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RESEARCH ARTICLE

A rational power function-based approach for solving rational fractional differential equations

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ABSTRACT

A highly efficient and accurate numerical method for systems of fractional differential equations (FDEs) with rational order is presented in this paper. Rational power functions and rational Taylor series projection are utilized to obtain approximate solutions. Rational semi-smooth spaces are introduced, and the regularity of solutions in these spaces is established. A series of theoretical results, such as the existence and uniqueness of solutions, properties of the rational Taylor series and its remainder term, and an operational matrix approach, are derived. It is proven that the numerical solution is exact when the exact solution is a rational power series, and the approximate solution is shown to be the rational Taylor series projection of the exact solution. The convergence of the method is analyzed. The efficiency of the proposed method is demonstrated through numerical experiments, which show significant improvements in computational time compared to existing methods.

1. Introduction

Systems of fractional differential equations make important contributions in the modeling of complex systems. The state of complex systems can be described by fading memory.[1] A variety of real-world physical or biological issues can be effectively represented using fractional differential equations, as illustrated by Mainardi[2] for a relaxation system (FDEs).

Fading memory in the definitions of fractional operators makes them an effective tool for modeling natural phenomena. Fading memory may involve singular and non-singular kernels; a recent review of fractional operators can be found in the study by Shiri and Baleanu.[3]

Fractional calculus frameworks and FDE applications are well-documented,[3–5] with recent studies further expanding their utility across diverse fields, such as cancer modeling [6], network vulnerability analysis,[7] and the study of nonlinear boundary value problems.[8] Recent studies have demonstrated the applications of FDEs in various systems, such as time–fractional systems,[9,10] fractional Klein–Gordon equations,[10] non-singular fractional operators,[11,12] and stochastic system controllability.[13–15]

The characteristics of a system of FDEs can be outlined as follows in Equation (1):

$$\begin{aligned} {}^cD^\alpha \mathfrak{Y}(t) &= A\mathfrak{Y}(t) + \mathfrak{f}(t), \quad t \in [0, \mathfrak{T}],\ \mathfrak{T} \in \mathbb{R}, \\ \mathfrak{Y}(0) &= \mathfrak{Y}_\circ \end{aligned} \tag{1}$$

where $\mathfrak{Y}_\circ \in \mathbb{R}^\nu$, ${}^cD^\alpha$ is the Caputo derivative, $\alpha \in (0,1)$, $\mathfrak{Y} : [0, \mathfrak{T}] \to \mathbb{R}^\nu$ composed of state

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functions $\left[y_1, y_2, ..., y_\nu\right]^T$, A is a matrix of dimension $\nu \times \nu$, $\mathfrak{f} : [0, \mathfrak{T}] \to \mathbb{R}^\nu$ is composed of source functions, that is $\mathfrak{f} = \left[\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\nu\right]^T$, and ν is the dimension of the system.

Despite the significant dynamics that characterize the solutions to System Equation (1), it is critical to seek analytical or numerical solutions, as highlighted by Gomez-Aguilar et al.[6] Notably, there is a research gap for efficient (fast and accurate) numerical methods for these systems, which has yet to be explored.

Due to the singularity in the kernel of ${}^cD^\alpha$, solutions of the system may not lie in C^1 or higher smooth function spaces. Thus, numerical methods using polynomials—even spline polynomials—fail to achieve higher-order accuracy. One approach to address this is to utilize piecewise polynomials with graded meshes.[16–18] However, to our knowledge, no comparable progress has been made for polynomial-based spectral methods.

We aim to introduce a faster operational matrix approximation that overcomes the problem of singularity in the solution of such equations. From Equation (1), we get

$$\mathfrak{Y}(t) = AI^\alpha \mathfrak{Y}(t) + \mathfrak{g}(t) \tag{2}$$

where $I^\alpha \mathfrak{f}(t) = \left[I^\alpha \mathfrak{f}_1(t), ..., I^\alpha \mathfrak{f}_\nu(t)\right]$ and $\mathfrak{g} = I^\alpha \mathfrak{f} + \mathfrak{Y}_0$. Consequently, $\mathfrak{Y}$ satisfies

$$\mathfrak{Y}(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)}\, \mathfrak{Y}(s)\, ds + \mathfrak{g}(t) \tag{3}$$

An interesting theorem of Chapter 6 of the book by Brunner.[19] reveals the regularity of the solutions of such integral equations. According to this theorem, the weakly singular integral Equation (3) accepts a unique solution of the form

$$\mathfrak{Y}_m(t) = \sum_{\substack{j,k \in \mathbb{N}_0 \\ j+k(1-\alpha)<m}} \gamma_{j,k}(\alpha)\, t^{j+k(1-\alpha)} + Y_m(t;\alpha) \tag{4}$$

and $Y_m(.;\alpha) \in C^m(I)$, if $\mathfrak{g} \in C^m([0,\mathfrak{T}])$ for a $m \in \mathbb{N}$.

Surprisingly, as far as we know, no one used such a representation to obtain a numerical solution. In this paper, our contribution is to introduce such methods.

The power functions $t^{j+k\alpha}$ include polynomials and fracto-nomials. They are used as an approximate basis in several research. For example, several spectral methods in other studies[20–24] applied a polynomial-driven basis for solving such equations. Zayernouri et al.[25–27] obtained the solution of fractional Sturm–Liouville problems in terms of orthogonal poly-fracto-nomials and used it for new numerical methods. Recently, such methods of numerical approach have found more attention among researchers[28–30]

Despite these advances, a key research gap persists: there is no efficient numerical method tailored for FDEs with rational orders. Existing approaches either sacrifice accuracy for computational speed or vice versa. For example, Mittag–Leffler function-based solvers are accurate but computationally expensive (Example 3 & 4), while polynomial interpolations fail to converge for non-smooth FDE solutions (e.g., $t^{1/3}\ e^t$, Table 1).

This study addresses this gap by introducing a rational power function-based framework, with three core objectives:

- Overcome singularity limitations: Replace traditional polynomial bases with rational power functions ($t^{i/m}$, where $\alpha = n/m$ is the rational fractional order) to naturally match the non-smooth structure of FDE solutions, ensuring convergence even for functions like $\sqrt{t}e^t$.
- Establish theoretical foundations: Introduce "rational semi-smooth spaces" ($C^{p,\alpha}[0,\mathfrak{T}]$), a larger space than standard smooth function spaces, to rigorously prove solution regularity, existence, and uniqueness for rational-order FDEs—filling a theoretical void in existing regularity analyses.[19]
- Develop an efficient numerical tool: Derive a rational Taylor series (RTS) projection method and corresponding operational matrix for the Riemann–Liouville integral. This method guarantees exactness when the exact solution is a rational power series, achieves fast convergence ($O((pm+1)!)^{-1}$), and cuts computational time drastically (e.g., 0.02 seconds vs. 15 seconds for Mittag–Leffler solvers, Table 2), addressing the speed accuracy tradeoff in current methods.

Our main idea in developing these methods is based on the fact that for rational orders $\alpha = \frac{n}{m}$ with $\gcd(n,m) = 1$, we can replace $t^{j+k\alpha}$ with rational power functions $t^{\frac{i}{m}}$, $(i = 0, 1, ...)$, without loss of generality. Then, we establish an approximation theory based on this new basis. We generalize Taylor's expansion theory to bigger spaces that reveal the regularity of the solutions for a larger class of the systems of FDEs.

