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RESEARCH ARTICLE

New horizons in analytic function classes induced by the Erdélyi–Kober fractional integral operators

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ABSTRACT

This study investigates a new subclass of analytic and univalent functions in the open unit disk, through the convolution of normalized analytic functions with a generalized Erdélyi-Kober fractional integral operator. The main objective is to define a new subclass $TS(\beta, \gamma)$ related with the normalized form of the Erdélyi–Kober fractional integral operator K_{δ}^{ϑ} and explore its geometrical properties. This study obtains sharp coefficient bounds and geometric characteristics such as growth, and distortion properties. Furthermore, convexity, close-to-convexity, radii of and starlikeness, extremal functions, and inclusion relations are determined. These results contribute to the broader understanding of defined subclass within geometric function theory and provide a mathematical foundation for modelling phenomena in fractional calculus, conformal mapping, and applied engineering contexts. The study also highlights limitations associated with the operator parameters and suggests extensions to numerical and control-based models for future investigation.

1. Introduction

The theory of analytic and univalent functions continues to play a pivotal role in geometric function theory, particularly due to its broad applicability in complex analysis, conformal mappings, and mathematical modeling. Within this framework, special attention has been given to developing and characterizing function subclass defined via differential and integral operators. These operators not only enhance the structure of classical function classes but also provide a bridge between fractional calculus and geometric function theory.[1–4]

A single-valued analytic function $f(\zeta)$ is said to be univalent in a domain $\mathcal{D} \subset \mathbb{C}$ if it is injective, that is,

$$f(\zeta_1) = f(\zeta_2) \Rightarrow \zeta_1 = \zeta_2, \quad \text{for all } \zeta_1, \zeta_2 \in \mathcal{D} \quad (1)$$

Univalent functions play a central role in geometric function theory, particularly when defined in the open unit disk $\mathbb{D}_\zeta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The class $\mathcal{A}$ consists of all functions analytic in $\mathbb{D}_\zeta$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ such that:

$$f(\zeta) = \zeta + \sum_{r=2}^{\infty} \varepsilon_r \zeta^r \quad (2)$$

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and $\mathcal{S}$ is a subclass $\mathcal{A}$ consisting of analytic and univalent functions of the form (2).[5–8]

The subclass of starlike and convex functions[9,10] were introduced in the early development of geometric function theory to describe analytic functions that preserve geometric properties such as radial. A starlike domain is a region that is starlike with respect to a point z_0 if every line segment from z_0 to any point in the region remains entirely inside it. If a function $f(\zeta) \in \mathcal{A}$ maps $\mathbb{D}_\zeta$ onto such a starlike domain, it is called starlike with respect to z_0. In particular, if $z_0 = 0$ we simply call $f(\zeta)$ a starlike function. A function f is said to be starlike in $\mathbb{D}_\zeta$, if it satisfies Equation (3):

$$\Re\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > 0 \tag{3}$$

Similarly, a domain is convex if every line segment joining any two points within the domain lies entirely inside the domain. A function $f(\zeta) \in \mathcal{A}$ mapping $\mathbb{D}_\zeta$ onto a convex domain is called a convex function. The convex functions defined by the following inequality:

$$\Re\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right) > 0 \tag{4}$$

These conditions enabled the splitting of analytic function subclass for convex and starlike functions. The set $\mathcal{S}^*(\tau)$ (starlike of order τ) and $\mathcal{K}(\tau)$ (convex of order τ) were defined and studied by W. Kaplan in the early 1950s. The function exhibits starlikeness of order τ $(0 \leq \tau < 1)$ provided that

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > \tau, \quad \zeta \in \mathbb{D}_\zeta \tag{5}$$

and this class is denoted by $\mathcal{S}^*(\tau)$. In the same way, if

$$\Re\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} > \tau, \quad \zeta \in \mathbb{D}_\zeta \tag{6}$$

The function f lies in the subclass of convex functions of order τ $(0 \leq \tau < 1)$, and $\mathcal{K}(\tau)$ serves to denote this particular class. Explicitly, the traditional starlike and convex function classes are given by $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, respectively.

In recent years, researchers have extended the classical theory of univalent and starlike functions using generalized differential operators, such as the Ruscheweyh[11] and Salagean operators,[12] as well as fractional calculus operators[13–17] including the Erdélyi–Kober and well-poised fractional derivative operators.[18] These operators have enabled the definition of new subclasses of univalent functions with rich geometric and analytic properties, and have been effectively used in deriving new transformation and summation formulas for hypergeometric functions.[19] Classical foundations are found in Duren[5] and Pommerenke,[6] while Mitra et al.[20] provided modern extensions connecting operator theory and univalence.

Fractional differential and integral operators have profoundly influenced modern geometric function theory, introducing parameters that describe memory and hereditary properties. Ibrahim[21] investigated a combined fractional differential–integral operator in the complex domain. Indushree[22] utilized the Prabhakar fractional operator to define a subclass of analytic univalent functions, deriving coefficient bounds and subordination results. Further contributions include Khan and Darus[23] on partial sums for normalized Mittag-Leffler–Prabhakar functions, and Murugusundaramoorthy et al.[24] on fractional bi-univalent functions. Alotaibi and Darus[25] studied generalized fractional derivative operators of Srivastava–Owa type, while Zainab and Lashin[26] examined fractional differential subordinations involving Caputo operators.

