak{P}}))\right]^{\frac{1}{2}}.$$

From, Ref. [24] for $0\leq\phi\leq1$, $1-H(\phi,1-\phi)\leq|\phi-(1-\phi)|$.

Thus, we obtain $\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})\leq\left[\frac{1}{2}(\mathfrak{E}+\mathfrak{P})|\frac{\mathfrak{E}}{\mathfrak{E}+\mathfrak{P}}-\frac{\mathfrak{P}}{\mathfrak{E}+\mathfrak{P}}|\right]^{\frac{1}{2}}=\left[\frac{1}{2}V(\mathfrak{E},\mathfrak{P})\right]^{\frac{1}{2}}$,

where V denotes the variational distance. From the results of, Ref. [24] $0\leq V(\mathfrak{E},\mathfrak{P})\leq2$. Therefore, we get:

$$0\leq\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})\leq1$$

Proof. (P4) $\mathfrak{D}_1(\mathfrak{Q},\mathfrak{E})+\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})\geq\mathfrak{D}_1(\mathfrak{Q},\mathfrak{P})$

Consider four assumptions as follows:

$\mathbb{A}1$: $\mathfrak{x}_{\mathfrak{Q}}\leq\mathfrak{x}_{\mathfrak{E}}\leq\mathfrak{x}_{\mathfrak{P}}$

$\mathbb{A}2$: $\mathfrak{x}_{\mathfrak{P}}\leq\mathfrak{x}_{\mathfrak{E}}\leq\mathfrak{x}_{\mathfrak{Q}}$

$\mathbb{A}3$: $\mathfrak{x}_{\mathfrak{E}}\leq min\{\mathfrak{x}_{\mathfrak{Q}},\mathfrak{x}_{\mathfrak{P}}\}$

$\mathbb{A}4$: $\mathfrak{x}_{\mathfrak{E}}\geq max\{\mathfrak{x}_{\mathfrak{Q}}\leq\mathfrak{x}_{\mathfrak{P}}\}$

By Assumptions $\mathbb{A}1$ and $\mathbb{A}2$, the triangular inequality is fulfilled, where $|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}}|\leq|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}|\leq|\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}|$.
Using $\mathbb{A}3$, we have $\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}\geq0$ and $\mathfrak{x}_{\mathfrak{P}}-\mathfrak{x}_{\mathfrak{E}}\geq0$.
Then:

$$\begin{aligned}&|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}|+|\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}|+|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}}|\\&\qquad=\begin{cases}\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}-\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{Q}}+\mathfrak{x}_{\mathfrak{P}} & if\ \mathfrak{x}_{\mathfrak{Q}}\geq\mathfrak{x}_{\mathfrak{P}}\\\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}-\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}} & if\ \mathfrak{x}_{\mathfrak{Q}}\leq\mathfrak{x}_{\mathfrak{P}}\end{cases}\\&\qquad=2(min\{\mathfrak{x}_{\mathfrak{Q}},\mathfrak{x}_{\mathfrak{P}}\}-\mathfrak{x}_{\mathfrak{E}})\geq0\end{aligned}$$

Under Assumption $\mathbb{A}4$, we conclude that,

$$\begin{aligned}&|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}|+|\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}|+|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}}|\\&\qquad=\begin{cases}\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{Q}}+\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}-\mathfrak{x}_{\mathfrak{Q}}+\mathfrak{x}_{\mathfrak{P}} & if\ \mathfrak{x}_{\mathfrak{Q}}\geq\mathfrak{x}_{\mathfrak{P}}\\\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{Q}}+\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}+\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}} & if\ \mathfrak{x}_{\mathfrak{Q}}\leq\mathfrak{x}_{\mathfrak{P}}\end{cases}\\&\qquad=2(\mathfrak{x}_{\mathfrak{E}}-max\{\mathfrak{x}_{\mathfrak{Q}},\mathfrak{x}_{\mathfrak{P}}\})\geq0.\end{aligned}$$

Thus, the triangle inequality is satisfied under Assumptions $\mathbb{A}3$ and $\mathbb{A}4$, where $|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{P}}|\leq|\mathfrak{x}_{\mathfrak{Q}}-\mathfrak{x}_{\mathfrak{E}}|\leq|\mathfrak{x}_{\mathfrak{E}}-\mathfrak{x}_{\mathfrak{P}}|$.
Similarly, we can obtain that:

$$|\mathfrak{y}_{\mathfrak{Q}}-\mathfrak{y}_{\mathfrak{P}}|\leq|\mathfrak{y}_{\mathfrak{Q}}-\mathfrak{y}_{\mathfrak{E}}|\leq|\mathfrak{y}_{\mathfrak{E}}-\mathfrak{y}_{\mathfrak{P}}|.$$
$$|\tau_{\mathfrak{Q}}-\tau_{\mathfrak{P}}|\leq|\tau_{\mathfrak{Q}}-\tau_{\mathfrak{E}}|\leq|\tau_{\mathfrak{E}}-\tau_{\mathfrak{P}}|.$$
$$|\upsilon_{\mathfrak{Q}}-\upsilon_{\mathfrak{P}}|\leq|\upsilon_{\mathfrak{Q}}-\upsilon_{\mathfrak{E}}|\leq|\upsilon_{\mathfrak{E}}-\upsilon_{\mathfrak{P}}|.$$
$$|\mathfrak{w}_{\mathfrak{Q}}-\mathfrak{w}_{\mathfrak{P}}|\leq|\mathfrak{w}_{\mathfrak{Q}}-\mathfrak{w}_{\mathfrak{E}}|\leq|\mathfrak{w}_{\mathfrak{E}}-\mathfrak{w}_{\mathfrak{P}}|.$$

Hence, $\mathfrak{D}_1(\mathfrak{Q},\mathfrak{E})+\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})\geq\mathfrak{D}_1(\mathfrak{Q},\mathfrak{P})$.

Definition 9. *Let $\mathfrak{E}$ and $\mathfrak{P}$ be two LDFN over the universe Λ. Then, the distance measure between $\mathfrak{E}$ and $\mathfrak{P}$ is depicted as:*

$$\mathfrak{D}_2(\mathfrak{E},\mathfrak{P}) = \left[\frac{1}{2}\left\{\begin{array}{l} \mathfrak{x}_{\mathfrak{E}} log\dfrac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}} log\dfrac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\dfrac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \\ \mathfrak{y}_{\mathfrak{P}} log\dfrac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\dfrac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\dfrac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \\ \upsilon_{\mathfrak{E}} log\dfrac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\dfrac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} \end{array}\right\}\right]^{\frac{1}{2}}$$

Theorem 2. *Consider that $\mathfrak{E}$ and $\mathfrak{P}$ be two LDFN over the universe Λ. Let $\mathfrak{D}_2(\mathfrak{E},\mathfrak{P})$ be the distance measure between $\mathfrak{E}$ and $\mathfrak{P}$. Then, $\mathfrak{D}_1(\mathfrak{E},\mathfrak{P})$ holds the following properties:*

(P1) Non − degeneracy : $\mathfrak{D}_2(\mathfrak{E},\mathfrak{P}) = 0$ *iff* $\mathfrak{E} = \mathfrak{P}$.
(P2) Symmetry : $\mathfrak{D}_2(\mathfrak{E},\mathfrak{P}) = \mathfrak{D}_2(\mathfrak{P},\mathfrak{E})$
(P3) Boundedness : $0 \leq \mathfrak{D}_2(\mathfrak{E},\mathfrak{P}) \leq 1$.
(P4) TriangularInequality : $\mathfrak{D}_2(\mathfrak{Q},\mathfrak{E}) + \mathfrak{D}_2(\mathfrak{E},\mathfrak{P}) \geq \mathfrak{D}_2(\mathfrak{Q},\mathfrak{P})$.

Proof. The proof is similar to the previous theorem.

Example 1. *Let*

$$\begin{aligned} \mathfrak{E}_1 &= \langle 0.6, 0.5\rangle, \langle 0.2, 0.8\rangle, \\ \mathfrak{E}_2 &= \langle 0.7, 0.4\rangle, \langle 0.5, 0.4\rangle, \\ \mathfrak{E}_3 &= \langle 0.6, 0.8\rangle, \langle 0.3, 0.4\rangle, \end{aligned}$$

Then, the distance measures
$\mathfrak{D}_1(\mathfrak{E}_1,\mathfrak{E}_2) = 0.050927$, $\mathfrak{D}_2(\mathfrak{E}_1,\mathfrak{E}_2) = 0.050927$;
$\mathfrak{D}_1(\mathfrak{E}_1,\mathfrak{E}_3) = 0.06026$; $\mathfrak{D}_2(\mathfrak{E}_1,\mathfrak{E}_3) = 0.037536$;
$\mathfrak{D}_1(\mathfrak{E}_2,\mathfrak{E}_3) = 0.064884$; $\mathfrak{D}_2(\mathfrak{E}_2,\mathfrak{E}_3) = 0.04216$.

4. Hybrid LOPCOW–PROMETHEE algorithm under the linear Diophantine fuzzy model

In this section, we discuss the LOPCOW and PROMETHEE methods, and we define a hybrid algorithm of LOPCOW-PROMETHEE for LDFS. The diagrammatic representation of LDF-LOPCOW-PROMETHEE is shown in Figure 1.

Algorithm:

Step 1: Assemble the decision matrix.
Begin by formulating a decision matrix that captures all available alternatives and their corresponding performance values under each criterion.

$$\mathfrak{M} = [\mathfrak{m}_{rs}]$$

$$\mathfrak{M} = \begin{bmatrix} \mathfrak{m}_{11} & \mathfrak{m}_{12} & \dots & \mathfrak{m}_{1s} \\ \mathfrak{m}_{21} & \mathfrak{m}_{22} & \dots & \mathfrak{m}_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{m}_{r1} & \mathfrak{m}_{r2} & \dots & \mathfrak{m}_{rs} \end{bmatrix}$$

where $\mathfrak{m}_{rs} = \langle \mathfrak{x}_{rs}, \mathfrak{y}_{rs}\rangle, \langle \tau_{rs}, \upsilon_{rs}\rangle$ is an LDFN of the alternative $\mathfrak{k}_r$ under criteria $\mathfrak{g}_s$.

Step 2: Normalize the decision matrix.
Normalize the values in the decision matrix to bring them onto a common scale, ensuring comparability across different criteria that may be of cost or benefit types.

$$\mathfrak{N} = [\mathfrak{n}_{rs}]$$

where,

$$\mathfrak{n}_{rs} = \begin{cases} \mathfrak{m}_{rs} & if\ s\ is\ benefit\ criteria, \\ \mathfrak{m}_{rs}^c & if\ s\ is\ cost\ criteria \end{cases}$$

Step 3: Determine the difference between the evaluation values.
Determine the difference between the assessment values of each pair of options. This phase helps

determine the degree to which one option deviates from another on each criterion.

$$\mathfrak{d}_s(\mathfrak{k}_r, \mathfrak{k}_t) = \begin{cases} \mathfrak{D}(\mathfrak{n}_{rs}, \mathfrak{n}_{ts}) & if\ \mathfrak{n}_{rs} > \mathfrak{n}_{ts}, \\ 0 & if\ \mathfrak{n}_{rs} < \mathfrak{n}_{ts} \end{cases}$$

where $\mathfrak{D}(\mathfrak{n}_{rs}, \mathfrak{n}_{ts})$ is the distance between two LDFN $\mathfrak{n}_{rs}$ and $\mathfrak{k}_{ts}$.
Step 4: Define the Performance Function.
Using the deviation values determined in the previous step, create a performance function that measures how strongly one alternative is preferred over another.

$$\mathfrak{W}_s(\mathfrak{k}_r, \mathfrak{k}_t) = f(\mathfrak{d}_s(\mathfrak{k}_r, \mathfrak{k}_t))$$

Step 5: Compute the weight of each criterion using the LDF–LOPCOW method as follows:
Step 5.1: Assemble an evaluation matrix.
Consider the evaluation matrix $\mathbb{E}$ of LDFNs, with alternatives in rows and criteria in columns.