Our contributions are as follow:

- We propose a rational power series for solving FDEs.

Table 1. Table of notations

Notation	Definition
α	Fractional order of FDEs, where $\alpha \in (0,1)$
n, m	Numerator/denominator of rational fractional order $\alpha = \frac{n}{m}$ ($\gcd(n,m)=1$, $n,m \in \mathbb{N}$)
p	Degree of rational power basis polynomials,
$\mathfrak{Y}(t)$	ν-dimensional state vector of FDE systems, $\mathfrak{Y} : [0,\mathfrak{T}] \to \mathbb{R}^\nu$, $\mathfrak{Y} = [y_1, y_2, \dots, y_\nu]^T$
${}^cD^\alpha$	Caputo fractional derivative of order α
I^α	RL fractional integral of order α
$C^{p,\alpha}[0,\mathfrak{T}]$	Rational semi-smooth space: $\{g : [0,\mathfrak{T}] \to \mathbb{R} \mid g(x^m) \in C^p[0,\mathfrak{T}^{\frac{1}{m}}]\}$
$\lVert\cdot\rVert_{p,\alpha}$	Norm of $C^{p,\alpha}[0,\mathfrak{T}]$: $\lVert g\rVert_{p,\alpha} = \sum_{i=0}^{p} \lVert(g(x^m))^{(i)}\rVert$ (supremum norm for vector functions)
$\Pi_{p,\alpha}$	Linear space spanned by rational power functions: $\Pi_{p,\alpha} = \mathrm{span}\{t^{\frac{i}{m}} \mid i = 0,1,\dots,pm\}$
$T_{p,\alpha}$	RTS projection operator, mapping $C^{pm+1,\alpha}[0,\mathfrak{T}]$ to $\Pi_{p,\alpha}$
$\Psi(t)$	Basis vector of rational power functions: $\Psi = [\psi_0, \psi_1, \dots, \psi_{pm}]^T$, $\psi_j(t) = t^{\frac{j}{m}}$
Λ	Operational matrix for RL integral I^α, satisfying $I^\alpha\Psi \approx \Lambda\Psi$ (with negligible error $\mathfrak{E} = \mathbf{O}(t^p)$)
$\mathfrak{T}$	Upper bounds of the time interval
ν	Dimension of the FDE system (number of state/source functions)
$\mathfrak{Y}_0$	Initial condition of FDE systems, $\mathfrak{Y}_0 \in \mathbb{R}^\nu$ Abbreviations: FDE: Fractional differential equation; RL: Riemann–Liouville; RTS: Rational Taylor series.

Abbreviations: FDE: Fractional differential equation; RL: Riemann–Liouville; RTS: Rational Taylor series.

- We introduce a rational semi-smooth space (larger than a smooth space) and obtain standard existence and regularization results for the solutions of FDEs in that space.
- We introduce the rational Taylor series and obtain its remainder term in a rational semi-smooth space.
- We construct the corresponding operational method from the analysis.
- We provide theorems related to an operational method and Taylor series projection.
- We show the exactness of the solution when the exact solution is a rational power series.
- We prove the approximated solution is a reflection of the exact solutions on rational power series.
- We provide a convergence analysis, showing that the order of convergence is $((pm+1)!)^{-1}$.

To ensure clarity and readability for readers, we first consolidate all key symbols and their definitions used throughout this work (Table 1).

In Section 2, we analyze the use of hybrid-fractional power functions as the basis of approximation. In Section 3, we obtain the regularity of the studied FDE on new spaces. In Section 4, we introduce RTS expansion and analyze the truncated error on the new spaces. In Section 5, we introduce an operational matrix approach for the Riemann–Liouville (RL) integral operator. In Section 6, we develop an RTS projection method for solving FDEs. In Section 7, we analyze the new method, including the condition for exactness of the proposed numerical solution, as well as its convergence and order of convergence. Finally, in Section 8, we support our research in terms of numerical experiments.

2. Rational power functions

Here and throughout the paper, we assume that $\alpha = \frac{n}{m}$, $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$.

In this paper, we suggest rational power functions (fracto-nomials) for solving rational FDEs. A rational power series is a series of the form

$$\mathfrak{Y}(t) = \sum_{i=0}^{pm} \eta_i t^{\frac{i}{m}} \tag{5}$$

where $\frac{n}{m} = \alpha$, and $\gcd(m,n) = 1$. It is easy to establish a uniqueness theorem for interpolations with such a series.

Theorem 1. *Let $\alpha = \frac{n}{m}$, $\gcd(m,n) = 1$, and t_i, ($i = 0, \cdots, pm$) for a nonnegative integer number p. Suppose Θ be a $pm \times pm$ matrix defined by*

$$\Theta_{i,j} = t_i^{\frac{j}{m}}. \tag{6}$$

Then, the interpolation problem

$$\Theta\eta = F$$

has a unique solution, where $F = [f(t_0), \cdots, f(t_{pm})]^T$.

Proof. Setting $x_i = t_i^{\frac{1}{m}}$, we get $\Theta_{i,j} = x_i^j$. But Θ is a Vandermonde matrix and invertible which completes the proof.

For smooth functions, polynomial interpolations converge rapidly to the solution in L^2 norm[31] and [32] But, this is not valid for functions in $C[0,\mathfrak{T}]$ with sup norm such as $\sqrt{t}$ and $\sqrt{t}\sin(t)$ as illustrated in the next example. With this superiority, convergence using the proposed basis will be guaranteed for such functions. To illustrate this superiority, we briefly examined it through interpolation.

Example 1. *Consider $f(t) = t^{\frac{1}{3}}e^t$. We interpolated $[0,2]$ with $m = 3$ and $t_i = \frac{i}{pm}$, $N = 0,\cdots,pm$.*

Table 2, reports the interpolation with polynomials and our introduced method. Notably, polynomial interpolation did not converge, but better results can be obtained with rational power functions.

Table 2. Rational power function interpolation converges for functions like $t^{\frac{1}{3}}e^t$, unlike polynomial interpolation

p	4	6	8	10
e_1	2.0e+0	6.5e-3	1.2e-3	4.6e-5
e_2	1.4e-1	1.2e-1	1.1e-1	8.5e-1

Note: e_1 = supremum error (rational power functions); e_2 = supremum error (polynomials).

3. Regularity of the solution in $C^{p,\alpha}[0,\mathfrak{T}]$

For the regularity of FDEs solutions, we started by defining rational semi-smooth spaces $C^{p,\alpha}[0,\mathfrak{T}]$:

Definition 1. *Rational semi-smooth space: Let $p \in \mathbb{N}\cup\{0\}$. The rational semi-smooth space $C^{p,\alpha}[0,\mathfrak{T}]$ is defined as:*

$$C^{p,\alpha}[0,\mathfrak{T}] := \{g : [0,\mathfrak{T}] \to \mathbb{R} \mid g(x^m) \in C^p[0,\mathfrak{T}^{\frac{1}{m}}]\} \tag{7}$$

where $C^p[0,\mathfrak{T}]$ denotes the standard space of p-times continuously differentiable functions on $[0,\mathfrak{T}]$.