Among fractional-type operators, the Erdélyi–Kober operator and its variants have received extensive attention due to their analytical versatility. This operator generalizes Riemann–Liouville[27,28] and Weyl forms, employing kernels such as $x^\gamma e^{-\alpha x}$, highly effective in complex–domain analyses. Bulut and Kumar[29] introduced Mittag–Leffler Poisson distribution series for defining new univalent subclasses. Kanwal, Raza, and Mubeen[30] analyzed fuzzy differential subordination with generalized Mittag-Leffler–Poisson convolution operators, extending Erdélyi–Kober frameworks to fuzzy systems. Abubaker, Kumar, and Bulut[31] studied four-parameter Mittag–Leffler functions. Recent related efforts include studies of Borel-type operators[32], fractional Hadamard operators[33], and generalized Poisson–Mittag-Leffler kernels[34], reinforcing the Erdélyi–Kober operator's flexibility. The Erdélyi–Kober operators possess robust mathematical structures and wide applicability. From numerical methods and operational calculus to geometric function theory and statistical modeling, these operators serve as a bridge between classical and fractional analysis[35,36]. Recent advancements have highlighted the broad applicability of Erdélyi–Kober fractional operators in various branches of mathematics[37–42]. This has been further demonstrated in works such as Ref.,[43–47] which explore new inequalities, subclasses of analytic functions, and applications of multiple Erdélyi–Kober operators in geometric function theory and fractional inequalities. The current research builds upon these foundational studies by investigating new subclasses of analytic univalent functions defined via Erdélyi–Kober operators, aiming to derive novel coefficient estimates, distortion theorems, and mapping properties that can extend existing results in fractional and geometric analysis.

In this terminology, we define the Erdélyi–Kober fractional integral operator acting on a function $f \in \mathcal{A}$ as follows:

Let $f(\zeta)$ be a Regular function throughout the unit disk:

$$\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \tag{7}$$

belonging to the complex plane.

Let $\delta > 0$ and $\vartheta > 0$ are parameters such that $\Re(\delta - \vartheta) > 0$, an Erdélyi–Kober type integral operator

$$I_\delta^\vartheta : \mathcal{A} \to \mathcal{A} \tag{8}$$

be defined by

$$I_\delta^\vartheta f(\zeta) = \frac{\zeta^{-\vartheta}}{\Gamma(\delta)} \int_0^\zeta (\zeta - t)^{\delta-1} t^{\vartheta-1} f(t)\, dt, \tag{9}$$

where $\Gamma(\delta)$ denotes the Gamma function.

A general-purpose method for generating new subclasses of analytic functions, this operator arises from fractional calculus and provides a generalization of numerous existing integral transforms. Specifically, we derive and investigate new classes of univalent functions that satisfy radius problems, distortion estimates, and coefficient inequalities, exhibiting a rich geometric structure when this operator is applied to functions in $\mathcal{A}$. Let $f(\zeta)$ be analytic in the open unit disk:

$$\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \tag{10}$$

Then, $f(\zeta)$ can be expressed by its Taylor expansion at 0 as:

$$f(\zeta) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} \zeta^r \tag{11}$$

If we interchange the sum and the integral and substitute the Taylor expansion of $f(t)$ into the Erdélyi–Kober operator, this yields:

$$I_\delta^\vartheta f(\zeta) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} \cdot \frac{\Gamma(r+\vartheta)}{\Gamma(r+\vartheta+\delta)} \zeta^{r+\delta-1} \tag{12}$$

The Erdélyi–Kober fractional integral operator $\mathcal{K}_\delta^\vartheta : \mathcal{A} \to \mathcal{A}$ is defined as follows:

$$\mathcal{K}_\delta^\vartheta f(\zeta) = \zeta + \sum_{r=2}^{\infty} \frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)} \cdot \frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)} \varepsilon_r \zeta^r \tag{13}$$

where $\delta > 0$, $\vartheta > 0$.

Let $f \in TS(\beta, \gamma)$, then

$$\left| \frac{(\mathcal{K}_\delta^\vartheta f)'(\zeta) + \zeta \left(\mathcal{K}_\delta^\vartheta f\right)''(\zeta)}{\beta\zeta \left(\mathcal{K}_\delta^\vartheta f\right)''(\zeta) + \left(\mathcal{K}_\delta^\vartheta f\right)'(\zeta)} \right| > \gamma, \quad \zeta \in \mathfrak{D} \tag{14}$$

where $\mathcal{K}_\delta^\vartheta f(\zeta)$ is given by Equation ((13)). In the expression Equation ((14)), the derivatives $(K_\delta^\vartheta f)'(\zeta)$ and $(K_\delta^\vartheta f)''(\zeta)$ are taken with respect to ζ. The parameter β serves as a weighting factor controlling the influence of the second derivative in the denominator. The restriction $\beta < 1$ ensures that the denominator in Equation (13) does not vanish within the unit disk, preserving analyticity and validity of the subordination condition. Similarly, the parameter γ determines the right half-plane $\{z \in \mathbb{C} : \Re(z) > \gamma\}$ to which the analytic expression is subordinate. The range $0 \leq \gamma < 1$ guarantees that the image domain lies in the right half-plane, thereby defining a subclass of analytic functions that retain geometric properties such as starlikeness or convexity, depending on the choice of the operator K_δ^ϑ.

2. Sharp coefficient bounds

The subsequent theorem establishes a characterization that is both necessary and sufficient membership in the class requires that a function $TS(\beta, \gamma)$.