$$\mathbb{E} = [\epsilon_{qs}]_{p\times m}$$

$$\mathcal{C} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \dots & \epsilon_{1s} \\ \epsilon_{21} & \epsilon_{22} & \dots & \epsilon_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_{r1} & \epsilon_{r2} & \dots & \epsilon_{qs} \end{bmatrix}$$

where $\epsilon_{qs} = \langle \mathfrak{x}_{qs}, \mathfrak{y}_{qs} \rangle, \langle \tau_{qs}, \upsilon_{qs} \rangle$ is an LDFN (q=1,2,3,..,p), (s = 1,2,3,...,m).
Step 5.2: Obtain an accuracy matrix.
By utilizing Equation (2), find an accuracy value of each LDFN

$$\mathbb{A} = [\rho_{qs}]_{p\times m}$$

where $\rho_{rs} = \mathfrak{A}(\epsilon_{qs})$.
Step 5.3: Compute the normalized accuracy matrix.
Normalize each elements by utilizing the formula:

$$\tilde{\mathbb{A}} = [\alpha_{qs}]_{p\times m}$$

where

$$\alpha_{ij} = \begin{cases} \frac{\rho_{qs}-\rho_{min}}{\rho_{max}-\rho_{min}} & if\ s \in\ benefit \\ \frac{\rho_{max}-\rho_{qs}}{\rho_{max}-\rho_{min}} & if\ s \in\ cost \end{cases} \tag{9}$$

Step 5.4: Determine the percentage value(δ_s) for each criterion.
In order to lessen the effects of extreme values, the log measure is calculated in this stage, which also determines the mean square value and the percentage of variance. For this, LOPCOW is utilized.

$$\delta_s = ln|\frac{\sqrt{\frac{\sum_{q=1}^{p}\alpha_{qs}^2}{p}}}{\sigma_s}| \times 100 \tag{10}$$

where

$$\sigma_s = \sqrt{\frac{1}{p-1}\sum_{i=1}^{p}(\alpha_{qs} - \frac{1}{m}\sum_{f=1}^{p}\alpha_{fs})^2}$$

Step 5.5: Calculate the objective weights of criteria as:

$$\omega_s = \frac{\delta_s}{\sum_{s=1}^{m}\delta_s} \tag{11}$$

Step 6: Obtain the global preference matrix.
Aggregate the performance function values across all criteria to generate a global preference matrix, which reflects the overall preference of one alternative over another.

$$\Pi(\mathfrak{k}_r, \mathfrak{k}_t) = \sum_{s=1}^{m}\omega_s\mathfrak{W}_s(\mathfrak{k}_r, \mathfrak{k}_t)$$

where ω_s is a weight for each criterion obtained by the LOPCOW method.
Step 7: Compute the leaving flow $\psi^+(\mathfrak{k}_r)$ and entering flow $\psi^-(\mathfrak{k}_r)$.

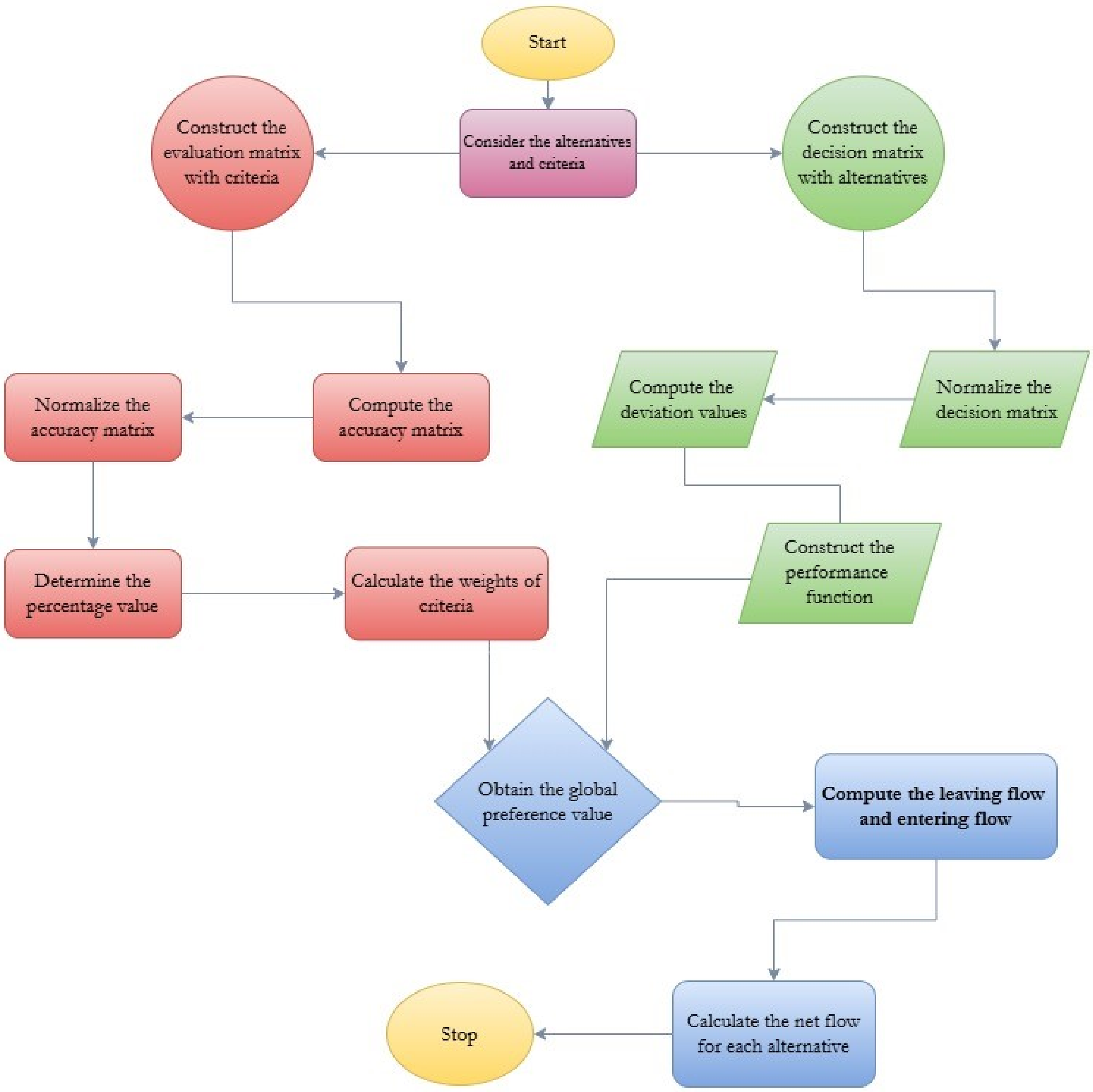

Figure 1. Schematic representation of LDF–LOPCOW-PROMETHEE.
Abbreviations: LDF: linear Diophantine fuzzy; LOPCOW: Logarithmic precursor chain-driven objective weighting; PROMETHEE: Preference ranking organization method for enrichment of evaluations.

Calculate the leaving (or positive) flow and entering (or negative) flow for each alternative. The leaving flow indicates the strength of preference of an alternative over all others, while the entering flow shows how much others dominate the alternative.

$$\psi^{+}(\mathfrak{l}_r) = \frac{1}{m-1} \sum_{\mathfrak{k} \in \Lambda} \Pi(\mathfrak{k}_r, \mathfrak{k})$$

$$\psi^{-}(\mathfrak{k}_r) = \frac{1}{m-1} \sum_{\mathfrak{k} \in \Lambda} \Pi(\mathfrak{k}, \mathfrak{k}_r)$$

Step 8: Determine the Net Flow and Establish the Final Ranking. Compute the net flow for each alternative by subtracting the entering flow from the leaving flow. Rank the alternatives based on their net flow values.

$$\psi(\mathfrak{k}_r) = \psi^{+}(\mathfrak{k}_r) - \psi^{-}(\mathfrak{k}_r)$$

The higher net flows indicate better performance.

5. Case study: Sustainable location selection for emergency service stations

The selection of appropriate locations for emergency service stations, such as fire stations, ambulance depots, and disaster relief centers, is a critical component of urban and regional planning. The physical location of the station has a direct impact on the speed at which services can reach impacted communities, which is a critical component of emergency response operations. Stations located in poor areas may result in more casualties, slower response times, and wasteful use of public funds. Thus, organizing locations strategically is essential to improving public safety, particularly in densely populated urban regions with complicated traffic patterns. The process of planning a sustainable strategic position of emergency service stations can improve coverage, save lives, prevent property damage, and significantly reduce response times. The selected location must satisfy operational needs as well as adhere to sustainability standards in light of increasing urbanization, climate change, and resource limitations. A sustainable location selection process strikes a balance between cost, accessibility, environmental impact, and long-term efficiency to urban and regional planners.
The concept of sustainability Ref. [33] emerged highly in recent years in the field of design and administration of public infrastructure, like emergency service stations. In selecting a location, a sustainable approach considers long-term aspects, particularly social equality, economic viability, and environmental impact. This ensures the minimization of ecological disturbance and economic strain in optimal sites. The method becomes more resilient and progressive by integrating sustainability into planning, as it improves adaptation to future urban expansion.

Description of criteria:

(1) Accessibility to High-Risk Areas ($\mathfrak{H}_1$): It evaluates how rapidly emergency services can get from the station to areas prone to disasters or have a high population density. The increased accessibility saves lives and enhances response.

(2) Environmental impact score ($\mathfrak{H}_2$): The score provides the environmental friendliness of the location by considering factors like carbon footprint, disturbance of natural habitats, and green cover. High score indicates high sustainability and less damage to the environment.

(3) Land acquisition cost ($\mathfrak{H}_3$): The approximate sum of money needed to purchase land at each potential location. This covers taxes, legal fees, and the purchase price. To reduce budget expenditures, lower expenses are preferred.

(4) Proximity to major roads and transportation networks ($\mathfrak{H}_4$): Assesses the location's proximity to major highways or roads to guarantee prompt deployment. Faster mobilization during emergencies is a result of improved connectivity.

(5) Maintenance and operational cost ($\mathfrak{H}_5$): Ongoing expenses for personnel pay, equipment maintenance, utilities, and station upkeep. Better long-term financial sustainability is implied by lower values.

(6) Future expansion potential ($\mathfrak{H}_6$): Indicates the possibility of future station expansion contingent on zoning laws and land availability. A higher grade encourages flexibility and long-term planning.

5.1. Numerical example

In a fast-growing urban area, a municipal administration is preparing to open new emergency service stations. Out of six potential locations $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6\}$, the goal is to choose the most sustainable one. Six criteria $\{\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \mathfrak{H}_4, \mathfrak{H}_5, \mathfrak{H}_6\}$ divided into cost-type and benefit-type categories are used by a decision-making committee. The objective is to guarantee long-term environmental and economic sustainability in addition to prompt emergency action. The six criteria is examined by five decision makers $\{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4, \mathfrak{D}_5\}$.

5.2. Mathematical computation of LDF–LOPCOW–PROMETHEE

In this subsection, the proposed LDF–LOPCOW–PROMETHEE algorithm is utilized to rank the SESSs.

Step 1: The LDF-decision matrix is constructed by taking rows as alternatives (SESS) and columns as criteria.

$$\mathfrak{M} = \begin{bmatrix} \langle 0.83, 0.42\rangle, & \langle 0.76, 0.45\rangle, & \langle 0.42, 0.76\rangle, & \langle 0.87, 0.34\rangle, & \langle 0.32, 0.78\rangle, & \langle 0.76, 0.33\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.76, 0.35\rangle, & \langle 0.85, 0.21\rangle, & \langle 0.56, 0.69\rangle, & \langle 0.79, 0.54\rangle, & \langle 0.49, 0.54\rangle, & \langle 0.58, 0.42\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.65, 0.34\rangle, & \langle 0.58, 0.46\rangle, & \langle 0.74, 0.37\rangle, & \langle 0.63, 0.49\rangle, & \langle 0.43, 0.85\rangle, & \langle 0.68, 0.43\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.93, 0.21\rangle, & \langle 0.67, 0.26\rangle, & \langle 0.59, 0.66\rangle, & \langle 0.58, 0.48\rangle, & \langle 0.49, 0.57\rangle, & \langle 0.74, 0.39\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.34, 0.78\rangle, & \langle 0.79, 0.27\rangle, & \langle 0.59, 0.43\rangle, & \langle 0.75, 0.49\rangle, & \langle 0.38, 0.68\rangle, & \langle 0.47, 0.60\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.49, 0.62\rangle, & \langle 0.79, 0.27\rangle, & \langle 0.31, 0.89\rangle, & \langle 0.83, 0.46\rangle, & \langle 0.32, 0.77\rangle, & \langle 0.52, 0.49\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.17, 0.81\rangle & \langle 0.85, 0.11\rangle & \langle 0.23, 0.76\rangle & \langle 0.83, 0.15\rangle \end{bmatrix}$$

Step 2: Given that the criteria $\mathfrak{H}_3$ and $\mathfrak{H}_5$ are cost-type, the decision matrix is normalized. The normalized decision matrix $\mathfrak{N}$ is computed.

$$\mathfrak{M} = \begin{bmatrix} \langle 0.83, 0.42\rangle, & \langle 0.76, 0.45\rangle, & \langle 0.76, 0.42\rangle, & \langle 0.87, 0.34\rangle, & \langle 0.78, 0.32\rangle, & \langle 0.76, 0.33\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.76, 0.35\rangle, & \langle 0.85, 0.21\rangle, & \langle 0.69, 0.56\rangle, & \langle 0.79, 0.54\rangle, & \langle 0.54, 0.49\rangle, & \langle 0.58, 0.42\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.65, 0.34\rangle, & \langle 0.58, 0.46\rangle, & \langle 0.39, 0.74\rangle, & \langle 0.63, 0.49\rangle, & \langle 0.85, 0.43\rangle, & \langle 0.68, 0.43\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.93, 0.21\rangle, & \langle 0.67, 0.26\rangle, & \langle 0.66, 0.59\rangle, & \langle 0.58, 0.48\rangle, & \langle 0.57, 0.49\rangle, & \langle 0.74, 0.39\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.34, 0.78\rangle, & \langle 0.79, 0.27\rangle, & \langle 0.43, 0.59\rangle, & \langle 0.75, 0.49\rangle, & \langle 0.68, 0.38\rangle, & \langle 0.47, 0.60\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \\ & & & & & \\ \langle 0.49, 0.62\rangle, & \langle 0.79, 0.27\rangle, & \langle 0.31, 0.89\rangle, & \langle 0.83, 0.46\rangle, & \langle 0.32, 0.77\rangle, & \langle 0.52, 0.49\rangle, \\ \langle 0.83, 0.16\rangle & \langle 0.87, 0.11\rangle & \langle 0.81, 0.17\rangle & \langle 0.85, 0.11\rangle & \langle 0.76, 0.23\rangle & \langle 0.83, 0.15\rangle \end{bmatrix}$$

Step 3: The difference between the evaluation values is computed in Table 2. Here, we are using our two proposed distance measures. Results of each distance measure are shown separately in the table.