It is straightforward to verify that the following equality holds:

$$C^{0,\alpha}[0,\mathfrak{T}] = C[0,\mathfrak{T}] \tag{8}$$

Example 2. *Let $\alpha = 0.5$. Then, $m = 2$ and not only $e^t \in C^{p,\alpha}[0,\mathfrak{T}]$, but also $\sqrt{t}e^t \in C^{p,\alpha}[0,\mathfrak{T}]$. For the latter,*

$$g(x^2) = \sqrt{x^2}e^{x^2} = xe^{x^2} \in C^p[0,\sqrt{\mathfrak{T}}] \tag{9}$$

for any non-negative integer number p.

Remark 1. *We know that C^p and $C^{p,\alpha}$ for $p \geq 1$ are not Banach spaces with supremum norm. Similarly, their Cartesian product $(C^p)^\nu$ and $(C^{p,\alpha})^\nu$ are not Banach. But, we can make them Banach with another norm. In this respect, the space $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$ is a Banach space with the norm:*

$$\|g\|_{p,\alpha} = \sum_{i=0}^{p} \|(g(x^m))^{(i)}\| \tag{10}$$

where $\|.\|$ is the maximum of the supremum norm of each vector function. It is important to emphasize that the ith derivative of a vector here is an element-wise operator.

Theorem 2. *$C^{p,\alpha}[0,\mathfrak{T}]$ with the Equation (10) is a Banach space.*

Proof. Let $\{g_k\}_{k=1}^{\infty} \subset C^{p,\alpha}[0,\mathfrak{T}]$ be Cauchy under $\|.\|_{p,\alpha}$. By definition,

$$\{h_{g_k}(x) := g_k(x^m)\}_{k=1} \tag{11}$$

is Cauchy in $C^p[0,\mathfrak{T}]$ (a known Banach space), Hence, there exists $h \in C^p[0,\mathfrak{T}]$ with $\lim_{k\to\infty}\|h_{g_k} - h\|_{C^p} = 0$ where

$$\|h\|_{C^p} = \sum_{i=0}^{p}\|h^{(i)}\|. \tag{12}$$

Define $g(x) := h(x^{1/m})$. Then, $g \in C^{p,\alpha}[0,\mathfrak{T}]$. Moreover,

$$\lim_{k\to\infty}\|g_k - g\|_{p,\alpha} = \lim_{k\to\infty}\|h_{g_k} - h\|_{C^p} = 0. \tag{13}$$

As a result, every Cauchy sequence in $C^{p,\alpha}[0,\mathfrak{T}]$ converges to an element of the space. Thus, $C^{p,\alpha}[0,\mathfrak{T}]$ is Banach.

The next theorem states that $C^{p,\alpha}[0,\mathfrak{T}] \subseteq C[0,\mathfrak{T}]$ is invariant subspace of I^α.

Theorem 3. *Let $y \in (C^{p,\alpha}[0,\mathfrak{T}], \|.\|_{p,\alpha})$. Then, $I^\alpha y \in (C^{p,\alpha}[0,\mathfrak{T}], \|.\|_{p,\alpha})$ and is a bounded operator.*

Proof. Define $w(t) := I^\alpha y(t)$. Then,

$$\begin{aligned} w(t^m) =& \frac{1}{\Gamma(\alpha)}\int_0^{t^m}\frac{y(\tau)}{(t^m-\tau)^{1-\alpha}}d\tau \\ =& \frac{mt^n}{\Gamma(\alpha)}\int_0^1\frac{y(t^m s^m)s^{m-1}}{(1-s^m)^{1-\alpha}}ds \end{aligned} \tag{14}$$

for $t \in [0,\mathfrak{T}^{\frac{1}{m}}]$. For $i = 0,\ldots,p$, the ith-derivative of Equation (14) (with respect to t) follows from the product rule:

$$(w(t^m))^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \frac{(mt^n)^{(k)}}{\Gamma(\alpha)} \int_0^1 \frac{(y(t^m s^m))^{(i-k)} s^{m-1}}{(1-s^m)^{1-\alpha}} ds. \tag{15}$$

We first analyze derivatives of $y\left(t^m s^m\right)$. Direct computations give:

$$\begin{aligned}(y\left(t^m s^m\right))^{(1)} =& m s^m t^{m-1} y'\left(t^m s^m\right),\\ (y\left(t^m s^m\right))^{(2)} =& m\left(m t^{m-1}\right)^2 y''\left(t^m s^m\right)\\ &+ m(m-1) s^m t^{m-2} y'\left(t^m s^m\right)\end{aligned} \tag{16}$$

and similarly,

$$(y\left(t^m s^m\right))^{(j)} = \sum_{k=1}^{j} \gamma_{k,j}(s,t) y^{(k)}\left(t^m s^m\right) \tag{17}$$

where $\gamma_{k,j}(s,t)$ (for $j = 1, \cdots, p$, $k = 1, \cdots, j$) are polynomials in t and s. Thus, $\gamma_{k,j}(s,t)$ are bounded on $[0,1] \times [0, \mathfrak{T}^{\frac{1}{m}}]$ and we define

$$C_1 := \max_{k,j} \left\{ \sup_{\substack{s\in[0,1] \\ t\in[0,\mathfrak{T}^{\frac{1}{m}}]}} |\gamma_{k,j}(s,t)| \right\}. \tag{18}$$

From Equation (17), it follows that

$$\| \left(y\left(t^m s^m\right)\right)^{(j)} \| \leq C \|y\|_{p,\alpha}. \tag{19}$$

Substituting this into Equation (15) and taking supremum norm over $[0, \mathfrak{T}^{\frac{1}{m}}]$, we get:

$$\begin{aligned}&\|(w(t^m))^{(i)}\|\\ \leq& \sum_{k=0}^{i} \binom{i}{k} \frac{(mt^n)^{(k)} C_1 \|y\|_{p,\alpha}}{\Gamma(\alpha)} \int_0^1 \frac{s^{m-1}}{(1-s^m)^{1-\alpha}} ds\\ =& \sum_{k=0}^{i} \binom{i}{k} \frac{(mt^n)^{(k)}}{m\Gamma(\alpha+1)} C_1 \|y\|_{p,\alpha}.\end{aligned} \tag{20}$$

Define

$$C_2 := \sup_{t\in[0,\mathfrak{T}^{\frac{1}{m}}]} \left\{ \sum_{k=0}^{i} \binom{i}{k} \frac{(mt^n)^{(k)}}{m\Gamma(\alpha+1)} \right\}. \tag{21}$$

Then, for each $i = 0, \ldots, p$

$$\|(w(t^m))^{(i)}\| \leq C_1 C_2 \|y\|_{p,\alpha} \tag{22}$$

Summing over i, we obtain

$$\|(w(t^m))\|_{p,\alpha} \leq \sum_{i=0}^{p} C_1 C_2 \|y\|_{p,\alpha} = p C_1 C_2 \|y\|_{p,\alpha}. \tag{23}$$

Finally, note that $ts \in [0, \mathfrak{T}^{\frac{1}{m}}]$ for $s \in [0,1]$, so $y(t^m s^m) = y((ts)^m) \in C^p[0, \mathfrak{T}^{\frac{1}{m}}]$ (with respect to t and s), since $y(x^m) \in C^p[0, \mathfrak{T}^{\frac{1}{m}}]$. Thus, $w \in C^{p,\alpha}[0, \mathfrak{T}]$, and the operator norm $\|I^\alpha\|$ is bounded by pC_1C_2. This completes the proof.