For the function f introduced in Equation (2), it holds that $f \in TS(\beta, \gamma)$ exactly when

$$\begin{aligned} &\sum_{r=2}^{\infty} [r\,(r - r\gamma\beta + \gamma\beta - \gamma) \\ &\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)|\varepsilon_r| \leq \Gamma(\vartheta) \\ &\Gamma(\vartheta+r+\delta-1)\,(1-\gamma) \end{aligned} \tag{15}$$

where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$

The result is sharp for the function

$$\begin{aligned} f(\zeta) = &\zeta + \frac{\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)\,(1-\gamma)}{[r\,(r - r\gamma\beta + \gamma\beta - \gamma)]} \\ &\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)\,\zeta^r, \quad r \geq 2 \end{aligned} \tag{16}$$

Proof. By the definition of the class $TS(\beta, \gamma)$, we require that

$$\Re\mathbb{F}(\zeta) > 0, \quad |\zeta| < 1$$

where

$$\mathbb{F}(\zeta) = \frac{\mathcal{K}_\delta^\vartheta f'(\zeta) + \zeta \left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)''}{\beta\zeta \left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)'' + \left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)'} - \gamma \tag{17}$$

It is well known that

$$\Re\mathbb{F}(\zeta)>0 \quad\Longleftrightarrow\quad \left|\frac{\mathbb{F}(\zeta)-1}{\mathbb{F}(\zeta)+1}\right|<1,$$

since the Möbius map $w\mapsto(w-1)/(w+1)$ sends the right half-plane onto the open unit disk. Thus, to establish the theorem, it suffices to show that

$$\left|\frac{\mathbb{F}(\zeta)-1}{\mathbb{F}(\zeta)+1}\right|\le 1,\quad |\zeta|=1. \tag{18}$$

The maximum modulus principle then ensures that this holds for all $|\zeta|<1$

Let

$$f(\zeta)=\zeta+\sum_{r=2}^{\infty}\varepsilon_r\zeta^r \tag{19}$$

and note that, by the definition of the operator $\mathcal{K}_\delta^\vartheta$,

$$\mathcal{K}_\delta^\vartheta f(\zeta)=\zeta+\sum_{r=2}^{\infty}\frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)}\frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)}\varepsilon_r\zeta^r \tag{20}$$

Hence,

$$\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)'=1+\sum_{r=2}^{\infty}r\frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)}\frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)}\varepsilon_r\zeta^{r-1},$$

$$\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)''=\sum_{r=2}^{\infty}r(r-1)\frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)}\frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)}\varepsilon_r\zeta^{r-2} \tag{21}$$

Substituting these into (17), and simplifying, yields

$$\frac{\mathbb{F}(\zeta)-1}{\mathbb{F}(\zeta)+1}=\frac{(1-\gamma\beta-\beta)\,\zeta\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)''-\gamma\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)'}{(1-\gamma\beta+\beta)\,\zeta\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)''+(2-\gamma)\left(\mathcal{K}_\delta^\vartheta f(\zeta)\right)'} \tag{22}$$

Now, for $|\zeta|=1$, applying the triangle inequality to the numerator and the reverse triangle inequality to the denominator gives

$$\gamma+\sum_{r=2}^{\infty}\left[(r^2-r^2\gamma\beta-r^2\beta-r+r\beta\gamma+r\beta-r\gamma)\right.\frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)}\frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)}|\varepsilon_r| \tag{23}$$

$$\le(2-\gamma)-\sum_{r=2}^{\infty}\left[(r^2-r^2\gamma\beta+r^2\beta+r+r\beta\gamma\right.-r\beta-r\gamma)\frac{\Gamma(\vartheta+\delta)}{\Gamma(\vartheta)}\frac{\Gamma(\vartheta+r-1)}{\Gamma(\vartheta+r+\delta-1)}|\varepsilon_r| \tag{24}$$

Rearranging terms and simplifying leads directly to the coefficient condition

$$\sum_{r=2}^{\infty}\left[r\left(r-r\gamma\beta+\gamma\beta-\gamma\right)\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)\right]|\varepsilon_r|\le\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)\,(1-\gamma) \tag{25}$$

which is exactly Equation (5)

Therefore, condition Equation (5) ensures that $|(\mathbb{F}(\zeta)-1)/(\mathbb{F}(\zeta)+1)|\le 1$ for $|\zeta|=1$, and by the maximum modulus principle, also for all $|\zeta|<1$. Hence, $\Re\mathbb{F}(\zeta)>0$ and $f\in TS(\beta,\gamma)$.

Finally, equality holds for the extremal function given in the theorem, demonstrating the sharpness of the bound.

The equivalence $\Re\mathbb{F}(\zeta)>0\iff\left|\frac{\mathbb{F}(\zeta)-1}{\mathbb{F}(\zeta)+1}\right|<1$ follows from the conformal mapping of the right half-plane onto the open unit disk. By the Herglotz representation theorem, any analytic function with $\Re\mathbb{F}(\zeta)>0$ in the unit disk admits the integral representation

$$\mathbb{F}(\zeta)=\int_0^{2\pi}\frac{1+\zeta e^{-it}}{1-\zeta e^{-it}}\,d\mu(t) \tag{26}$$

for some positive measure $\mu(t)$ on $[0,2\pi]$. This provides a geometric interpretation of the coefficient inequality (27) as enforcing the positivity of the real part of $\mathbb{F}(\zeta)$ (Extremal function) Let $f\in TS(\beta,\gamma)$, then

$$\sum_{r=2}^{\infty}\left[r\left(r-r\gamma\beta+\gamma\beta-\gamma\right)\Gamma(\vartheta+\delta)\right.\Gamma(\vartheta+r-1)|\varepsilon_r|\le\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)\,(1-\gamma) \tag{27}$$

$$f(\zeta)=\zeta+\frac{(1-\gamma)\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{\left(r(r-r\gamma\beta+\gamma\beta-\gamma)\right)\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\zeta^r \tag{28}$$