Step 4: The preference function is skipped in this case.

Step 5.1: The evaluation matrix $\mathbb{E}$ is computed as the LDFN values for each criterion are provided

Table 2. The deviation between alternatives $\mathfrak{D}_1$ and $\mathfrak{D}_2$

	$\mathfrak{D}_1$						$\mathfrak{D}_2$					
$\mathfrak{d}_1$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.04432	0.0574	0	0.1868	0.1176	0	0.0443	0.0880	0	0.2263	0.1524
$\mathfrak{R}_2$	0	0	0.0308	0.0755	0.1900	0.1210	0	0	0.0475	0.0935	0.2136	0.1382
$\mathfrak{R}_3$	0	0	0	0	0.1732	0.1070	0	0	0	0	0.1839	0.1113
$\mathfrak{R}_4$	0.0915	0.0755	0	0	0.2624	0.1950	0.0947	0.0935	0	0	0.3030	0.2295
$\mathfrak{R}_5$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_6$	0	0	0	0	0.0704	0	0	0	0	0	0.0770	0
$\mathfrak{d}_2$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0	0.0514	0	0.0545	0	0	0.0790	0	0.0839	0	
$\mathfrak{R}_2$	0.1012	0	0.1263	0.0538	0.1227	0.0325	0.1039	0	0.1513	0.0831	0.1505	0.0386
$\mathfrak{R}_3$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_4$	0	0.0538	0	0	0	0.0331	0	0.0831	0	0	0	0.0538
$\mathfrak{R}_5$	0	0	0.0076	0	0	0	0	0	0.0085	0	0	0
$\mathfrak{R}_6$	0.0707	0	0.0946	0.0331	0.0913	0	0.0708	0	0.1135	0.0538	0.1130	0
$\mathfrak{d}_3$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0505	0.1512	0.0623	0.1148	0	0	0.0522	0.1746	0.0657	0.1429	0
$\mathfrak{R}_2$	0	0	0.1090	0.0125	0.0818	0	0	0	0.1317	0.0146	0.1089	0
$\mathfrak{R}_3$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_4$	0	0	0.0973	0	0.0729	0	0	0	0.1178	0	0.0972	0
$\mathfrak{R}_5$	0	0	0.0453	0	0	0	0	0	0.0454	0	0	0
$\mathfrak{R}_6$	0.0540	0.1033	0.2037	0.1155	0.1655	0	0.0666	0.1157	0.2392	0.1300	0.2064	0
$\mathfrak{d}_4$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0735	0.0845	0.0946	0.0627	0.0454	0	0.0766	0.1152	0.1330	0.0737	0.0465
$\mathfrak{R}_2$	0	0	0.0472	0.0624	0	0	0	0	0.0738	0.0954	0	0
$\mathfrak{R}_3$	0	0	0	0.0153	0	0	0	0	0	0.0218	0	0
$\mathfrak{R}_4$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_5$	0	0.0194	0.0337	0.0488	0	0	0	0.0259	0.0521	0.0736	0	0
$\mathfrak{R}_6$	0	0.0283	0.0556	0.0699	0.0233	0	0	0.0306	0.0880	0.1090	0.0370	0
$\mathfrak{d}_4$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0931	0.0457	0.0864	0.0361	0.0026	0	0.1069	0.0568	0.0973	0.0434	0.0040
$\mathfrak{R}_2$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_3$	0	0.0894	0	0.0804	0.0489	0	0	0.1241	0	0.1123	0.0756	0
$\mathfrak{R}_4$	0	0.0094	0	0	0	0	0	0.0122	0	0	0	0
$\mathfrak{R}_5$	0	0.0571	0	0.0507	0	0	0	0.0639	0	0.0548	0	0
$\mathfrak{R}_6$	0	0.0912	0.0468	0.0846	0.0341	0	0	0.1038	0.0595	0.0943	0.0401	0
$\mathfrak{d}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0617	0.0438	0.0239	0.1270	0.0912	0.0801	0.0483	0.0241	0.1462	0.1116	
$\mathfrak{R}_2$	0	0	0	0	0.0687	0.0307	0	0	0	0	0.0720	0.0334
$\mathfrak{R}_3$	0	0.0296	0	0	0.0851	0.0524	0	0.0426	0	0	0.0998	0.0683
$\mathfrak{R}_4$	0	0.0472	0.0221	0	0.1071	0.0737	0	0.0673	0.0280	0	0.1275	0.0957
$\mathfrak{R}_5$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{R}_6$	0	0	0	0	0.0385	0	0	0	0	0	0.0394	0

by five decision makers.

$$\mathbb{E} = \begin{bmatrix} \langle 0.81, 0.32\rangle, & \langle 0.9, 0.22\rangle, & \langle 0.75, 0.23\rangle, & \langle 0.49, 0.59\rangle, & \langle 0.55, 0.62\rangle, & \langle 0.68, 0.42\rangle, \\ \langle 0.64, 0.31\rangle & \langle 0.51, 0.43\rangle & \langle 0.77, 0.2\rangle & \langle 0.70, 0.25\rangle & \langle 0.43, 0.56\rangle & \langle 0.73, 0.23\rangle \\ & & & & & \\ \langle 0.62, 0.43\rangle, & \langle 0.89, 0.41\rangle, & \langle 0.83, 0.32\rangle, & \langle 0.62, 0.56\rangle, & \langle 0.85, 0.33\rangle, & \langle 0.56, 0.44\rangle, \\ \langle 0.53, 0.44\rangle & \langle 0.54, 0.41\rangle & \langle 0.72, 0.27\rangle & \langle 0.65, 0.32\rangle & \langle 0.65, 0.33\rangle & \langle 0.74, 0.22\rangle \\ & & & & & \\ \langle 0.75, 0.64\rangle, & \langle 0.56, 0.74\rangle, & \langle 0.86, 0.59\rangle, & \langle 0.88, 0.3\rangle, & \langle 0.87, 0.48\rangle, & \langle 0.89, 0.45\rangle, \\ \langle 0.46, 0.51\rangle & \langle 0.81, 0.14\rangle & \langle 0.65, 0.32\rangle & \langle 0.65, 0.33\rangle & \langle 0.86, 0.12\rangle & \langle 0.78, 0.2\rangle \\ & & & & & \\ \langle 0.55, 0.75\rangle, & \langle 0.83, 0.53\rangle, & \langle 0.88, 0.34\rangle, & \langle 0.54, 0.5\rangle, & \langle 0.75, 0.37\rangle, & \langle 0.97, 0.1\rangle, \\ \langle 0.39, 0.6\rangle & \langle 0.65, 0.27\rangle & \langle 0.59, 0.4\rangle & \langle 0.73, 0.25\rangle & \langle 0.65, 0.31\rangle & \langle 0.6, 0.38\rangle \\ & & & & & \\ \langle 0.43, 0.55\rangle, & \langle 0.47, 0.63\rangle, & \langle 0.57, 0.66\rangle, & \langle 0.67, 0.52\rangle, & \langle 0.85, 0.21\rangle, & \langle 0.68, 0.56\rangle, \\ \langle 0.45, 0.51\rangle & \langle 0.93, 0.04\rangle & \langle 0.73, 0.22\rangle & \langle 0.66, 0.31\rangle & \langle 0.74, 0.24\rangle & \langle 0.59, 0.42\rangle \end{bmatrix}$$

Step 5.2: In the next step, the accuracy matrix $\mathbb{A}$ is constructed with the help of Equation (2).

$$\mathbb{A} = \begin{bmatrix} 0.7575 & 0.75 & 0.73 & 0.745 & 0.7875 & 0.755 \\ 0.7475 & 0.80 & 0.7825 & 0.78 & 0.785 & 0.73 \\ 0.8325 & 0.8 & 0.8475 & 0.785 & 0.8275 & 0.825 \\ 0.82 & 0.8 & 0.8 & 0.75 & 0.76 & 0.7575 \\ 0.725 & 0.76 & 0.7825 & 0.7825 & 0.755 & 0.815 \end{bmatrix}$$

Step 5.3: Then, the normalized accuracy matrix is computed by utilizing Equation (9).

$$\tilde{\mathbb{A}} = \begin{bmatrix} 0.302 & 0.000 & 1.000 & 0.000 & 0.552 & 0.263 \\ 0.209 & 1.000 & 0.553 & 0.875 & 0.586 & 0.000 \\ 1.000 & 1.000 & 0.000 & 1.000 & 0.000 & 1.000 \\ 0.884 & 1.000 & 0.404 & 0.125 & 0.931 & 0.289 \\ 0.000 & 0.200 & 0.553 & 0.937 & 1.000 & 0.895 \end{bmatrix}$$

Step 5.4: After that, we enumerated the percantage value δ_s for each criterion by Equation (10).

$$\delta_1 = 45.67, \delta_2 = 55.99, \delta_3 = 61.90, \delta_4 = 52.31, \delta_5 = 69.12, \delta_6 = 47.49.$$

Step 5.5: Finally, the objective weights are calculated (using 11) as follows:

$$\begin{aligned} \omega_1 &= 0.1374; \\ \omega_2 &= 0.1684; \\ \omega_3 &= 0.1862; \\ \omega_4 &= 0.1573; \\ \omega_5 &= 0.2079; \\ \omega_6 &= 0.1428. \end{aligned}$$

Step 6: The global preference matrix is computed based on two distance measures, $\mathfrak{D}_1$ and $\mathfrak{D}_2$, in Tables 3 and 4 respectively.

Table 3. Global preference using $\mathfrak{D}_1$

Π	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0552	0.0737	0.0479	0.0917	0.0369
$\mathfrak{R}_2$	0.0170	0	0.0532	0.0316	0.0718	0.0265
$\mathfrak{R}_3$	0	0.0228	0	0.0191	0.0461	0.0222
$\mathfrak{R}_4$	0.0126	0.0281	0.0213	0	0.0649	0.0429
$\mathfrak{R}_5$	0	0.0149	0.0150	0.0182	0	0
$\mathfrak{R}_6$	0.0220	0.0426	0.0723	0.0556	0.0721	0

Table 4. Global preference using $\mathfrak{D}_2$

Π	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$
$\mathfrak{R}_1$	0	0.0615	0.0947	0.0568	0.1133	0.0450
$\mathfrak{R}_2$	0.0175	0	0.0681	0.0446	0.0852	0.0303
$\mathfrak{R}_3$	0	0.0319	0	0.0268	0.0552	0.0251
$\mathfrak{R}_4$	0.0130	0.0390	0.0259	0	0.0779	0.0543
$\mathfrak{R}_5$	0	0.0174	0.0181	0.0230	0	0
$\mathfrak{R}_6$	0.0243	0.0479	0.0899	0.0700	0.0878	0

Step 7: The leaving flow ψ^+, ψ^- and net flow ϕ are calculated in Table 5 and 6 for $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively.