From Equation (2)

$$(I - AI^\alpha)\mathfrak{Y}(t) = I^\alpha \mathfrak{f}(t) + \mathfrak{Y}_0 \tag{24}$$

where I is an identity operator. Space $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$ is invariant for $(I - AI^\alpha) : (C([0,\mathfrak{T}]))^\nu \to (C([0,\mathfrak{T}]))^\nu$, i.e.,

$$(I - AI^\alpha)(C^{p,\alpha}[0,\mathfrak{T}])^\nu \subseteq (C^{p,\alpha}[0,\mathfrak{T}])^\nu. \tag{25}$$

Proof. Each component of the operator $I - AI^\alpha$ is a linear combination of the 1D identity operator and I^α. Thus, the assertion of this theorem follows Theorem 3.

Let $\mathfrak{f} \in (C^p[0,\mathfrak{T}])^\nu$. Then, $I^\alpha \mathfrak{f} + \mathfrak{Y}_0 \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$.

Proof. It follows Theorem 3.

Remark 2. *To obtain regularity in $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$, we need to prove that the solution belongs to this space. Such regularity can be obtained by properties of the resolved kernel similar to Chapter 6 of the book by Brunner,[19] which may take several pages. We avoid repeating such complex analyses by using Banach's fixed-point theorem. Another approach can be continued on Hilbert spaces and using compact operator theories, particularly Fredholm's alternative theorem. Such approaches were obtained in Hilbert spaces such as $L^2[0,\mathfrak{T}]$. Other regular Hilbert spaces have been introduced and studied.[16,17]*

Regularity of Equation (2) on Banach space $C([0,\mathfrak{T}])$ and Hilbert space L^2 is well-studied.[19] Here, we establish the regularity of the solution by Banach's fixed point theorem in the Banach space $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$.

Theorem 4. *Suppose $\mathfrak{f} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$ and $\|AI^\alpha\|_{p,\alpha} < 1$, where $\|.\|_{p,\alpha}$ in induced operator norm on $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$. Then, Equation (24) has a unique solution on $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$.*

Proof. We show $\mathcal{G}$ is a contraction. Define $\mathcal{G} : (C^{p,\alpha}[0,\mathfrak{T}])^\nu \to (C^{p,\alpha}[0,\mathfrak{T}])^\nu$ by

$$\mathcal{G}\mathfrak{Y} = AI^\alpha \mathfrak{Y} + I^\alpha \mathfrak{f}(t) + \mathfrak{Y}_0. \tag{26}$$

Therefore,

$$\begin{aligned}\|\mathcal{G}\mathfrak{Y} - \mathcal{G}\mathfrak{Z}\|_{p,\alpha} =& \|AI^\alpha(\mathfrak{Y} - \mathfrak{Z})\|_{p,\alpha}\\ \leq& \|AI^\alpha\|_{p,\alpha} . \|\mathfrak{Y} - \mathfrak{Z}\|_{p,\alpha}\\ \leq& \|\mathfrak{Y} - \mathfrak{Z}\|_{p,\alpha}.\end{aligned} \tag{27}$$

Thus, $\mathcal{G}$ is a contraction on $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$.

Although using Banach fixed points is sometimes inevitable, it does not provide an optimal result. We can obtain an optimal analysis by another approach. We improve the above theorem by removing the condition $\|AI^\alpha\|_{p,\alpha} < 1$. For nonlinear FDEs in continuous spaces, such relaxations have been established in recent publications.[33,34] Here, we leverage the advantages of the index rule and the linearity of the fractional operator AI^α, specifically:

$$(AI^\alpha)^n = A^n I^{\alpha n}. \tag{28}$$

Supposing

$$\mathfrak{Y}_1 = I^\alpha \mathfrak{f}(t) + \mathfrak{Y}_0$$

and applying the recursive iteration

$$\mathfrak{Y}_{n+1} = AI^\alpha \mathfrak{Y}_n + \mathfrak{Y}_1,$$

we obtain

$$\begin{aligned}\mathfrak{Y}_n(t) &= \sum_{i=0}^{n-1} A^i I^{\alpha(i+1)}\mathfrak{f} + \sum_{i=0}^{n-1} A^i I^{\alpha(i)}\mathfrak{Y}_0 \\ &= \int_0^t \frac{\sum_{i=0}^{n-1}\frac{(t-z)^{\alpha i}A^i}{\Gamma(\alpha i+\alpha)}\mathfrak{f}(z)}{(t-z)^{1-\alpha}}dz \\ &\quad + \sum_{i=0}^{n-1}\frac{A^i t^{i\alpha}}{\Gamma(i\alpha+1)}\mathfrak{Y}_0.\end{aligned} \tag{29}$$

If $\mathfrak{Y}_n$ converges, then $\lim_{n\to\infty}\mathfrak{Y}_n$ is a solution of the system

$$\mathfrak{Y} = AI^\alpha\mathfrak{Y} + \mathfrak{Y}_1 \tag{30}$$

which is a rewrite of Equation (24). Thus, supposing the convergence of

$$E_{\alpha,\alpha}((t-z)^\alpha A) := \sum_{i=0}^{\infty}\frac{(t-z)^{\alpha i}A^i}{\Gamma(\alpha i+\alpha)} \tag{31}$$

and

$$E_\alpha(At^\alpha) = \sum_{i=0}^{\infty}\frac{A^i t^{i\alpha}}{\Gamma(i\alpha+1)} \tag{32}$$

a solutions of Equation (24) satisfies

$$\mathfrak{Y}(t) = \int_0^t \frac{E_{\alpha,\alpha}((t-z)^\alpha A)\mathfrak{f}(z)}{(t-z)^{1-\alpha}}dz + E_\alpha(At^\alpha)\mathfrak{Y}_0. \tag{33}$$

Theorem 5. *Suppose* $\mathfrak{f} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$ *and the series in definition of* $E_\alpha(Ax^\alpha)$ *and* $E_{\alpha,\alpha}(Ax^\alpha)$ *converges uniformly on* $[0,\mathfrak{T}]$. *Then,* $\mathfrak{Y} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$.

Proof. According to Theorem 3, $I^{\alpha(i+1)}\mathfrak{f} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$ and $I^{\alpha(i)}\mathfrak{Y}_0 \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$, thus $\mathfrak{Y}_n \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$. Since $(C^{p,\alpha}[0,\mathfrak{T}])^\nu$ is a compact space, $\mathfrak{Y} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$ from Equation (33).

Theorem 6. $E_\alpha(Ax^\alpha)$ *and* $E_{\alpha,\alpha}(Ax^\alpha)$ *converges uniformly on* $[0,\mathfrak{T}]$, *if* $\mathrm{Re}\alpha > 0$.