If suitable values of parameters are replaced in the above theorem, then the following corollaries are obtained Using the assumption that r=1, $\delta=1$ and replace the parameters r=n, $\gamma=\alpha,\beta=\lambda$ in (1.10) the outcome obtained a corollary kindred to the result of Porwal[48]

A function f(ζ), defined by Equation (2), is in the class C(λ, α) if, and only if,

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha)|a_n| \leqslant 1-\alpha \tag{29}$$

where $\alpha > 0, 0 \leq \lambda < 1$, $\zeta \in \mathrm{U}$

The extremal function (Equation [28]) for the subclass $TS(\beta,\gamma)$ is given by

$$f(\zeta) = \zeta + \frac{(1-\gamma)\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{r(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\,\zeta^r \tag{30}$$

where $r \geq 2$
For the parameter values:

$$\delta = 1, \quad \gamma = 0.5, \quad \vartheta = 1, \quad \beta = 0.5 \tag{31}$$

the function becomes

$$f(\zeta) = \zeta + 0.4\,\zeta^2 + 0.25\,\zeta^3 + 0.1818\,\zeta^4 + \cdots \tag{32}$$

The extremal Equation (28) for the subclass $TS(\beta,\gamma)$ is given by

$$f(\zeta) = \zeta + \frac{(1-\gamma)\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{r(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\,\zeta^r \tag{33}$$

where $r \geq 2$.
For the parameter values:

$$\delta = 1, \quad \gamma = 0.6, \quad \vartheta = 4, \quad \beta = 0.25 \tag{34}$$

the corresponding series expansion becomes approximately

$$f(\zeta) = \zeta + 0.2\,\zeta^2 + 0.1212\,\zeta^3 + 0.0853\,\zeta^4 + \cdots \tag{35}$$

where each coefficient corresponds to successive values $r = 2, 3, 4, \ldots$ computed using the extremal formula.

3. Geometric properties: Distortion and covering

We obtain estimates related to the magnitude and conformal distortion for functions of the given class $TS(\beta,\gamma)$.

If $f \in TS(\beta,\gamma)$. Then for $|\zeta| = k^*$,

$$k^* - \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2 \leq |f(\zeta)| \leq k^* + \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2 \tag{36}$$

where $c_2 = 2(2-\gamma\beta-\gamma))$

Proof. Since

$$|\varepsilon_r| \leq \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)} \tag{37}$$

then,

$$|f(\zeta)| \leq k^* + \sum_{r=2}^{\infty} \varepsilon_r (k^*)^r \tag{38}$$

$$|f(\zeta)| \leq k^* + \sum_{r=2}^{\infty} \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\,(k^*)^r \tag{39}$$

$$\leq k^* + \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{2(2-\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2 \tag{40}$$

$$= k^* + \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2, \tag{41}$$

where $\mathrm{c}_2 = 2(2-\gamma\beta-\gamma)$

Similarly,

$$|f(\zeta)| \geq k^* - \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2, \tag{42}$$

where $\mathrm{c}_2 = 2(2-\gamma\beta-\gamma)$

Hence,

$$k^* - \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2, \leq |f(\zeta)| \leq k^* + \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}\,(k^*)^2 \tag{43}$$

Where $\mathrm{c}_2 = 2(2-\gamma\beta-\gamma)$.

Let the function $f \in TS(\beta,\gamma)$. Then,

$$1 - \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}k^* \leq |f'(\zeta)| \leq 1 + \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)}k^* \tag{44}$$