Table 5. The entering flow, leaving flow, and net flow based on $\mathfrak{D}_1$

Alternatives	ψ^+	ψ^-	ψ
$\mathfrak{R}_1$	0.0133	0.0611	0.0478
$\mathfrak{R}_2$	0.0328	0.0400	0.0073
$\mathfrak{R}_3$	0.0477	0.0220	-0.0257
$\mathfrak{R}_4$	0.0308	0.0340	0.0031
$\mathfrak{R}_5$	0.0693	0.0096	-0.0597
$\mathfrak{R}_6$	0.0257	0.0529	0.0273

Table 6. The entering flow, leaving flow, and net flow based on $\mathfrak{D}_2$

Alternatives	ψ^+	ψ^-	ψ
$\mathfrak{R}_1$	0.0144	0.0743	0.0598
$\mathfrak{R}_2$	0.0397	0.0491	0.0095
$\mathfrak{R}_3$	0.0603	0.0278	-0.0325
$\mathfrak{R}_4$	0.0396	0.0420	0.0024
$\mathfrak{R}_5$	0.0839	0.0117	-0.0722
$\mathfrak{R}_6$	0.0309	0.0640	0.0331

Step 8: Based on the values of net flow, the ranking of alternatives is shown in Table 7.

Table 7. Ranking of the sustainable emergency service station based on two distance measures, $\mathfrak{D}_1$ and $\mathfrak{D}_2$

Distance measure	ranking
$\mathfrak{D}_1$	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
$\mathfrak{D}_2$	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$

5.3. Comparative analysis

The effectiveness of the suggested LDF–LOPCOW–PROMETHEE method was assessed through an extensive comparison with a number of current LDF–based algorithms, such as LDF–COPRAS, LDF–MACROS, LDF–MEREC, and LDF–CODAS, in addition to LDF averaging, geometric operators, and their weighted variants (Table (8)). For evaluation consistency, all approaches were used on the same decision matrix.

With alternative $\mathfrak{R}_1$ continuously ranking highest in every technique, as shown in Figure 2, the results show a high degree of agreement across all methodologies. This strong alignment demonstrates the ranking process's consistency and dependability. The consistency of the results was confirmed by the fact that, although there was a minor variance in the rankings of alternatives $\mathfrak{R}_2$ and $\mathfrak{R}_4$ in several techniques, the remaining rankings were the same. A unique feature of the suggested LDF–LOPCOW–PROMETHEE technique is its integrated framework, which combines two potent MCDM algorithms: PROMETHEE for preference ranking and LOPCOW for objective weight calculation. On the other hand, a single algorithm is used by all other approaches considered in this study. This dual-algorithm integration makes the suggested method more robust and discriminative, resulting in a more complete and efficient decision-making tool.

5.4. Sensitivity analysis

A sensitivity analysis was performed to evaluate the impact of variations in criterion weights on the ranking results. Five distinct cases were examined, each assigning the highest weight to a different criterion. The motive of this analysis was to determine whether these weight variations will affect the final ranking of alternatives.

The results in Table 9 indicate that the ranking of alternatives remains the same across all proposed cases. This consistency reveals that the outcomes are independent of individual weight alterations, and it confirms that the method is

Table 8. Comparative analysis on existing linear diophantine fuzzy (LDF) algorithms

Methods	Alternatives						Ranking
	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	
LDF-COPRAS	–0.1275	–0.2196	–0.3535	–0.2080	–0.4031	–0.1834	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-MEREC	0.2061	0.1764	0.1424	0.1808	0.1108	0.1835	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-CODAS	0.7872	0.1152	–0.2969	0.3042	–1.2628	0.3531	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-MACROS	1.1708	0.9914	0.9608	0.9935	0.8752	1.1742	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF -AO	0.9985	0.9974	0.9939	0.9983	0.9852	0.9980	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-GO	–0.5110	–0.5920	–0.6280	–0.5928	–0.6493	–0.5873	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-WAO	0.5464	0.4884	0.4312	0.4981	0.3631	0.5290	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
LDF-WGO	0.5350	0.4501	0.3821	0.4504	0.3188	0.4824	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_4 > \mathfrak{R}_2 > \mathfrak{R}_3 > \mathfrak{R}_5$
Proposed	0.0478	0.0073	–0.0257	0.0031	–0.0597	0.0273	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
Proposed	0.0598	0.0095	–0.0325	0.0024	–0.0722	0.0331	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$

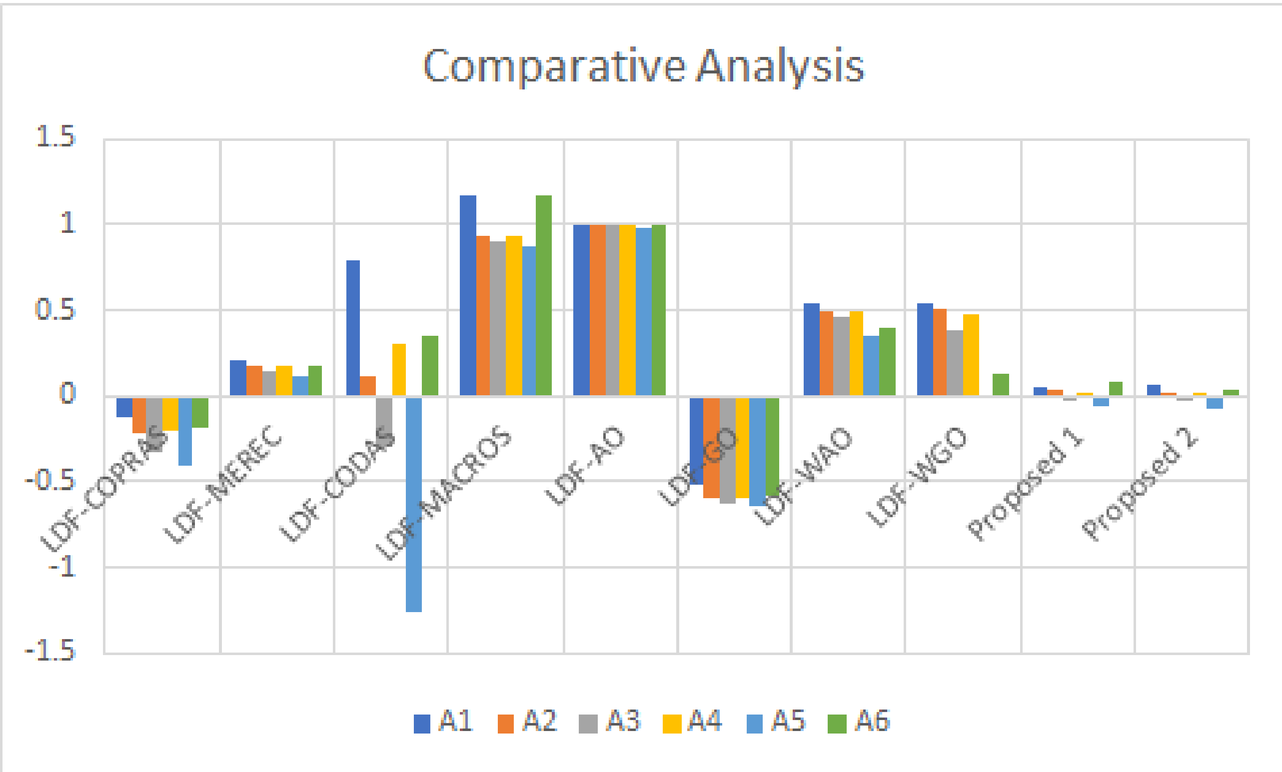

Figure 2. Comparative Analysis on different LDF-Algorithms

resilient to changes in the weights of the criteria. Figure 3 shows the results obtained from the distance measure $\mathfrak{D}_1$, and Figure 4 shows the results from the distance measure $\mathfrak{D}_2$ for each of the five cases. The efficacy of the proposed LDF–LOPCOW–PROMETHEE approach is demonstrated by showing that even substantial shifts in the focus of criteria do not influence the justification

Table 9. Sensitivity analysis on weights with two distance measures $\mathfrak{D}_1$ and $\mathfrak{D}_2$

Case	Alternatives							Ranking
	$\mathfrak{D}$	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_3$	$\mathfrak{R}_4$	$\mathfrak{R}_5$	$\mathfrak{R}_6$	
Case 1	$\mathfrak{D}_1$	0.0509	0.01357	-0.0174	0.01607	-0.0718	0.0086	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
	$\mathfrak{D}_2$	0.0641	0.0166	-0.0243	0.0173	-0.0848	0.0111	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
Case 2	$\mathfrak{D}_1$	0.0510	0.0099	-0.0165	0.0147	-0.0704	0.0112	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
	$\mathfrak{D}_2$	0.0638	0.0119	-0.0222	0.0164	-0.0835	0.0137	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
Case 3	$\mathfrak{D}_1$	0.0463	0.0146	-0.0242	0.0076	-0.0626	0.0183	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
	$\mathfrak{D}_2$	0.0638	0.0119	-0.0222	0.0164	-0.0835	0.0137	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
Case 4	$\mathfrak{D}_1$	0.0505	0.0068	-0.0253	0.0073	-0.0646	0.0253	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
	$\mathfrak{D}_2$	0.0626	0.0088	-0.0320	0.0079	-0.0779	0.0305	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
Case 5	$\mathfrak{D}_1$	0.0445	0.0144	-0.0200	0.0130	-0.0659	0.0139	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$
	$\mathfrak{D}_2$	0.0565	0.0178	-0.0270	0.0140	-0.0785	0.0172	$\mathfrak{R}_1 > \mathfrak{R}_6 > \mathfrak{R}_2 > \mathfrak{R}_4 > \mathfrak{R}_3 > \mathfrak{R}_5$

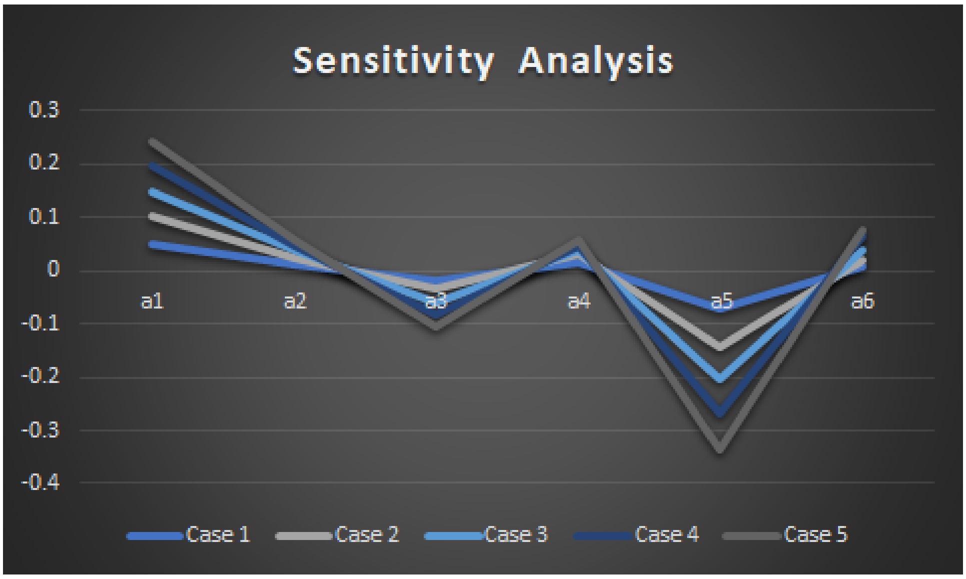

Figure 3. Sensitivity analysis on weights with distance $\mathfrak{D}_1$

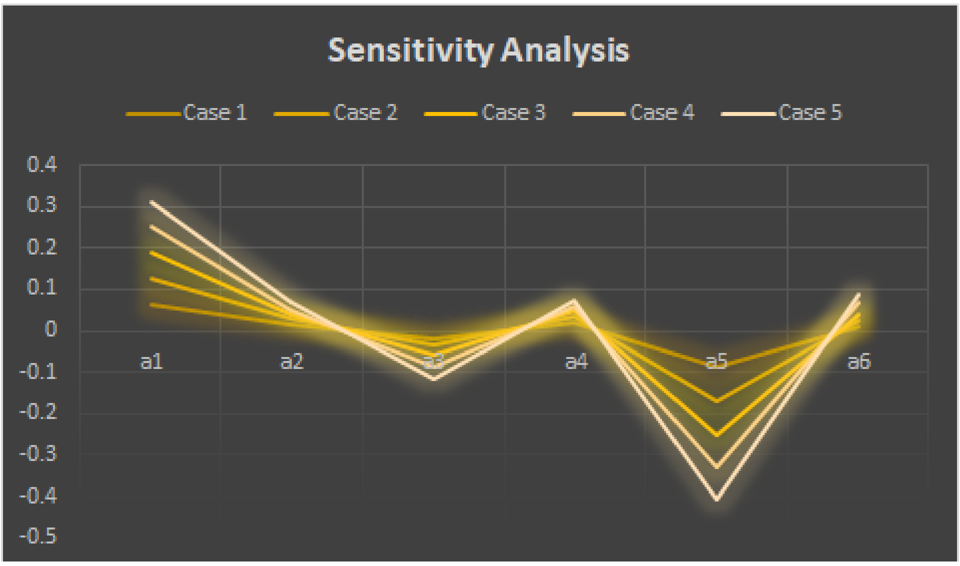

Figure 4. Sensitivity analysis on weights with distance $\mathfrak{D}_2$

overall ranking of the SESS alternatives. The stability over five case scenarios of weights, as well as two distance measures, guarantees the robustness and dependability of the proposed approach. The ranking of the alternatives was constant across all scenarios, suggesting that the input variations in the weights of the criteria weights do not affect the final decision.