Proof. For one dimensional case, the assertion is already well studied.[35] The higher-dimensional case follows from the one-dimensional case. Let

$$\mathcal{S}_k(x) := (E_\alpha(Ax^\alpha))_k = \sum_{i=0}^{k}\frac{1}{\Gamma(\alpha i+1)}A^i x^{i\alpha}. \tag{34}$$

Thus,

$$\begin{aligned}\frac{d^j}{dx^j}\mathcal{S}_k(x) &= \sum_{i=j}^{k}\frac{1}{\Gamma(\alpha i-j+1)}A^i x^{\alpha i-j} \\ &= A^j x^{j(\alpha-1)}\sum_{i=0}^{k}\frac{A^i x^{\alpha i}}{\Gamma(\alpha i+\alpha j-j+1)} \\ &= A^j x^{j(\alpha-1)}(E_{\alpha,\alpha j-j+1}(Ax^\alpha))_k\end{aligned} \tag{35}$$

and

$$\frac{d^j}{dx^j}\mathcal{S}_k(x^m) = mA^j x^{nj-1}(E_{\alpha,\alpha j-j+1}(Ax^n))_k. \tag{36}$$

Therefore, $\mathcal{S}_k(x) \in (C^{p,\alpha}[0,\mathfrak{T}])^{\nu\times\nu}$. On one hand,

$$\begin{aligned}&\|\frac{d^j}{dx^j}\mathcal{S}_k(x^m)\| \\ &\le m\|A\|^j.|x|^{nj-1}(E_{\alpha,\alpha j-j+1}(\|A\|.|x|^n))_k\end{aligned} \tag{37}$$

since $\|A^n\| \le \|A\|^n$. Equation (37) is composed of a one-dimensional Mittag–Leffler function multiplied by a uniformly continuous function. Thus, it converges uniformly as $k\to\infty$. Consequently, $\frac{d^j}{dx^j}\mathcal{S}_k(x^m)$ converges on Banach space $(C[0,\mathfrak{T}])^{\nu\times\nu}$. As a final result, $\mathcal{S}_k$ uniformly converges on $(C^{p,\alpha}[0,\mathfrak{T}])^{\nu\times\nu}$, which complete the proof for one parameter Mittag–Lefller functions. A similar analysis holds for the two-parameter Mittag–Leffler function.

Theorem 7. *Suppose* $\mathfrak{f} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$. *Then,* $\mathfrak{Y} \in (C^{p,\alpha}[0,\mathfrak{T}])^\nu$.

Proof. It is a combination of Theorems (5) and (6).

4. Fractional taylor series projection

Let $g \in C^{mp+1,\alpha}[0,\mathfrak{T}]$ and define $h : [0,\mathfrak{T}^{\frac{1}{m}}] \to \mathbb{R}$ by

$$h(x) = g(x^m). \tag{38}$$

Then, the pmth degree Taylor series is

$$T_{pm}h(x) = \sum_{i=0}^{pm}\frac{h^{(i)}(x)}{i!}x^i. \tag{39}$$

This can define the RTS by substituting $x = t^{\frac{1}{m}}$

$$T_{pm}g(t) = T_{pm}g(x^m) = T_{pm}h(x) = \sum_{i=0}^{pm} \frac{h^{(i)}(0)}{i!} t^{\frac{i}{m}}. \tag{40}$$

In this respect, we can define the rational Taylor projection $T_{p,\alpha} : C^{pm+1,\alpha}[0, \mathfrak{T}] \to \Pi_{p,\alpha}$ by

$$T_{p,\alpha}g(t) = \sum_{i=0}^{pm} \frac{h^{(i)}(0)}{i!} t^{\frac{i}{m}} \tag{41}$$

where

$$\Pi_{p,\alpha} = \text{span}\{t^{\frac{i}{m}} : i = 0, \cdots, pm\}.$$

Now, we can state a very nice theorem. We can extend the mean value theorem for the space $C^{pm+1,\alpha}[0, \mathfrak{T}]$.

Theorem 8. *Let $g \in C^{mp+1,\alpha}[0, \mathfrak{T}]$, and p be a non-negative integer number. Then,*

$$R_p(g)(t) = g(t) - T_{p,\alpha}g(t) = \frac{h^{(pm+1)}(\xi)}{(pm+1)!} t^{p+\frac{1}{m}} \tag{42}$$

where $h(x) = g(x^m)$.

Proof. According to the classical mean value theorem,

$$R_p h(x) := h(x) - T_{pm}h(x) = \frac{h^{(pm+1)}(\xi)}{(pm+1)!} x^{pm+1} \tag{43}$$

where $\xi \in (0, x)$. Since $h(x) = g(x^m) \in C^{pm+1}[0, \mathfrak{T}^{\frac{1}{m}}]$,

$$g(t) - T_{p,\alpha}g(t) = \frac{h^{(pm+1)}(\xi)}{(pm+1)!} t^{p+\frac{1}{m}}. \tag{44}$$

5. Operational matrix approach

Let $\mathfrak{Y} = [y_1, \cdots, y_\nu]^T$ be a ν-dimensional vector-function. Let $\eta_i = [\eta_{i,0}, \cdots, \eta_{i,pm}]$ be ith row of the $\nu \times (pm+1)$ matrix H and $\Psi = [\psi_0, \cdots, \psi_{pm}]^T$, where $\psi_j(t) := t^{\frac{j}{m}}$. We suppose $y_i = \eta_i \Psi$. Then, we can write

$$\mathfrak{Y} = H\Psi. \tag{45}$$

It is easy to check

$$\mathfrak{Y}_0 = \mathfrak{O}\Psi \tag{46}$$

where $\mathfrak{O}$ is a $\nu \times (pm+1)$ matrix described by

$$\mathfrak{O} = \begin{pmatrix} \mathfrak{Y}_{01} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{Y}_{0\nu} & 0 & \cdots & 0 \end{pmatrix}. \tag{47}$$

For the RL integral, we know

$$I^\alpha t^{\frac{j}{m}} = \frac{\Gamma(1+\frac{j}{m})}{\Gamma(1+\alpha+\frac{j}{m})} t^{\frac{j+n}{m}}. \tag{48}$$

Thus, setting $\rho = pm - n + 1$

$$C_{\rho\times\rho} = \text{diag}\left[\frac{\Gamma(1)}{\Gamma(1+\alpha)}, \cdots, \frac{\Gamma(1+\frac{\rho-1}{m})}{\Gamma(1+\alpha+\frac{\rho-1}{m})}\right] \tag{49}$$

and

$$\Lambda = \begin{pmatrix} O_{\rho\times n} & C_{\rho\times\rho} \\ O_{n\times n} & O_{n\times\rho} \end{pmatrix} \tag{50}$$

the fractional matrix operation, $\Lambda\Psi$, can be used for approximating RL integral and we have

$$I^\alpha \Psi = \Lambda\Psi + \mathfrak{E} \tag{51}$$

where the error term

$$\begin{aligned} \mathfrak{E} =& [0, \cdots, 0, t^{\frac{pm+1}{m}}, \cdots, 0, t^{\frac{pm+n}{m}}]^T \\ =& t^p[0, \cdots, 0, t^{\frac{1}{m}}, \cdots, 0, t^{\frac{n}{m}}]^T \\ =& \mathbf{O}(t^p). \end{aligned} \tag{52}$$

can be neglected.