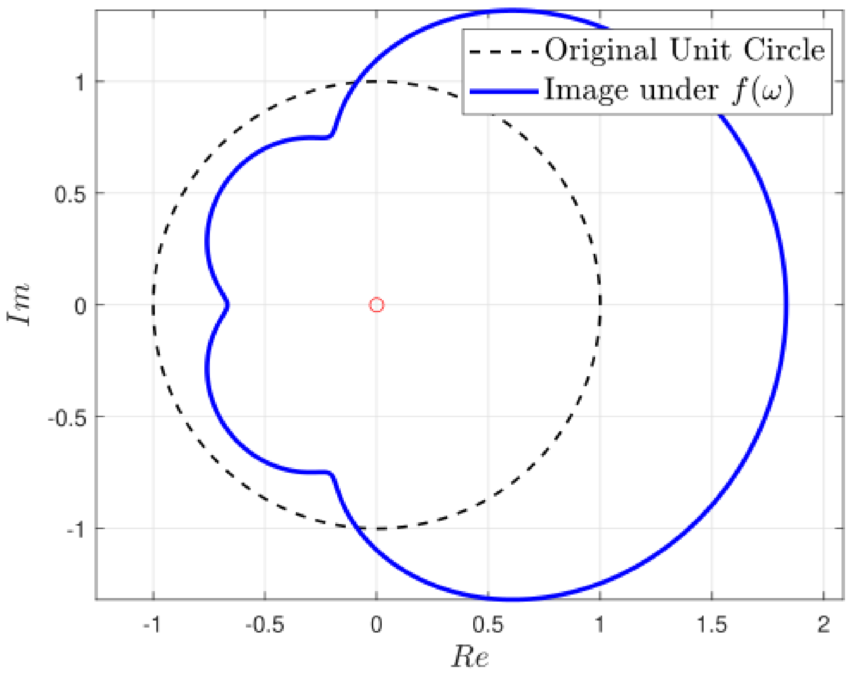

Figure 1. Image of the unit disk $|\zeta| < 1$ under the mapping $f(\zeta) = \zeta + 0.4\,\zeta^2 + 0.25\,\zeta^3 + 0.1818\,\zeta^4 + \cdots$. The figure illustrates the geometric distortion and boundary behavior of $f(\zeta)$ in the complex plane. Axes represent the real and imaginary components of $f(\zeta)$.

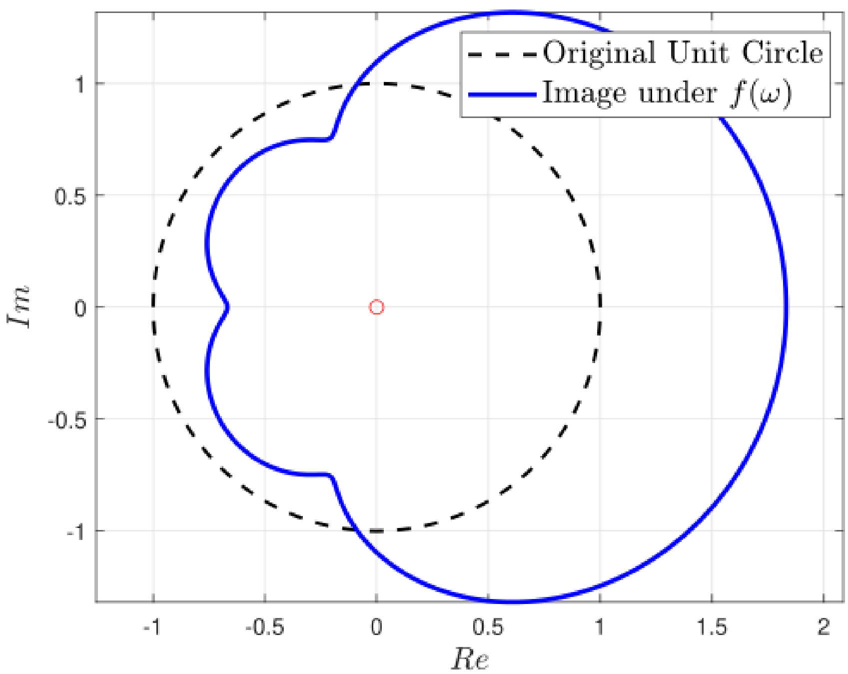

Figure 2. Image of the unit disk $|\zeta| < 1$ under the mapping $f(\zeta) = \zeta + 0.2\,\zeta^2 + 0.1212\,\zeta^3 + 0.0853\,\zeta^4 + \cdots$. The mapping demonstrates the deformation of the unit circle and the effect of parameters $\beta, \gamma, \delta, \vartheta$ on the resulting image domain. Axes indicate the real and imaginary parts of $f(\zeta)$.

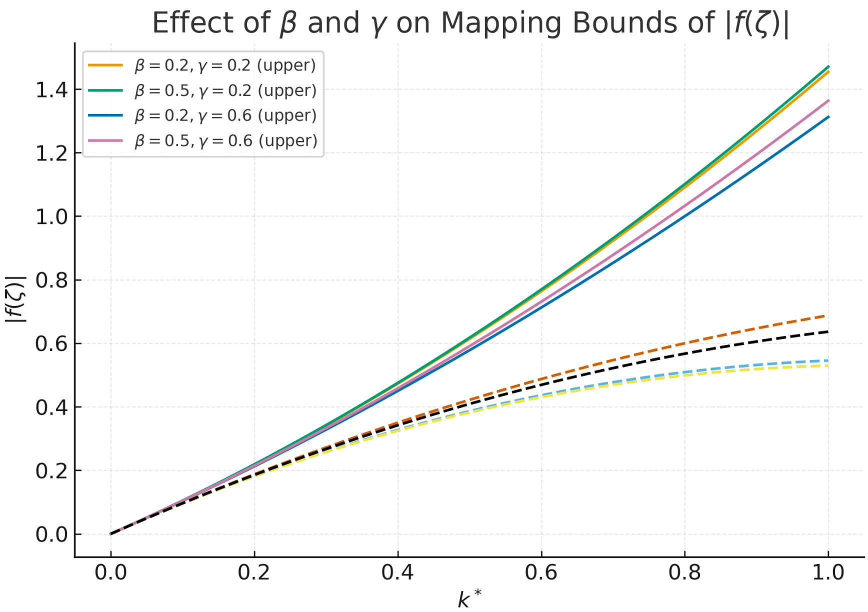

Figure 3. Effect of varying β and γ on the mapping bounds of $|f(\zeta)|$ for $f \in TS(\beta, \gamma)$. Higher γ values reduce the distortion range, indicating a stronger contraction in the image domain.