6. Case study: Investment portfolio selection

The investment portfolio selection is a process of allocating financial resources across a variety of assets, including mutual funds, stocks, and real estates, to accomplish specific financial objectives. While dealing with limitations like time horizon, budget, and market volatility, most investors prefer to maximize the gains and minimize risks. In addition to guaranteeing capital growth, a well-built portfolio offers resilience and stability in the face of market volatility. Poor choices in portfolio selection may lead to significant losses. Hence, the selection of the best investment portfolio is a crucial component for both individual investors and institutions. In recent years, data-driven and sustainable methods of portfolio selection have become more significant. MCDM methods provide an organized approach to assessing a variety of investment possibilities according to several criteria. This leads to a balanced, optimized portfolio selection that matches with the investor's financial objectives and risk tolerance thereby improving long-term results.

6.1. Numerical example

An investment advisor assists a mid-level investor in creating a sustainable and diversified investment portfolio. Six criteria, $\{\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4, \mathfrak{J}_5\}$, are being used to analyze six different investment options, $\{\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4, \mathfrak{W}_5, \mathfrak{W}_6\}$. The investor seeks to maximize profits, control risk, and account for growth and sustainability. There are three benefit-type criteria (to be maximized) and two cost-type criteria (to be minimized). The six criteria are examined by five decision makers, $\{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4, \mathfrak{D}_5\}$. Description of criteria:

(1) **Expected return rate ($\mathfrak{J}_1$):** The anticipated yearly return on investment, as determined by forecasts or historical data. Growth-oriented portfolios are better off with higher return rates.
(2) **Environmental, social, and governance (ESG) score ($\mathfrak{J}_2$):** It assesses the investment's ethical implications and sustainability. Better adherence to social and environmental norms is indicated by higher scores.
(3) **Level of risk ($\mathfrak{J}_3$):** This criterion, which is measured by the beta coefficient or volatility, indicates the degree of return uncertainty. A safer investment is indicated by a lower value.
(4) **Liquidity ($\mathfrak{J}_4$):** The ease of converting an asset into cash without experiencing a significant drop in value. Higher liquidity investments offer more financial flexibility.
(5) **Management/Transaction cost ($\mathfrak{J}_5$):** This category covers fees related to purchasing or keeping the investment, such as brokerage fees or fund management charges. It is better to have lower values.

6.2. Mathematical computation of LDF-LOPCOW-PROMETHEE

In this subsection, the proposed LDF-LOPCOW-PROMETHEE algorithm is utilized to rank the SIPS.

Step 1: The LDF–decision matrix $\mathfrak{M}$ is constructed by taking rows as alternatives (investment portfolios) and columns as criteria.

$$\mathfrak{M} = \begin{bmatrix} \langle 0.56, 0.40\rangle, & \langle 0.83, 0.33\rangle, & \langle 0.46, 0.53\rangle, & \langle 0.76, 0.28\rangle, & \langle 0.14, 0.89\rangle, \\ \langle 0.38, 0.61\rangle & \langle 0.85, 0.13\rangle & \langle 0.11, 0.84\rangle & \langle 0.56, 0.41\rangle & \langle 0.28, 0.69\rangle \\ & & & & \\ \langle 0.69, 0.33\rangle, & \langle 0.68, 0.39\rangle, & \langle 0.39, 0.75\rangle, & \langle 0.58, 0.44\rangle, & \langle 0.57, 0.49\rangle, \\ \langle 0.57, 0.40\rangle & \langle 0.67, 0.13\rangle & \langle 0.31, 0.68\rangle & \langle 0.38, 0.61\rangle & \langle 0.05, 0.93\rangle \\ & & & & \\ \langle 0.38, 0.70\rangle, & \langle 0.79, 0.26\rangle, & \langle 0.46, 0.59\rangle, & \langle 0.38, 0.59\rangle, & \langle 0.85, 0.43\rangle, \\ \langle 0.48, 0.50\rangle & \langle 0.89, 0.01\rangle & \langle 0.50, 0.49\rangle & \langle 0.83, 0.11\rangle & \langle 0.61, 0.38\rangle \\ & & & & \\ \langle 0.44, 0.59\rangle, & \langle 0.83, 0.22\rangle, & \langle 0.60, 0.49\rangle, & \langle 0.67, 0.38\rangle, & \langle 0.28, 0.64\rangle, \\ \langle 0.78, 0.21\rangle & \langle 0.74, 0.25\rangle & \langle 0.20, 0.78\rangle & \langle 0.48, 0.60\rangle & \langle 0.49, 0.49\rangle \\ & & & & \\ \langle 0.63, 0.47\rangle, & \langle 0.58, 0.43\rangle, & \langle 0.39, 0.63\rangle, & \langle 0.79, 0.26\rangle, & \langle 0.51, 0.48\rangle, \\ \langle 0.83, 0.10\rangle & \langle 0.57, 0.04\rangle & \langle 0.28, 0.69\rangle & \langle 0.68, 0.30\rangle & \langle 0.24, 0.68\rangle \\ & & & & \\ \langle 0.80, 0.24\rangle, & \langle 0.75, 0.29\rangle, & \langle 0.18, 0.73\rangle, & \langle 0.48, 0.71\rangle, & \langle 0.14, 0.83\rangle, \\ \langle 0.79, 0.20\rangle & \langle 0.62, 0.35\rangle & \langle 0.22, 0.76\rangle & \langle 0.70, 0.27\rangle & \langle 0.42, 0.55\rangle \end{bmatrix}$$

Step 2: Given that the criteria $\mathfrak{J}_3$ and $\mathfrak{J}_4$ are cost-type, the decision matrix is normalized accordingly. The normalized decision matrix $\mathfrak{N}$ is computed using appropriate formulas.

$$\mathfrak{N} = \begin{bmatrix} \langle 0.56, 0.40\rangle, & \langle 0.83, 0.33\rangle, & \langle 0.53, 0.46\rangle, & \langle 0.76, 0.28\rangle, & \langle 0.89, 0.14\rangle, \\ \langle 0.38, 0.61\rangle & \langle 0.85, 0.13\rangle & \langle 0.84, 0.11\rangle & \langle 0.56, 0.41\rangle & \langle 0.69, 0.28\rangle \\ & & & & \\ \langle 0.69, 0.33\rangle, & \langle 0.68, 0.39\rangle, & \langle 0.75, 0.39\rangle, & \langle 0.58, 0.44\rangle, & \langle 0.49, 0.57\rangle, \\ \langle 0.57, 0.40\rangle & \langle 0.67, 0.13\rangle & \langle 0.68, 0.31\rangle & \langle 0.38, 0.61\rangle & \langle 0.93, 0.05\rangle \\ & & & & \\ \langle 0.38, 0.70\rangle, & \langle 0.79, 0.26\rangle, & \langle 0.59, 0.46\rangle, & \langle 0.38, 0.59\rangle, & \langle 0.43, 0.85\rangle, \\ \langle 0.48, 0.50\rangle & \langle 0.89, 0.01\rangle & \langle 0.49, 0.50\rangle & \langle 0.83, 0.11\rangle & \langle 0.38, 0.61\rangle \\ & & & & \\ \langle 0.44, 0.59\rangle, & \langle 0.83, 0.22\rangle, & \langle 0.49, 0.60\rangle, & \langle 0.67, 0.38\rangle, & \langle 0.64, 0.28\rangle, \\ \langle 0.78, 0.21\rangle & \langle 0.74, 0.25\rangle & \langle 0.78, 0.20\rangle & \langle 0.48, 0.60\rangle & \langle 0.49, 0.49\rangle \\ & & & & \\ \langle 0.63, 0.47\rangle, & \langle 0.58, 0.43\rangle, & \langle 0.63, 0.39\rangle, & \langle 0.79, 0.26\rangle, & \langle 0.48, 0.51\rangle, \\ \langle 0.83, 0.10\rangle & \langle 0.57, 0.04\rangle & \langle 0.69, 0.28\rangle & \langle 0.68, 0.30\rangle & \langle 0.68, 0.24\rangle \\ & & & & \\ \langle 0.80, 0.24\rangle, & \langle 0.75, 0.29\rangle, & \langle 0.73, 0.18\rangle, & \langle 0.48, 0.71\rangle, & \langle 0.83, 0.14\rangle, \\ \langle 0.79, 0.20\rangle & \langle 0.62, 0.35\rangle & \langle 0.76, 0.22\rangle & \langle 0.70, 0.27\rangle & \langle 0.55, 0.42\rangle \end{bmatrix}$$

Step 3: Under each criterion, the difference between the evaluation values of every alternative pair is computed in Table 10.

Table 10. Deviation between alternatives $\mathfrak{D}_1$ and $\mathfrak{D}_2$.

$\mathfrak{d}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0	0.1233	0	0	0	0	0	0.1258	0	0	0
$\mathfrak{W}_2$	0.1055	0	0.1634	0.1557	0	0	0.1078	0	0.1634	0.1569	0	0
$\mathfrak{W}_3$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{W}_4$	0.2073	0	0.1502	0	0	0	0.2073	0	0.1518	0	0	0
$\mathfrak{W}_5$	0.2527	0.1708	0.2315	0.0983	0	0	0.2555	0.1715	0.2319	0.1042	0	0
$\mathfrak{W}_6$	0.2204	0.1170	0.2546	0.1676	0.1193	0	0.2343	0.1303	0.2605	0.1842	0.1265	0
$\mathfrak{d}_2$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0	0.0392	0.0861	0.1672	0.1265	0	0	0.0402	0.0927	0.1921	0.1460
$\mathfrak{W}_2$	0	0	0	0	0.1327	0.1164	0	0	0	0	0.1327	0.1182
$\mathfrak{W}_3$	0.0392	0.0868	0	0.0958	0.1916	0.1467	0.0402	0.1212	0	0.0991	0.2099	0.1601
$\mathfrak{W}_4$	0	0.1068	0	0	0.1361	0.0668	0	0.1221	0	0	0.1490	0.0772
$\mathfrak{W}_5$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{W}_6$	0	0	0	0	0.0770	0	0	0	0	0	0.0800	0
$\mathfrak{d}_3$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0.1317	0.1994	0.0728	0.1072	0	0	0.1401	0.1996	0.0729	0.1084	0
$\mathfrak{W}_2$	0	0	0.1047	0.1191	0.0362	0	0	0	0.1121	0.1275	0.0480	0
$\mathfrak{W}_3$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{W}_4$	0	0	0.1571	0	0	0	0	0	0.1572	0	0	0
$\mathfrak{W}_5$	0	0	0.1063	0.0940	0	0	0	0	0.1066	0.0950	0	0
$\mathfrak{W}_6$	0.1474	0.1038	0.1841	0.1766	0.1027	0	0.1513	0.1047	0.1860	0.1795	0.1043	0
$\mathfrak{d}_4$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0.1207	0.2281	0.0827	0	0.1809	0	0.1220	0.2337	0.0828	0	0.1810
$\mathfrak{W}_2$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{W}_3$	0	0.2590	0	0.2539	0	0.1059	0	0.2612	0	0.2596	0	0.1157
$\mathfrak{W}_4$	0	0.0496	0	0	0	0	0	0.0536	0	0	0	0
$\mathfrak{W}_5$	0.0571	0.1721	0.2044	0.1355	0	0.1791	0.0631	0.1777	0.2186	0.1376	0	0.1817
$\mathfrak{W}_6$	0	0.1818	0	0.1813	0	0	0	0.1823	0	0.1814	0	0
$\mathfrak{d}_5$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0.2572	0.2421	0.1401	0.1954	0.0708	0.2642	0.2577	0.1569	0.2080	0.0866	
$\mathfrak{W}_2$	0	0	0.1774	0.2707	0.1389	0	0	0	0.1797	0.2709	0.1394	0
$\mathfrak{W}_3$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathfrak{W}_4$	0	0	0.1648	0	0	0	0	0	0.1658	0	0	0
$\mathfrak{W}_5$	0	0	0.0617	0.1507	0	0	0	0	0.0639	0.1507	0	0
$\mathfrak{W}_6$	0	0.2946	0.2361	0.0938	0.2036	0	0	0.2948	0.2392	0.0961	0.2048	0