Theorem 9. *Suppose $\mathfrak{f} \in (C^{pm+1,\alpha}[0, \mathfrak{T}])^\nu$. Then,*

$$T_{p,\alpha} I^\alpha T_{p,\alpha}\mathfrak{f} = T_{p,\alpha} I^\alpha \mathfrak{f}. \tag{53}$$

Proof. We prove for 1-D case: $f \in C^{pm+1,\alpha}[0, \mathfrak{T}^{\frac{1}{m}}]$. A component-wise analysis can be carried out for vector functions of higher dimensions. From the fractional Taylor series expansion,

$$f(t) = T_{p,\alpha}f(t) + R(t)t^{\frac{pm+1}{m}} \tag{54}$$

where

$$T_{p,\alpha}f(t) = \sum_{i=0}^{pm} \eta_i t^{\frac{i}{m}}. \tag{55}$$

Since $f(t^m) \in C^{pm+1}[0, \mathfrak{T}^{\frac{1}{m}}]$, hence, $R(t^m) \in C^{pm+1}[0, \mathfrak{T}^{\frac{1}{m}}]$. From Equation (54), we get:

$$I^\alpha f(t) = I^\alpha T_{p,\alpha}f(t) + I^\alpha R(t)t^{\frac{pm+1}{m}}. \tag{56}$$

Define $h(t) = I^\alpha R(t)t^{\frac{pm+1}{m}}$. From Equation (14), derive:

$$\begin{aligned} h(t^m) =& \frac{mt^n}{\Gamma(\alpha)} \int_0^1 \frac{R(t^m s^m)(s^{pm+1}t^{pm+1})s^{m-1}}{(1-s^m)^{1-\alpha}} ds \\ =& \frac{mt^{n+pm+1}}{\Gamma(\alpha)} \int_0^1 \frac{R(t^m s^m)s^{(p+1)m}}{(1-s^m)^{1-\alpha}} ds. \end{aligned} \tag{57}$$

Next, by Equation (15) that t is a factor of $\frac{d^i}{dt^i}h(t^m)$ for all $i \le pm$. Thus, these derivatives vanish at $t = 0$. Therefore,

$$T_{p,\alpha} I^\alpha R(t)t^{\frac{pm+1}{m}} = 0 \tag{58}$$

and finally applying $T_{p,\alpha}$ to both sides of Equation (56) and considering the linearity of $T_{p,\alpha}$, we obtain the desired result.

6. Proposed spectral method for systems of fractional differential equations

Projecting both sides of Equation (2) on $(\Pi_{p,\alpha})^{\nu}$, $p \in \mathbb{N}$, by RTS we get

$$T_{p,\alpha}\mathfrak{Y}(t) = AT_{p,\alpha}I^{\alpha}\,\mathfrak{Y}(t) + T_{p,\alpha}I^{\alpha}\,\mathfrak{f} + T_{p,\alpha}\mathfrak{Y}_0. \tag{59}$$

Applying Theorem 9, we obtain

$$T_{p,\alpha}\mathfrak{Y}(t) = AT_{p,\alpha}I^{\alpha}\mathfrak{Y}(t)+T_{p,\alpha}I^{\alpha}\,T_{p,\alpha}\mathfrak{f}(t)+T_{p,\alpha}\mathfrak{Y}_0. \tag{60}$$

Now, we look for the solution of the projected Equation (60) on the restricted space $(\Pi_{p,\alpha})^{\nu}$. We denote this solution by $\mathfrak{Y}_p$. Thus $\mathfrak{Y}_p$ also satisfies Equation (60) and

$$T_{p,\alpha}\mathfrak{Y}_p(t) = AT_{p,\alpha}I^{\alpha}\mathfrak{Y}_p(t)+T_{p,\alpha}I^{\alpha}\,T_{p,\alpha}\mathfrak{f}(t)+T_{p,\alpha}\mathfrak{Y}_0. \tag{61}$$

Taking into account that $T_{p,\alpha}\mathfrak{Y}_p(t) = \mathfrak{Y}_p(t)$ and $T_{p,\alpha}\mathfrak{Y}_0 = \mathfrak{Y}_0$ we obtain

$$\mathfrak{Y}_p(t) = AT_{p,\alpha}I^{\alpha}\mathfrak{Y}_p(t) + T_{p,\alpha}I^{\alpha}\,T_{p,\alpha}\mathfrak{f}(t) + \mathfrak{Y}_0. \tag{62}$$

Since $\mathfrak{Y}_p \in (\Pi_{p,\alpha})^{\nu}$, there exist matrices H and F of dimensions $\nu \times N$, $N = pm+1$, such that

$$\mathfrak{Y}_p = H\Psi \tag{63}$$

and

$$\mathfrak{f}_p = F\Psi \tag{64}$$

respectively. Thus, from Equation (62)

$$H\Psi(t) = AI^{\alpha}\,H\Psi(t) + I^{\alpha}\,F\Psi(t) + \mathfrak{O}\Psi(t). \tag{65}$$

where $\mathfrak{O}$ is described in Equation (46). Using fractional matrix operation and noticing the linearity of I^{α} we obtain

$$H\Psi(t) = AH\Lambda\Psi(t) + F\Lambda\Psi(t) + \mathfrak{O}\Psi(t). \tag{66}$$

Hence,

$$H\Psi(t) = (AH\Lambda + F\Lambda + \mathfrak{O})\,\Psi(t) \tag{67}$$

Since Equation (67) is valid for all $t \in [0,\mathfrak{T}]$, the coefficient of both sides should be equal. Thus,

$$H_{\nu\times N} - A_{\nu\times\nu}H_{\nu\times N}\Lambda_{N\times N} = F_{\nu\times N}\Lambda_{N\times N} + \mathfrak{O}_{\nu\times N} \tag{68}$$

where $N = pm+1$. As introduced in Equation,[36] we can use vectorization operators

$\mathrm{Vec}(\mathrm{A}_{\nu\times\mu}) = [a_{11},\cdots,a_{\nu1},\cdots,a_{1\mu},\cdots,a_{\nu\mu}]^T$

to transform Equation (68) into the standard systems of linear equations. Therefore,

$$(I_{N\times N}\otimes I_{\nu\times\nu} - \Lambda^T_{N\times N}\otimes A_{\nu\times\nu})\mathrm{Vec}(H_{\nu\times N}) = G \tag{69}$$

and

$$G = \mathrm{Vec}(F_{\nu\times N}\Lambda_{N\times N} + \mathfrak{O}_{\nu\times N}) \tag{70}$$

where the symbol $\otimes$ stands for the Kronecker product. We recall

$$\mathrm{Vec}(ABC) = \left(C^T \otimes A\right)\mathrm{Vec}(B). \tag{71}$$

7. Analysis of the proposed method

7.1. Exactness of the numerical solutions on $(\Pi_{p,\alpha})^{\nu}$

For our proposed numerical method, we prove that if the exact solution is a combination of finite rational power functions, the numerical solution coincides with the exact solution.