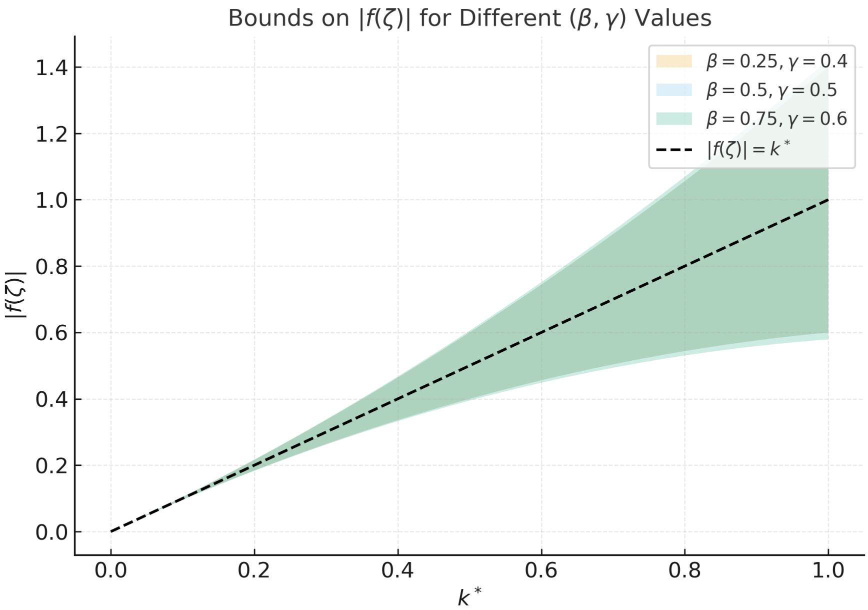

Figure 4. Variation in the bounds of $|f(\zeta)|$ with respect to parameters β and γ for $f \in TS(\beta, \gamma)$. The shaded region indicates the range of possible values of $|f(\zeta)|$ as k^* varies in the unit disk.

where $c_2 = 2(2 - \gamma\beta - \gamma)$

Proof. Since,

$$|\varepsilon_r| \leq \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta + r + \delta - 1)}{(r(r - r\gamma\beta + \gamma\beta - \gamma))\,\Gamma(\vartheta + \delta)\,\Gamma(\vartheta + r - 1)} \tag{45}$$

Let,

$$f(\zeta) = \zeta + \sum_{r=2}^{\infty} \varepsilon_r \zeta^r \tag{46}$$

Then,

$$f'(\zeta) = 1 + \sum_{r=2}^{\infty} \varepsilon_r r \zeta^{r-1} \tag{47}$$

So,

$$|f'(\zeta)| \le 1 + \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{2(2-\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)} r^* \tag{48}$$

and similarly,

$$|f'(\zeta)| \ge 1 - \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{2(2-\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)} k^* \tag{49}$$

Hence,

$$1 - \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)} k^* \le |f'(\zeta)| \le 1 + \frac{2(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+\delta+1)}{c_2\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+1)} k^* \tag{50}$$

where $c_2 = 2(2-\gamma\beta-\gamma)$.

For typical parameters $\vartheta = 1$, $\delta = 1$, and $k^* = 0.8$, the bounds on $|f'(\zeta)|$ for various (β,γ) values are computed as follows:

Table 1. The bounds on $|f'(\zeta)|$ for various (β,γ)

β	γ	Lower Bound	Upper Bound
0.25	0.4	0.783	1.217
0.50	0.5	0.824	1.176
0.75	0.6	0.864	1.136

These results show that increasing γ reduces the deviation from unity, indicating less distortion in the derivative magnitude and a more stable mapping behavior.

4. Radius problems for Star-domain and Convex-domain mappings and geometrically close-to-convex transformations

The subsequent results specify the radial limits for star-shaped, convex, and quasi-convex properties within the framework of the studied class $TS(\beta,\gamma)$.

Consider the function $f \in TS(\beta,\gamma)$. Consequently, f satisfies the starlikeness condition in $|\zeta| < R_1$ of order τ, $0 \le \tau < 1$, where

$$R_1 = \inf_{r\ge 2} \left\{ \left(\frac{(1-\tau)\left((r(r-r\gamma\beta+\gamma\beta-\gamma))\right)\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(r-\tau)(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \right)^{\frac{1}{r-1}} \right\} \tag{51}$$

Proof. The function f is considered starlike of order τ, $0 \le \tau < 1$, if

$$\Re\left\{ \frac{\zeta f'(\zeta)}{f(\zeta)} \right\} > \tau \tag{52}$$

Thus, it is adequate to confirm that

$$\left| \frac{\zeta f'(\zeta)}{f(\zeta)} - 1 \right| = \left| \frac{\sum_{r=2}^{\infty}(r-1)\varepsilon_r \zeta^{r-1}}{1+\sum_{r=2}^{\infty}\varepsilon_r\zeta^{r-1}} \right| \le \frac{\sum_{r=2}^{\infty}(r-1)|\varepsilon_r||\zeta|^{r-1}}{1+\sum_{r=2}^{\infty}|\varepsilon_r||\zeta|^{r-1}} \tag{53}$$

So,

$$\left| \frac{\zeta f'(\zeta)}{f(\zeta)} - 1 \right| \le 1-\tau \quad \text{if} \quad \sum_{r=2}^{\infty}\frac{(r-\tau)}{1-\tau}|\varepsilon_r||\zeta|^{r-1} \le 1 \tag{54}$$

By Theorem 2, the inequality above holds if

$$\frac{(r-\tau)}{1-\tau}|\zeta|^{r-1} \le \frac{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \tag{55}$$

or equivalently, if

$$|\zeta| \le \left(\frac{(1-\tau)(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(r-\tau)(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \right)^{\frac{1}{r-1}}, \quad r \ge 2 \tag{56}$$

The theorem follows easily from the above.