Step 4: The preference function is skipped in this case.
noindent**Step 5:** The weights of the criteria are calculated as as shown below.
noindent**Step 5.1:** The evaluation matrix $\mathbb{E}$ is computed as the LDFN values for each criterion are provided by five decision makers.

$$\mathbb{E} = \begin{bmatrix} \langle 0.67, 0.47\rangle, & \langle 0.79, 0.26\rangle, & \langle 0.67, 0.44\rangle, & \langle 0.82, 0.34\rangle, & \langle 0.87, 0.22\rangle, & \langle 0.67, 0.42\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.10\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.9, 0.08\rangle & \langle 0.78, 0.2\rangle \\ & & & & & \\ \langle 0.83, 0.34\rangle, & \langle 0.69, 0.33\rangle, & \langle 0.52, 0.5\rangle, & \langle 0.54, 0.55\rangle, & \langle 0.94, 0.12\rangle, & \langle 0.74, 0.31\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.1\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.9, 0.08\rangle & \langle 0.78, 0.2\rangle \\ & & & & & \\ \langle 0.86, 0.22\rangle, & \langle 0.59, 0.32\rangle, & \langle 0.53, 0.55\rangle, & \langle 0.61, 0.49\rangle, & \langle 0.78, 0.37\rangle, & \langle 0.84, 0.29\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.10\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.90, 0.08\rangle & \langle 0.78, 0.20\rangle \\ & & & & & \\ \langle 0.68, 0.51\rangle, & \langle 0.44, 0.62\rangle, & \langle 0.49, 0.62\rangle, & \langle 0.49, 0.62\rangle, & \langle 0.68, 0.37\rangle, & \langle 0.59, 0.48\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.10\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.9, 0.08\rangle & \langle 0.78, 0.20\rangle \\ & & & & & \\ \langle 0.59, 0.42\rangle, & \langle 0.53, 0.45\rangle, & \langle 0.58, 0.41\rangle, & \langle 0.74, 0.30\rangle, & \langle 0.38, 0.59\rangle, & \langle 0.33, 0.69\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.10\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.90, 0.08\rangle & \langle 0.78, 0.20\rangle \\ & & & & & \\ \langle 0.89, 0.31\rangle, & \langle 0.69, 0.42\rangle, & \langle 0.47, 0.53\rangle, & \langle 0.56, 0.48\rangle, & \langle 0.69, 0.20\rangle, & \langle 0.44, 0.57\rangle, \\ \langle 0.84, 0.12\rangle & \langle 0.87, 0.10\rangle & \langle 0.86, 0.12\rangle & \langle 0.79, 0.19\rangle & \langle 0.90, 0.08\rangle & \langle 0.78, 0.20\rangle \end{bmatrix}$$

Step 5.2: An accuracy matrix is derived from the evaluation matrix by applying the LDFN accuracy function 2.

$$\mathbb{A} = \begin{bmatrix} 0.765 & 0.7475 & 0.7675 & 0.78 & 0.7625 \\ 0.7725 & 0.74 & 0.745 & 0.7625 & 0.755 \\ 0.75 & 0.7125 & 0.76 & 0.765 & 0.7775 \\ 0.7775 & 0.75 & 0.7675 & 0.7675 & 0.7525 \\ 0.7325 & 0.73 & 0.7375 & 0.75 & 0.7325 \\ 0.78 & 0.7625 & 0.74 & 0.75 & 0.7125 \end{bmatrix}$$

Step 5.3: Using an appropriate normalizing formula, the accuracy matrix is normalized.

$$\begin{bmatrix} 0.684 & 0.538 & 0.000 & 1.000 & 0.231 \\ 0.842 & 0.308 & 0.750 & 0.417 & 0.346 \\ 0.368 & -0.538 & 0.250 & 0.500 & 0.000 \\ 0.947 & 0.615 & 0.000 & 0.583 & 0.385 \\ 0.000 & 0.000 & 1.000 & 0.000 & 0.692 \\ 1.000 & 1.000 & 0.917 & 0.000 & 1.000 \end{bmatrix}$$

Step 5.4: To represent the relative contribution of each criterion, the percentage value(δ) is computed.

$$\delta_1 = 81.81, \delta_2 = 26.96, \delta_3 = 51.91, \delta_4 = 53.76, \delta_5 = 61.80;$$

Step 5.5: The objective weights of each criterion are calculated as follows:

$$\begin{aligned} \omega_1 &= 0.2960; \\ \omega_2 &= 0.0976; \\ \omega_3 &= 0.1879; \\ \omega_4 &= 0.1946; \\ \omega_5 &= 0.2240. \end{aligned}$$

Step 6: The global preference matrix is derived using the weighted normalized decision matrix and is shown in Tables 11 and 12 using two proposed distance measures.

Table 11. Global preference using $\mathfrak{D}_1$

Π	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0.1059	0.1764	0.0696	0.0802	0.0634
A2	0.0312	0	0.1078	0.1291	0.0509	0.0114
$\mathfrak{W}_2$	0.0038	0.0589	0	0.0588	0.0187	0.0349
$\mathfrak{W}_3$	0.0613	0.0201	0.1109	0	0.0133	0.0065
$\mathfrak{W}_4$	0.0859	0.0840	0.1421	0.1069	0	0.0348
$\mathfrak{W}_5$	0.0929	0.1555	0.1629	0.1391	0.1077	0

Table 12. Global preference using $\mathfrak{D}_1$

Π	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$
$\mathfrak{W}_1$	0	0.0997	0.1578	0.0630	0.0639	0.0613
$\mathfrak{W}_2$	0.0240	0	0.0888	0.1091	0.0401	0.0035
$\mathfrak{W}_3$	0.0012	0.0584	0	0.0574	0.0063	0.0291
$\mathfrak{W}_4$	0.0462	0.0149	0.0881	0	0.0045	0.0023
$\mathfrak{W}_5$	0.0702	0.0755	0.1239	0.0961	0	0.0381
$\mathfrak{W}_6$	0.0697	0.1441	0.1320	0.1209	0.0876	0

Step 7: To determine how much an alternative outranks or is outranked by others, the leaving flow and entering flow are calculated for each alternative in Tables 13 and 14.

Table 13. Entering flow, leaving flow, and net flow by $\mathfrak{D}_1$

Alternatives	ψ^+	ψ^-	ψ
$\mathfrak{W}_1$	0.0547	0.0991	0.0444
$\mathfrak{W}_2$	0.0965	0.0661	-0.0304
$\mathfrak{W}_3$	0.1330	0.0350	-0.0980
$\mathfrak{W}_4$	0.0793	0.0424	-0.0369
$\mathfrak{W}_5$	0.0611	0.0908	0.0296
$\mathfrak{W}_6$	0.0232	0.1316	0.1084

Table 14. Entering flow, leaving flow, and net flow by $\mathfrak{D}_2$

Alternatives	ψ^+	ψ^-	ψ
$\mathfrak{W}_1$	0.0433	0.0891	0.0458
$\mathfrak{W}_2$	0.0882	0.0531	-0.0350
$\mathfrak{W}_3$	0.1126	0.0305	-0.0822
$\mathfrak{W}_4$	0.0701	0.0313	-0.0388
$\mathfrak{W}_5$	0.0481	0.0808	0.0327
$\mathfrak{W}_6$	0.0193	0.1108	0.0916

Step 8: Based on the values of net flow, the ranking of alternatives is shown in Table 15.

Table 15. Ranking of IPs based on two distance measures $\mathfrak{D}_1$ and $\mathfrak{D}_2$

Distance measure	Ranking
$\mathfrak{D}_1$	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
$\mathfrak{D}_2$	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$

Table 16. Comparative analysis with existing linear diophantine fuzzy (LDF) algorithms

Methods	Alternatives						Ranking
	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	
LDF-COPRAS	-0.2203	-0.3437	-0.5949	-0.4345	-0.3121	-0.2030	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
LDF- MEREC	0.2183	0.1481	0.1292	0.1270	0.1592	0.2183	$\mathfrak{W}_6 = \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_3 > \mathfrak{W}_4$
LDF-CODAS	0.2130	-0.0827	-0.7045	-0.9266	0.0150	1.4859	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_3 > \mathfrak{W}_4$
LDF- MACROS	1.2821	1.0237	0.8594	0.9583	1.0563	1.2826	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
LDF -AO	0.9968	0.9898	0.9732	0.9898	0.9901	0.9966	$\mathfrak{W}_1 > \mathfrak{W}_6 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
LDF-GO	-0.7329	-0.8074	-0.8570	-0.8318	-0.7441	-0.7001	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
LDF-WAO	0.4038	0.3580	0.1541	0.2746	0.3745	0.4681	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
LDF-WGO	0.2731	0.2326	0.0440	0.1799	0.3303	0.3845	$\mathfrak{W}_6 > \mathfrak{W}_5 > \mathfrak{W}_1 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Proposed	0.0444	-0.0304	-0.0980	-0.0369	0.0296	0.1084	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Proposed	0.0458	-0.0350	-0.0822	-0.0388	0.0327	0.0916	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$

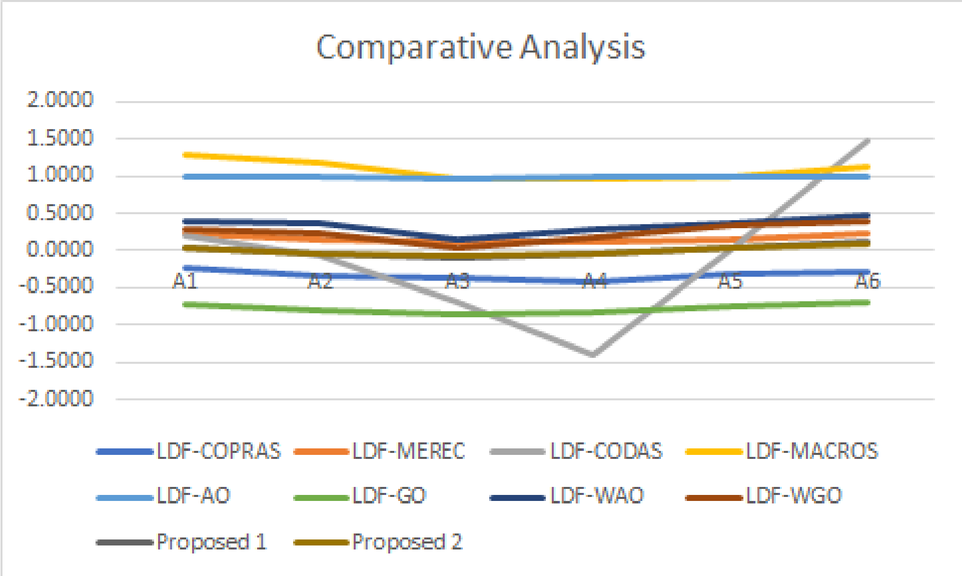

Figure 5. Comparative analysis on different linear Diophantine fuzzy (LDF) algorithms

Table 17. Sensitivity analysis on weights with two distance measures $\mathfrak{D}_1$ and $\mathfrak{D}_2$

Case	$\mathfrak{D}$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_5$	$\mathfrak{W}_6$	Ranking
Case 1	$\mathfrak{d}_1$	0.0432	-0.0296	-0.0981	-0.0362	0.0295	0.1085	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
	$\mathfrak{d}_2$	0.0466	-0.0316	-0.0986	-0.0379	0.0285	0.1108	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Case 2	$\mathfrak{d}_1$	0.0557	-0.0270	-0.0455	-0.0258	-0.0060	0.0619	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
	$\mathfrak{d}_2$	0.0608	-0.0310	-0.0427	-0.0257	-0.0091	0.0614	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Case 3	$\mathfrak{d}_1$	0.0597	-0.0169	-0.0833	-0.0280	-0.0106	0.0899	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
	$\mathfrak{d}_2$	0.0643	-0.0190	-0.0821	-0.0288	-0.0138	0.0903	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Case 4	$\mathfrak{d}_1$	0.0529	-0.0481	-0.0469	-0.0380	0.0229	0.0701	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
	$\mathfrak{d}_2$	0.0560	-0.0509	-0.0451	-0.0385	0.0216	0.0705	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
Case 5	$\mathfrak{d}_1$	0.0904	-0.0370	-0.0711	-0.0435	-0.0103	0.0791	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$
	$\mathfrak{d}_2$	0.0964	-0.0397	-0.0701	-0.0443	-0.0125	0.0781	$\mathfrak{W}_6 > \mathfrak{W}_1 > \mathfrak{W}_5 > \mathfrak{W}_2 > \mathfrak{W}_4 > \mathfrak{W}_3$