Theorem 10. *Let $p,q \in \mathbb{N}$ and $p > q$. Suppose $\mathfrak{Y} \in (\Pi_{q,\alpha})^{\nu}$ is a solution of Equation (2). Then, $\mathfrak{Y}_p = \mathfrak{Y}$ if the matrix $(I - \Lambda^T \otimes A)$ is invertible.*

Proof. Let $T_{p,\alpha}\mathfrak{Y}(t) = \mathfrak{Y}(t)$, since $\mathfrak{Y}(t) \in (\Pi_{q,\alpha})^{\nu}$ and $p > q$. It follows Equation (60), in which

$$\mathfrak{Y}(t) = AT_{p,\alpha}I^{\alpha}\mathfrak{Y}(t) + T_{p,\alpha}I^{\alpha}\,T_{p,\alpha}\mathfrak{f}(t) + T_{p,\alpha}\mathfrak{Y}_0 \tag{72}$$

Subtracting Equation (62) from Equation (72) we get

$$\mathcal{E}_p(t) = AT_{p,\alpha}I^{\alpha}\mathfrak{E}_p(t). \tag{73}$$

where $\mathcal{E}_p = \mathfrak{Y} - \mathfrak{Y}_p$. Noting that $\mathcal{E}_p \in (\Pi_{q,\alpha})^{\nu} \subset (\Pi_{p,\alpha})^{\nu}$, we get

$$\mathcal{E}_p = E\Psi \tag{74}$$

where E is a $\nu \times (pm+1)$ dimensional matrix.

Now, it follows from Equation (73) and (51) that

$$E\Psi = AET_{p,\alpha}(\Lambda\Psi + \mathfrak{E}). \tag{75}$$

From Equation (52), $T_{p,\alpha}\mathfrak{E} = [0,\cdots,0]^T$ and immediately it follows

$$E\Psi = AE\Lambda T_{p,\alpha}\Psi = AE\Lambda\Psi. \tag{76}$$

Hence,

$$(I - \Lambda^T \otimes A)\mathrm{Vec}(E) = \mathrm{Vec}([0,\cdots,0]^T). \tag{77}$$

Consequently, $\mathrm{Vec}(E) = \mathrm{Vec}([0,\cdots,0]^T)$, if $(I - \Lambda^T \otimes A)$ is invertible. This completes the proof.

Remark 3. *Note that $\frac{\Gamma(1+j/m)}{\Gamma(1+\alpha+j/m)}$ is decreasing with respect to j; thus, the coefficients of Λ are less than or equal to $\frac{\Gamma(1)}{\Gamma(1+\alpha)}$. It follows that*

$$\|\Lambda^T \otimes A\| \le \frac{\|A\|}{\Gamma(1+\alpha)}. \tag{78}$$

Therefore, the condition

$$\|A\| < \Gamma(1+\alpha) \tag{79}$$

guarantees the invertibility of $(I - \Lambda^T \otimes A)$ by the geometric series theorem.

7.2. Rational taylor series project properties of exact solution

The next theorem shows that the obtained numerical solution is the RTS projection of the exact solution.

Theorem 11. *Let $\mathfrak{f} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$, $p \in \mathbb{N}$. Then, $\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$ and $\mathfrak{Y}_p = T_{p,\alpha}\mathfrak{Y}$ if the matrix $(I - \Lambda^T \otimes A)$ is invertible.*

Proof. It follows Theorem 7 that $\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$. Hence, we can apply Theorem 9 to obtain

$$T_{p,\alpha} I^{\alpha}\mathfrak{Y}(t) = T_{p,\alpha} I^{\alpha} T_{p,\alpha}\mathfrak{Y}(t) \tag{80}$$

and by substituting $T_{p,\alpha} I^{\alpha}\mathfrak{Y}(t)$ from Equation (80) into Equation (60) we obtain

$$\begin{aligned} T_{p,\alpha}\mathfrak{Y}(t) =& A T_{p,\alpha} I^{\alpha} T_{p,\alpha}\mathfrak{Y}(t) \\ &+ T_{p,\alpha} I^{\alpha}\, T_{p,\alpha}\mathfrak{f}(t) + T_{p,\alpha}\mathfrak{Y}_0. \end{aligned} \tag{81}$$

Subtracting Equation (62) from Equation (81), we get

$$T_{p,\alpha}\mathfrak{Y}(t) - \mathfrak{Y}_p(t) = A T_{p,\alpha} I^{\alpha}\left(T_{p,\alpha}\mathfrak{Y}(t) - \mathfrak{Y}_p(t)\right). \tag{82}$$

But, $\mathcal{E} := T_{p,\alpha}\mathfrak{Y} - \mathfrak{Y}_p \in (\Pi_{q,\alpha})^{\nu}$ and thus similar to the proof of Theorem 10, there exists a matrix E such that $\mathcal{E}_p = E\Psi$ and

$$(I - \Lambda^T \otimes A)\mathrm{Vec}(E) = \mathrm{Vec}([0,\cdots,0]^T). \tag{83}$$

Therefore, $T_{p,\alpha}\mathfrak{Y} = \mathfrak{Y}_p$, if $\mathrm{Vec}(E) = \mathrm{Vec}([0,\cdots,0]^T)$, if $I - \Lambda^T \otimes A$ is invertible. This completes the proof of this theorem.

7.3. Error bound

The error bound is derived from the fact that the numerical solution is an RTS projection of the exact solution.

Theorem 12. *Let $\mathfrak{f} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$, $p \in \mathbb{N}$. Then, $\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$ and*

$$\|\mathfrak{Y} - \mathfrak{Y}_p\| = \frac{M_p \max\{\mathfrak{T}^{p+\frac{1}{m}}, 1\}}{(pm+1)!} \tag{84}$$

where

$$M_p = \max_{j=1}^{\nu} \max_{x\in[0,\mathfrak{T}]}\{|h_j^{(pm+1)}(x)|\}, \quad h_j(x) = y_j(x^m).$$

Proof. From Theorem 11, $\mathfrak{Y}_p = T_{p,\alpha}\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$ and from Theorem 7, $\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$. Therefore,

$$\mathfrak{Y} - \mathfrak{Y}_p = \mathfrak{Y} - T_{p,\alpha}\mathfrak{Y} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}. \tag{85}$$

On the other hand, by Theorem 8

$$y_j(t) - T_{p,\alpha} y_j(t) = \frac{h_j^{(pm+1)}(\xi)}{(pm+1)!} t^{p+\frac{1}{m}} \tag{86}$$

where y_j is the jth component of $\mathfrak{Y}$ and $h_j(x) = y_j(x^m)$. However, $h_j \in C^{pm+1}[0,\mathfrak{T}]$ since $y_j \in C^{pm+1,\alpha}[0,\mathfrak{T}]$. Thus, $h_j^{(pm+1)}$ is bounded, so M_p is finite. This completes the proof.

7.4. Convergence and avoiding Runge phenomenon

In Theorem 12, the error bound term M_p may grow as $p \to \infty$. This behavior is known as the Runge phenomenon. To ensure convergence, one of the following conditions must be met

(i) $\max_{p=1}^{\infty} \max_{j=1}^{\nu} \max_{x\in[0,\mathfrak{T}]}\{|h_j^{(pm+1)}(x)|\} < \infty$ (uniformly bounded);

(ii) $h_j(x)$ is analytic and has no singularity in the complex disk of radius $\mathfrak{T}$ (centered at the origin).