If $f \in TS(\beta,\gamma)$. Then, f is convex in $|\zeta| < R_2$ of order τ, $0 \le \tau < 1$, where

$$R_2 = \inf_{r\ge 2} \left\{ \left(\frac{(1-\tau)(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(r-\tau)(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \right)^{\frac{1}{r-1}} \right\} \tag{57}$$

Proof. The function f is considered convex of order τ, $0 \le \tau < 1$, if

$$\Re\left\{ 1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right\} > \tau \tag{58}$$

Thus, it is sufficient to show that

$$\left| \frac{\zeta f''(\zeta)}{f'(\zeta)} \right| = \left| \frac{\sum_{r=2}^{\infty} r(r-1)\varepsilon_r\zeta^{r-1}}{1+\sum_{r=2}^{\infty}k\varepsilon_r\zeta^{r-1}} \right| \le \frac{\sum_{r=2}^{\infty} r(r-1)|\varepsilon_r||\zeta|^{r-1}}{1+\sum_{r=2}^{\infty} r|\varepsilon_r||\zeta|^{r-1}} \tag{59}$$

Therefore,

$$\left| \frac{\zeta f''(\zeta)}{f'(\zeta)} \right| \le 1-\tau \quad \text{if} \quad \sum_{r=2}^{\infty}\frac{r(r-\tau)}{1-\tau}|\varepsilon_r||\zeta|^{r-1} \le 1 \tag{60}$$

By Theorem 2.1, the inequality holds if

$$\frac{r(r-\tau)}{1-\tau}|\zeta|^{r-1} \le \frac{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \tag{61}$$

or equivalently, if

$$|\zeta| \leq \left(\frac{(1-\tau)(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(r-\tau)(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}\right)^{\frac{1}{r-1}}, \quad r \geq 2 \tag{62}$$

The theorem follows immediately.

Assume that the function $f \in TS(\beta,\gamma)$. Then, f belongs to the class of the functions close-to-convex in $|\zeta| < R_3$ of order τ, $0 \leq \tau < 1$, in which

$$R_3 = \inf_{r\geq 2} \left\{\left(\frac{(1-\tau)(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}\right)^{\frac{1}{r-1}}\right\} \tag{63}$$

Proof. A function f belongs to the class of close-to-convex functions of order τ, $0 \leq \tau < 1$, if

$$\Re\left\{f'(\zeta)\right\} > \tau \tag{64}$$

Consequently, it is adequate to establish that

$$|f'(\zeta)-1| = \left|\sum_{r=2}^{\infty} r\varepsilon_r \zeta^{r-1}\right| \leq \sum_{r=2}^{\infty} r|\varepsilon_r||\zeta|^{r-1} \tag{65}$$

So,

$$|f'(\zeta)-1| \leq 1-\tau \quad \text{if} \quad \sum_{r=2}^{\infty} \frac{r}{1-\tau}|\varepsilon_r||\zeta|^{r-1} \leq 1 \tag{66}$$

By Theorem 2.1, this inequality holds if

$$\frac{r}{1-\tau}|\zeta|^{r-1} \leq \frac{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \tag{67}$$

or equivalently, if

$$|\zeta| \leq \left(\frac{(1-\tau)(r-r\gamma\beta+\gamma\beta-\gamma)\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}\right)^{\frac{1}{r-1}}, \quad r \geq 2 \tag{68}$$

The theorem follows directly.

5. Extremal functions associated with the class

In the theorem below, we characterize the boundary elements of the class $TS(\beta,\gamma)$.

Let $f_1(\zeta) = \zeta$
and

$$f_r(\zeta) = \zeta + \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\zeta^r, \quad r = 2,3,\ldots \tag{69}$$

Then, $f \in TS(\beta,\gamma)$ precisely when it can be represented as

$$f(\zeta) = \sum_{r=1}^{\infty} \theta_r f_k(\zeta), \quad \text{where } \theta_r \geq 0 \text{ and } \sum_{r=1}^{\infty} \theta_r = 1 \tag{70}$$

Proof. Let us suppose

$$f(\zeta) = \sum_{r=1}^{\infty} \theta_r f_k(\zeta) \tag{71}$$

then, we get

$$f(\zeta) = \zeta + \sum_{r=2}^{\infty} \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\theta_r\zeta^r \tag{72}$$

Now, $f \in TS(\beta,\gamma)$, since

$$\sum_{r=2}^{\infty} \frac{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)} \cdot \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}\theta_r = \sum_{r=2}^{\infty} \theta_r \leq 1-\theta_1 \leq 1$$

Conversely, suppose that $f \in TS(\beta,\gamma)$, then we show that f can be formulated as

$$f(\zeta) = \sum_{r=1}^{\infty} \theta_r f_k(\zeta) \tag{73}$$

Since $f \in TS(\beta,\gamma)$, we have the coefficient estimate:

$$\varepsilon_r \leq \frac{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)} \tag{74}$$

Let,

$$\theta_r = \frac{(r(r-r\gamma\beta+\gamma\beta-\gamma))\,\Gamma(\vartheta+\delta)\,\Gamma(\vartheta+r-1)}{(1-\gamma)\,\Gamma(\vartheta)\,\Gamma(\vartheta+r+\delta-1)}\varepsilon_r, \quad \text{for } r \geq 2 \tag{75}$$

and define

$$\theta_1 = 1 - \sum_{r=2}^{\infty} \theta_r \tag{76}$$

Then, clearly $\theta_r \geq 0$, $\sum_{r=1}^{\infty} \theta_r = 1$, and

$$f(\zeta) = \sum_{r=1}^{\infty} \theta_r f_r(\zeta) \tag{77}$$

6. Theorem on inclusion criterion

Let $0 \leq \beta_1 < \beta_2 < 1$ and $0 \leq \gamma_1 < \gamma_2 < 1$. Then the inclusion relation $TS(\beta_2, \gamma_2) \subseteq TS(\beta_1, \gamma_1)$ holds.