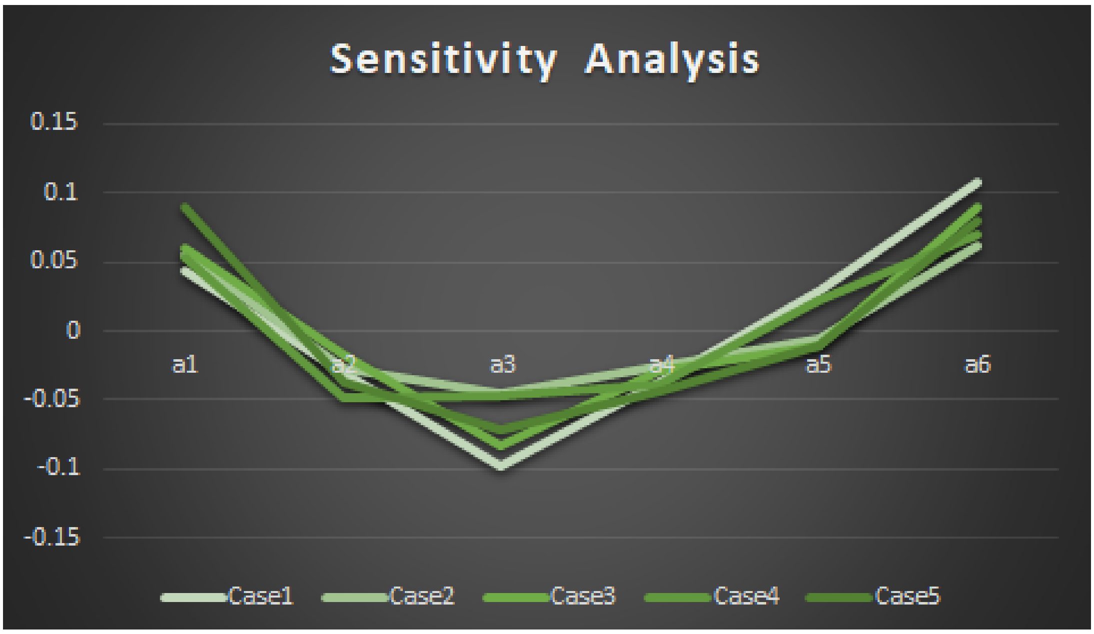

Figure 6. Sensitivity analysis on weights with distance $\mathfrak{D}_1$

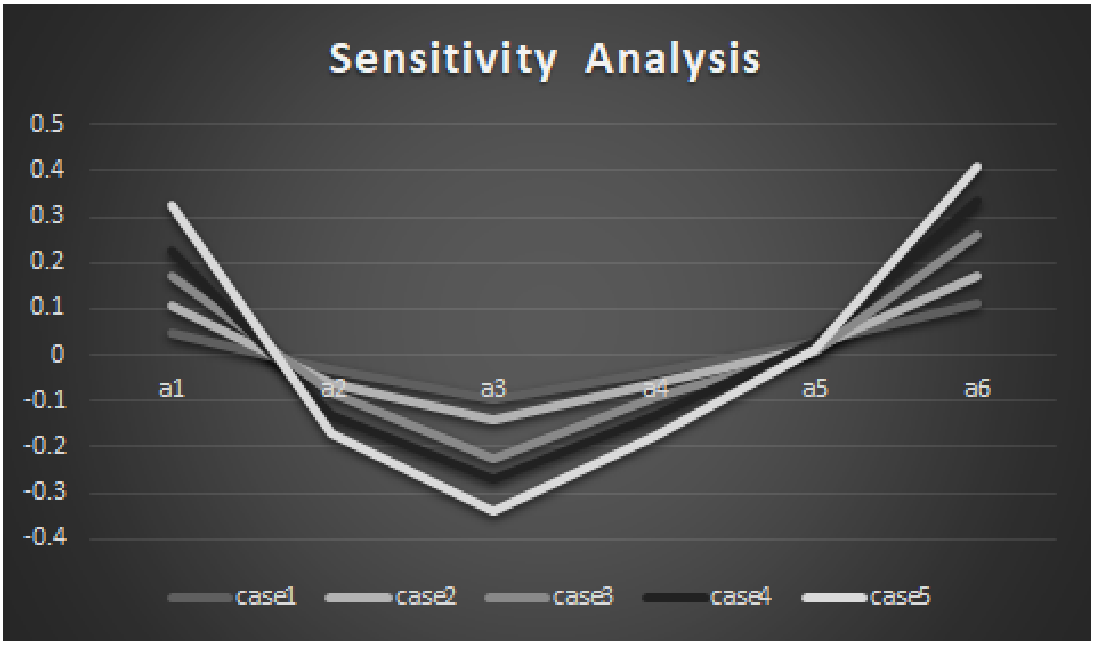

Figure 7. Sensitivity analysis on weights with distance $\mathfrak{D}_2$

6.3. Sensitivity analysis and comparative analysis

The performance of the proposed LDF–LOPCOW–PROMETHEE approach was compared against a number of other LDF–based algorithms currently in use, such as LDF-COPRAS, LDF–MACROS, LDF-MEREC, LDF–CODAS, LDF–averaging, geometric operators, and their weighted variants. Among all techniques, alternative $\mathfrak{W}_6$ consistently obtained the highest preference rank, according to the results, shown in Table 16 and Figure 5. All other rankings remained the same, except slight difference in how alternatives B2 and B4 were ranked in two algorithms, including the proposed one. This strong alignment demonstrates to the proposed method's robustness, consistency, and reliability. Furthermore, the hybrid structure of the proposed strategy, which incorporates two algorithms (LOPCOW and PROMETHEE), in contrast to other single-method models, makes it unique and improves its decision-making ability.

To evaluate the stability of the rankings under various weight conditions, a sensitivity analysis was conducted using two distinct distance measures $\mathfrak{D}_1$ and $\mathfrak{D}_2$ (refer Table 17). Five examples were tested for each distance metric by assigning the highest weight to a different criterion. The results reveal that the ranking order of the alternatives remained the same in all cases, as shown in Figures 6 ($\mathfrak{D}_1$) and 7 ($\mathfrak{D}_2$). The robustness and stability of the proposed approach across various weighting scenarios are confirmed by this consistency, indicating that changes in criterion weights do not affect the ranking outcomes.

7. Conclusion

The inherent uncertainty in multi-criteria choice issues is addressed by this study's innovative decision-making framework, which combines the LOPCOW-–PROMETHEE approach with Linear Diophantine Fuzzy (LDFS) sets. The creation of two new distance measures for LDF sets, which allow for the creation of two algorithmic processes based on each measure, is a significant contribution of this work. Our proposed algorithms were deeply tested on two real-world issues, including the selection of investment portfolios and the location of SESS to show their efficacy. Both algorithms showed stability in their rankings of alternatives. This crucial result demonstrated that the proposed model is reliable and strong. Further, the results of the sensitivity analyses also exhibited that the variations in the weights of the criteria do not influence the ranking of the alternatives. Moreover, the ranking under two different distance measures was stable and unique, and this highlights the originality of the proposed LDF distance measures. The LOPCOW–PROMETHEE approach provides a flexible and viable method to handle uncertainty in the LDF framework. Future research will expand this strategy in neural networks with various fuzzy extensions.

Acknowledgments

None.

Funding

None.

Conflict of interest

The authors declare they have no competing interests.

Author contributions

Conceptualization: Jeevitha Kannan
Investigation: Vimala Jayakumar, Dragan Pamucar
Methodology: Jeevitha Kannan, S. Rajareega
Formal analysis: Vimala Jayakumar, Dragan Pamucar
Writing–original draft: Jeevitha Kannan
Writing–review & editing: Jeevitha Kannan, S. Rajareega

Availability of data

Not applicable.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

References

1. Zadeh LA. Fuzzy sets. *Inf Control* 1965;8(3):338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* 1986;20(1):87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
3. Yager RR. Pythagorean fuzzy subsets. *Proc Joint IFSA World Congr NAFIPS Annu Meet.* 2013;57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608378
4. Yager RR. Generalized orthopair fuzzy sets. *IEEE Trans Fuzzy Syst.* 2017;25(5):1222–1230. https://doi.org/10.1109/TFUZZ.2016.2582684
5. Riaz M, Hashmi MR. Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems. *J Intell Fuzzy Syst.* 2019;37(4):5417–5439.
6. Kannan J, Jayakumar V, Kausar N, Pamucar D, Simic V. Enhancing decision-making with linear diophantine multi-fuzzy set: application of novel information measures in medical and engineering fields. *Sci Rep.* 2024;14(1). https://doi.org/10.1038/s41598-024-79725-0
7. Kannan J, Jayakumar V, Mohideen AK, Tamilvizhi M. Entropy-based analysis using linear diophantine multi fuzzy soft sets: A DEA approach for improved decision systems. *Yugoslav J Oper Res.* 2024;00:55. https://doi.org/10.2298/yjor240315055k
8. Kannan J, Jayakumar V, Pethaperumal M, Banu KA. An intensified linear diophantine fuzzy combined DEMATEL framework for the assessment of climate crisis. *Stoch Environ Res Risk Assess.* 2024. https://doi.org/10.1007/s00477-023-02618-7
9. Jayakumar V, Kannan J, Kausar N, Deveci M, Wen X. Multicriteria group decision making for prioritizing IoT risk factors with linear diophantine fuzzy sets and MARCOS method. *Granul Comput.* 2024;9:56. https://doi.org/10.1007/s41066-024-00480-8
10. Kannan J, Jayakumar V, Saeed M, Alballa T, Khalifa HAE-W, Gomaa HG. Linear Diophantine fuzzy clustering algorithm based on correlation coefficient and analysis on logistic efficiency of food products. *IEEE Access* 2024. https://doi.org/10.1109/ACCESS.2024.3371986
11. Kannan J, Jayakumar V, Pethaperumal M. Advanced Fuzzy-Based Decision-Making: The Linear Diophantine Fuzzy CODAS Method for Logistic Specialist Selection. *Spectr Oper Res.* 2025;2(1):41-60. https://doi.org/10.31181/sor2120259
12. Zia MD, Yousafzai F, Abdullah S, Hila K. Complex linear Diophantine fuzzy sets and their applications in multi-attribute decision making. *Eng Appl Artif Intell.* 2024;132:107953.
13. Gul R, Al-Shami TM, Ayub S, Shabir M, Hosny M. Development of Aczel-Alsina t-norm Based Linear Diophantine Fuzzy Aggregation Operators and their Applications in Multi-Criteria Decision-Making with Unknown Weight Information. *Heliyon* 2024;10(16):e35942.
14. Asif M, Zeb A, Ishtiaq U, Ahmad W, Hou M. Aczel-Alsina aggregation operators for linear diophantine fuzzy set and their application to multiple-attribute decision making problems. *Expert Syst Appl.* 2025;126552. https://doi.org/10.1016/j.eswa.2025.126552
15. Sri SN, Vimala J, Kausar N, Ozbilge E, Özbilge E, Pamucar D. An MCDM approach on Einstein aggregation operators under Bipolar Linear Diophantine Fuzzy Hypersoft Set. *Heliyon*, 2024;10(9):e29863.
16. Khan SU, Nasir A, Marwat M. Investigation of Threats to Data States in Data Loss Prevention Using the Concept of Complex Linear Diophantine Fuzzy Relations. *Spectr Oper Res.* 2025;3(1):128-152. https://doi.org/10.31181/sor31202638
17. Aydoğdu A, Gül S, Alniak T. New information measures for linear Diophantine fuzzy sets and their applications with LDF-ARAS on data storage system selection problem. *Expert Syst Appl.* 2024;252:124135. https://doi.org/10.1016/j.eswa.2024.124135
18. Anjum R, Mirza MU, Kausar N, Ali R. Decision-making Framework for Urban Transportation Using Linear Diophantine Fuzzy Z-numbers with

Dombi Aggregation, TOPSIS and VIKOR Methods. *Spectr Oper Res.* 2025;4(1):1-34. https://doi.org/10.31181/sor4155