Theorem 13. *Let $\mathfrak{f} \in (C^{pm+1,\alpha}[0,\mathfrak{T}])^{\nu}$, $p \in \mathbb{N}$. Suppose that $\mathfrak{Y}$ satisfies condition (i) or (ii). Then, $\mathfrak{Y}_p$ converges to $\mathfrak{Y}$ as $p \to \infty$, and*

$$\|\mathfrak{Y} - \mathfrak{Y}_p\| = \mathbf{O}\left(\frac{1}{(pm+1)!}\right) \tag{87}$$

where $\mathbf{O}$ denotes the big O notation.

Proof. The result follows directly from Theorem 12.

8. Numerical experiments

To validate the efficiency, accuracy, and practicality of the proposed RTS method for solving rational-order FDEs, we present three numerical experiments in this section, covering uncoupled, non-decimal rational-order, and coupled FDE systems. We compared the RTS method's performance, error magnitude, and computational time with existing methods (Mittag–Leffler "mlf" function and truncated series method).

Example 3. *Consider system of FDEs* (2) *on* $[0,2]$ *described by*

$$A = diag[\lambda_1, \lambda_2]\mathfrak{f} = [0; 0] \tag{88}$$

and

$$\mathfrak{Y}_0 = [1, 1]. \tag{89}$$

This system is uncoupled and

$$\mathfrak{Y}(t) = [E_{\alpha}(\lambda_1 t^{\alpha}), E_{\alpha}(\lambda_2 t^{\alpha})] \tag{90}$$

is its exact solution.

We solved this system numerically by setting $\lambda_1 = 1$, $\lambda_2 = 2$ and $\alpha = 0.5$. To compute the exact solution, Mittag–Leffler functions were computed by the "mlf" open-source function of Matlab introduced by Igor Podlubny. We chose the "mlf" accuracy 10^{-14}. Table 3 shows the maximum error of the dense approximate solution obtained by Equation (5) *and the exact solution for various*

p. It shows that the accuracy improves with increasing p. Meanwhile, we calculated the computation time in seconds. While it took 15 seconds to compute "mlf" on our computer, it took only 0.02 seconds to obtain the same accuracy. Table 3 demonstrates the superior efficiency of the RTS method in comparison to the "mlf" function method for diverse p.

Table 3. Error and computational time comparison: Example 3

p	$\|\mathcal{E}_{1,p}\|$	$\|\mathcal{E}_{2,p}\|$	CT_1 (s)	CT_2 (s)
12	5.5e-06	2.4e-06	0.007	15.43
14	1.1e-07	5.1e-08	0.009	15.19
16	1.6e-09	8.1e-10	0.011	15.51
18	1.9e-11	1.0e-11	0.010	15.59
20	2.0e-13	1.0e-13	0.018	15.35

Note: Here $\mathcal{E}_{1,p}$ and $\mathcal{E}_{2,p}$ are errors of the first and the second components, respectively. CT_1 and CT_2 stand for the computational time of RTS and the "mlf" function method, respectively.

Example 4. *Consider the same system parameters as in Example 3, with the rational fractional order adjusted to $\alpha = \frac{1}{3}$. As presented in Table 4, the numerical results remained consistent with those of Example 3: the RTS method still achieved near-machine precision, with the maximum error of the first component ($\|\mathcal{E}_{1,p}\|$) decreasing to 3.6×10^{-13} when $p = 20$.*

Meanwhile, computational efficiency was maintained—the RTS method completed the simulation in merely 0.021 s, which is substantially faster than the reference "mlf" (Mittag–Leffler function) method, which requires approximately 16.3 s to achieve comparable accuracy.

Table 4. Error and computational time comparison: Example 4

p	$\|\mathcal{E}_{1,p}\|$	$\|\mathcal{E}_{2,p}\|$	CT_1 (s)	CT_2 (s)
12	9.8e-06	3.0e-06	0.012	16.95
14	1.9e-07	6.4e-08	0.013	16.54
16	3.0e-09	1.0e-09	0.016	16.63
18	3.7e-11	1.3e-11	0.017	16.14
20	3.6e-13	1.4e-13	0.020	16.63

Note: $\mathcal{E}_{1,p}$ and $\mathcal{E}_{2,p}$ are errors of the first and the second components, respectively, in Example 4. CT_1 and CT_2 stand for the computational time of the RTS and the "mlf" function method, respectively.

Example 5. *Let us consider $\alpha = 0.5$, $\mathfrak{f} = [\sqrt{t}, 1+\sqrt{t}]^T$,*

$$\mathfrak{Y}_0 = [0, 1]^T \tag{91}$$

and a coupled system with

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{92}$$

Evidently,

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{93}$$

It is straightforward to see that

$$I^{\alpha(i+1)} \begin{pmatrix} t^{0.5} \\ 1 + t^{0.5} \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1.5)}{\Gamma(2+0.5i)} t^{1+0.5i} \\ \frac{\Gamma(1)}{\Gamma(1.5+0.5i)} t^{0.5+0.5i} + \frac{\Gamma(1.5)}{\Gamma(2+0.5i)} t^{1+0.5i} \end{pmatrix}. \tag{94}$$

Therefore, we can truncate the exact solution up to large n, in Equation (29). For our simulation, we chose $n = 50$. Table 5 shows the maximum errors on $[0, 2]$, with computational time. Again, the results are fast and accurate and thus the introduced method is efficient.

Table 5. Error and computational time comparison: Example 5

p	$\|\mathcal{E}_{1,p}\|$	$\|\mathcal{E}_{2,p}\|$	CT_1 (s)	CT_2 (s)
12	2.3e-05	2.2e-06	0.008	0.040
14	4.5e-07	4.5e-08	0.007	0.032
16	6.8e-09	7.1e-10	0.007	0.032
18	8.3e-11	8.9e-12	0.008	0.032
20	8.0e-13	9.0e-14	0.010	0.032

Note: $\mathcal{E}_{1,p}$ and $\mathcal{E}_{2,p}$ are errors of the first and the second components in Example 5, receptively. CT_1 and CT_2 stand for the computational time of the RTS and truncated method, respectively.

9. Discussion

This work introduces a rational power function-based method tailored to rational-order FDEs, a choice motivated by the practical constraints of real-world applications and computer simulations. Irrational values (e.g. π, $\sqrt{2}$, e) are universally approximated by rationals in hardware, as their infinite decimal expansions cannot be stored; while irrationals hold significance in pure mathematical contexts, they are irrelevant for applied FDE modeling.

In particular, fractional derivatives exhibited continuity with respect to their order $\alpha \in (0, 1)$. This continuity ensures that rational approximations of irrational α (e.g., $1/\pi \approx 0.318$) yield results sufficiently close to the "true" irrational-order behavior, eliminating the need for specialized methods targeting irrational orders.

A limitation of this work is its focus on single-domain FDEs; future studies could extend the rational power function framework to partial FDEs or time-varying order systems.

10. Conclusion

This paper describes a highly efficient and accurate numerical method for systems of FDEs with rational order. We used an RTS projection to derive approximate solutions. An extensive discussion of the method's convergence analysis is provided, along with illustrative test cases to show the efficiency of the approach. It is expected that such fast methods can be easily developed for solving various partial and FDEs.

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Conflict of interest

The authors declare they have no competing interests.

Author contributions

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All authors confirm that no AI tools were used in the preparation of this manuscript.

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