19. Tufail F, Gul R, Shabir M, Alharbi SK, Khalifa HAE-W. An enhanced Machine Learning Covering-Based Bipolar L-Fuzzy Rough Set PROMETHEE model for battery storage systems in renewable energy. *Expert Syst Appl.*, (in press) 2025;127951.
20. Xu W, Li Y. Enhancing information fusion and feature selection efficiency via the PROMETHEE method for multi-source dynamic decision data sets. *Knowl Based Syst.*, (in press) 2024;112781.
21. Yedjour D, Yedjour H, Amri MB, Senouci A. Rule extraction based on PROMETHEE-assisted multi-objective genetic algorithm for generating interpretable neural networks. *Appl Soft Comput.* 2023;151:111160. https://doi.org/10.1016/j.asoc.2023.111160
22. Li H, Yao X, Lu J. Optimal investment decision-making of energy storage technologies on renewable energy generation side based on the Internal Type-2 trapezoidal fuzzy sets-PROMETHEE-II method. *Int J Electr Power Energy Syst.* 2025;170:110861.
23. Bakshi S, Molla MU, Giri BC. Neutrosophic Hesitant Fuzzy PROMETHEE and its application to location selection of migrating swarm. *Appl Soft Comput.* 2025;180:113290.
24. Li Z, Pan Q, Wang D, Liu P. An extended PROMETHEE II method for multi-attribute group decision-making under Q-rung orthopair 2-tuple linguistic environment. *Int J Fuzzy Syst.* 2022;24(7):3039–3056. https://doi.org/10.1007/s40815-022-01321-z
25. Ecer F, Pamucar D. A novel LOPCOW-DOBI multi-criteria sustainability performance assessment methodology: An application in developing country banking sector. *Omega* 2022;112:102690. https://doi.org/10.1016/j.omega.2022.102690
26. Ulutaş A, Topal A, Görçün ÖF, Ecer F. Evaluation of third-party logistics service providers for car manufacturing firms using a novel integrated grey LOPCOW-PSI-MACONT model. *Expert Syst Appl.* 2023;241:122680.
27. Ecer F, Tanrıverdi G, Yaşar M, Görçün ÖF. Sustainable aviation fuel supplier evaluation for airlines through LOPCOW and MARCOS approaches with interval-valued fuzzy neutrosophic information. *J Air Transp Manag.* 2025;123:102705. https://doi.org/10.1016/j.jairtraman.2024.102705
28. Simic V, Dabic-Miletic S, Tirkolaee EB, Stević Ž, Ala A, Amirteimoori A. Neutrosophic LOPCOW-ARAS model for prioritizing industry 4.0-based material handling technologies in smart and sustainable warehouse management systems. *Appl Soft Comput.* 2023;143:110400.
29. Gülmez B. A novel hybrid MCDM framework combining TOPSIS, PROMETHEE II, and VIKOR for peach drying method selection. *Curr Res Food Sci.* 2025;10:101034. https://doi.org/10.1016/j.crfs.2025.101034
30. Gül S, Aydoğdu A. Novel distance and entropy definitions for linear diophantine fuzzy sets and an extension of TOPSIS (LDF-TOPSIS). *Expert Syst.* 2022;40(1). https://doi.org/10.1111/exsy.13104
31. Kamacı H, Marinkovic D, Petchimuthu S, Riaz M, Ashraf S. Novel distance-measures-based extended TOPSIS method under linguistic linear diophantine fuzzy information. *Symmetry* 2022;14(10):2140. https://doi.org/10.3390/sym14102140
32. Aydoğdu A. Novel linear diophantine fuzzy information measures based decision making approach using extended VIKOR method. *IEEE Access* 2023;11:95526–95544. https://doi.org/10.1109/access.2023.3309913
33. Jameel T, Yasin Y, Riaz M. An Integrated Hybrid MCDM Framework for Renewable Energy Prioritization in Sustainable Development. *Spectr Decis Mak Appl.* 2025;3(1):124-150. https://doi.org/10.31181/sdmap31202640
34. Wang D, Yuan Y, Liu Z, Zhu S, Sun Z. Novel distance measures of Q-Rung Orthopair Fuzzy sets and their applications. *Symmetry* 2024;16(5):574. https://doi.org/10.3390/sym16050574

Jeevitha Kannan *is working as a Research Scientist in SRM Institute of Science and Technology, Ramapuram, Chennai, India. She received the Ph.D and M.Sc. degree from the Department of Mathematics, Alagappa University, Karaikudi, Tamilnadu, India. She published 15+ research articles in an international peer-reviewed SCIE, ESCI, SCOPUS journals. She has presented many papers at several International Conferences. Her research areas are Fuzzy algebra, Fuzzy lattice ordered structures, Soft Computing, and Multi-Criteria Decision-Making Problems. She is a gold medalist in her Postgraduate.*

https://orcid.org/0000-0001-7846-9173

Vimala Jayakumar *currently serves as an Associate Professor in the Department of Mathematics at Alagappa University, Karaikudi, Tamil Nadu, India. She brings with her over 20 years of teaching experience. Her primary research areas include Algebra, Lattice Theory, Fuzzy Mathematics, Algebraic Hyperstructures, Fuzzy Algebraic Hyperstructures, Group Theory, Soft Computing, Neutrosophic Sets, and Multicriteria Decision Making. She has presented numerous research papers at various international conferences and has published over 90 research articles in reputed journals, including more than 30 papers in SCI-indexed international journals. She is also the author of 3 books in her domain of expertise. She has travelled professionally to more than five countries, contributing to global academic exchange and collaborations. Dr. J. Vimala currently serves on the editorial board of the Applied Soft Computing Journal and is also an Editor*

of the Journal of Fuzzy Extension and Applications.
https://orcid.org/0000-0003-3138-9365

Dragan Pamucar *is a Full Professor at University of Belgrade, Faculty of Organizational Sciences. Dr. Pamucar received a PhD in Applied Mathematics with specialization of Multi criteria modeling and soft computing techniques, from University of Defence in Belgrade, Serbia in 2013 and an MSc degree from the Faculty of Transport and Traffic Engineering in Belgrade, 2009. His research interest are in the field of Computational Intelligence, Multi-criteria decision-making problems, Neuro-fuzzy systems, fuzzy, rough and intuitionistic fuzzy set theory, neutrosophic theory. Application areas include wide range of logistics and engineering problems. Dr. Pamucar has five books and over 300 research papers published in SCI indexed International Journals including Experts Systems with Applications, Applied Soft Computing, Soft Computing, Computational intelligence, Computers and industrial engineering, Engineering Applications of Artificial Intelligence, IEEE Transactions on Intelligent Transportation Systems, IEEE Transactions of Fuzzy Systems, IEEE Transactions on Transportation Electrification, Information Sciences and research and so on, and many more. According to Scopus and Stanford University, he is among the World top 2% of scientists as of 2020 to present. According to WoS and Clarivate, he is among top 1% of highly cited researchers in 2022.*
https://orcid.org/0000-0003-3713-5849

S. Rajareega *is an Assistant Professor in the Department of Mathematics at GMR Institute of Technology, Rajam, Andhra Pradesh, India. Her research interests include Multi-Criteria Decision Making (MCDM), Fuzzy Set Theory, Fuzzy Algebra, and Soft Computing, with a particular focus on Fuzzy MCDM techniques. She has published a substantial number of research papers in reputed international journals and continues to contribute actively to advancements in the field of mathematical decision-making and computational intelligence.*
https://orcid.org/0009-0003-9081-0313

Appendix

Given two probability distributions $\mathbb{A}$ and $\mathbb{B}$, the Jensen-Shannon divergence [34] between them can be mathematically described follows:

$$\mathfrak{J}(\mathbb{A},\mathbb{B}) = \frac{1}{2}\Big[K(\mathbb{A},\frac{\mathbb{A}+\mathbb{B}}{2}) + K(\mathbb{B},\frac{\mathbb{A}+\mathbb{B}}{2})\Big]$$

with $K(\mathbb{A},\frac{\mathbb{A}+\mathbb{B}}{2}) = \sum_{i=1}^{n} a_i log\frac{a_i}{q_i}$ represents the Kullback–Leibler divergence.

$\mathfrak{J}(\mathbb{A},\mathbb{B})$ can also formulated by the following equation:

$$\begin{aligned}\mathfrak{J}(\mathbb{A},\mathbb{B}) =& H(\frac{\mathbb{A}+\mathbb{B}}{2}) - \frac{1}{2}H(\mathbb{A}) - \frac{1}{2}H(\mathbb{B})\\ =& \frac{1}{2}\Big(\sum_{i=1}^{n} a_i log\frac{2a_i}{a_i+b_i} + \sum_{i=1}^{n} b_i log\frac{2b_i}{a_i+b_i}\Big)\end{aligned}$$

with $H(\mathbb{A}) = -\sum_{i=1}^{n} a_i log a_i$, $H(\mathbb{B}) = -\sum_{i=1}^{n} b_i log b_i$, $H(\mathbb{A}+\mathbb{B}) = -\sum_{i=1}^{n}\frac{a_i+b_i}{2}log\frac{a_i+b_i}{2}$, where H denotes the Shannon entropy. Based on the above definition, the LDF divergence measure is obtained as follows:

Assume that

$$\mathfrak{E} = \langle \mathfrak{x}_{\mathfrak{E}}, \mathfrak{y}_{\mathfrak{E}}\rangle, \langle \tau_{\mathfrak{E}}, \upsilon_{\mathfrak{E}}\rangle$$

and

$$\mathfrak{P} = \langle \mathfrak{x}_{\mathfrak{P}}, \mathfrak{y}_{\mathfrak{P}}\rangle, \langle \tau_{\mathfrak{P}}, \upsilon_{\mathfrak{P}}\rangle$$

be two LFDN with hesitance $\mathfrak{w}_{\mathfrak{E}} = 1-(\mathfrak{x}_{\mathfrak{E}}\tau_{\mathfrak{E}} + \mathfrak{y}_{\mathfrak{E}}\upsilon_{\mathfrak{E}})$ and $\mathfrak{w}_{\mathfrak{P}} = 1-(\mathfrak{x}_{\mathfrak{P}}\tau_{\mathfrak{P}} + \mathfrak{y}_{\mathfrak{P}}\upsilon_{\mathfrak{P}})$, respectively. Then, the LDF divergence measure between $\mathfrak{E}$ and $\mathfrak{P}$ is depicted as:

$$\mathfrak{J}(\mathfrak{E},\mathfrak{P}) = \frac{1}{2}\Big[K(\mathfrak{E},\frac{\mathfrak{E}+\mathfrak{P}}{2}) + K(\mathfrak{P},\frac{\mathfrak{E}+\mathfrak{P}}{2})\Big]$$

with

$$\begin{aligned}K(\mathfrak{E},\mathfrak{P}) =& \mathfrak{x}_{\mathfrak{E}} log\frac{\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\frac{\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\frac{\tau_{\mathfrak{E}}}{\tau_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{E}} log\frac{\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{P}}}\\ &+ \upsilon_{\mathfrak{E}} log\frac{\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\frac{\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{P}}}\end{aligned}$$

The $\mathfrak{J}(\mathfrak{E},\mathfrak{P})$ can also be formulated follows:

$$\mathfrak{J}(\mathfrak{E},\mathfrak{P}) = \frac{1}{2}\left\{\begin{array}{l}\mathfrak{x}_{\mathfrak{E}} log\frac{2\mathfrak{x}_{\mathfrak{E}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{x}_{\mathfrak{P}} log\frac{2\mathfrak{x}_{\mathfrak{P}}}{\mathfrak{x}_{\mathfrak{E}}+\mathfrak{x}_{\mathfrak{P}}} + \mathfrak{y}_{\mathfrak{E}} log\frac{2\mathfrak{y}_{\mathfrak{E}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} +\\ \mathfrak{y}_{\mathfrak{P}} log\frac{2\mathfrak{y}_{\mathfrak{P}}}{\mathfrak{y}_{\mathfrak{E}}+\mathfrak{y}_{\mathfrak{P}}} + \tau_{\mathfrak{E}} log\frac{2\tau_{\mathfrak{E}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} + \tau_{\mathfrak{P}} log\frac{2\tau_{\mathfrak{P}}}{\tau_{\mathfrak{E}}+\tau_{\mathfrak{P}}} +\\ \upsilon_{\mathfrak{E}} log\frac{2\upsilon_{\mathfrak{E}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \upsilon_{\mathfrak{P}} log\frac{2\upsilon_{\mathfrak{P}}}{\upsilon_{\mathfrak{E}}+\upsilon_{\mathfrak{P}}} + \mathfrak{w}_{\mathfrak{E}} log\frac{2\mathfrak{w}_{\mathfrak{E}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}} +\\ \mathfrak{w}_{\mathfrak{P}} log\frac{2\mathfrak{w}_{\mathfrak{P}}}{\mathfrak{w}_{\mathfrak{E}}+\mathfrak{w}_{\mathfrak{P}}}\end{array}\right\}.$$

An International Journal of Optimization and Control: Theories & Applications
(https://accscience.com/journal/ijocta)