Proof. Define

$$G_{\beta,\gamma}(f;\zeta) := \left| \frac{(\mathcal{K}_\delta^\vartheta f)'(\zeta) + \zeta(\mathcal{K}_\delta^\vartheta f)''(\zeta)}{\beta\zeta(\mathcal{K}_\delta^\vartheta f)''(\zeta) + (\mathcal{K}_\delta^\vartheta f)'(\zeta)} \right| \tag{78}$$

Suppose $f \in TS(\beta_2, \gamma_2) \Rightarrow G_{\beta_2}(f;\zeta) > \gamma_2$. Since $\beta_1 < \beta_2$, the denominator

$$\beta_1\zeta(\mathcal{K}_\delta^\vartheta f)''(\zeta) + (\mathcal{K}_\delta^\vartheta f)'(\zeta) \tag{79}$$

is larger than that for β_2, assuming positivity and monotonicity of the involved terms. Hence the full expression decreases:

$$G_{\beta_1}(f;\zeta) \geq G_{\beta_2}(f;\zeta) > \gamma_2 > \gamma_1 \tag{80}$$

Therefore, $f \in TS(\beta_1, \gamma_1)$, proving the inclusion

$$TS(\beta_2, \gamma_2) \subseteq TS(\beta_1, \gamma_1) \tag{81}$$

The subclass $TS(\beta, \gamma)$ imposes geometric constraints governed by the operator $\mathcal{K}_\delta^\vartheta$. Increasing β emphasizes the contribution of the second derivative, enhancing curvature sensitivity. A higher γ restricts the region further, enforcing tighter sectorial control. Consequently, raising either parameter reduces the function space, resulting in a reverse inclusion relation.

6.1. Corollaries and example

For fixed β, if $\gamma_2 > \gamma_1$, then,

$$TS(\beta, \gamma_2) \subset TS(\beta, \gamma_1).$$

For fixed γ, if $\beta_2 > \beta_1$, then,

$$TS(\beta_2, \gamma) \subset TS(\beta_1, \gamma).$$

Let

$$\beta_1 = 0.2, \quad \gamma_1 = 0.3; \qquad \beta_2 = 0.6, \quad \gamma_2 = 0.5 \tag{82}$$

Then we have,

$$TS(0.6, 0.5) \subset TS(0.2, 0.3).$$

Here, f and g are representative functions in the class $TS(\beta_2, \gamma_2)$, and the arrows depict their respective images under the action of the Erdélyi–Kober operator. The diagram illustrates that the images $f(\mathfrak{D})$ and $g(\mathfrak{D})$ lie within the broader class $TS(\beta_1, \gamma_1)$, confirming the inclusion property.

6.2. Applications on inclusion relation

The inclusion property of the class $TS(\beta, \gamma)$ is central in geometric function theory and its applications in modeling and computational analysis. The relation

$$TS(\beta_2, \gamma_2) \subset TS(\beta_1, \gamma_1) \quad \text{for } \beta_2 > \beta_1,\ \gamma_2 > \gamma_1 \tag{83}$$

defines a hierarchy of subclasses characterized by increasingly strict geometric behaviors. This facilitates generalizations of results such as growth, covering, and coefficient bounds.

For instance, if a function in $TS(0.6, 0.5)$ satisfies a bound like $|f(\zeta)| \leq \frac{1}{(1-|\zeta|)^k}$, the same bound applies to all $TS(\beta_1, \gamma_1)$ with $\beta_1 < 0.6$, $\gamma_1 < 0.5$. In computational geometry and conformal mapping (e.g., via Schwarz–Christoffel techniques), adjusting β and γ tunes the angular and convexity constraints, promoting both analytic tractability and numerical stability. From an applied perspective, such inclusion results are especially valuable in engineering models involving fractional-order operators. Consider a viscoelastic constitutive relation:

$$\sigma(t) = E\, \mathcal{K}_\delta^\vartheta[\varepsilon(t)] \tag{84}$$

with $\varepsilon(t) \in TS(\beta_2, \gamma_2)$. By the inclusion relation, one may safely assume $\varepsilon(t) \in TS(\beta_1, \gamma_1)$ for lower parameter values, expanding the solution set without sacrificing desired analytic features. This is crucial in modeling adaptive materials or designing damping systems where flexibility, robustness, and physical realism must be harmonized.

7. Discussion

In this study, we introduced and examined a new subclass of analytic and univalent functions within the open unit disk $\mathbb{D}_\zeta$, formulated through the application of the normalized Erdélyi–Kober fractional integral operator. This subclass, denoted by $TS(\beta, \gamma)$, serves as a generalization of several known function classes associated with classical and modern differential operators. The proposed framework builds upon and extends the foundational work of researchers such as Tremblay, Salagean, Ruscheweyh, Hohlov, and Komatu, offering a comprehensive operator-based perspective.

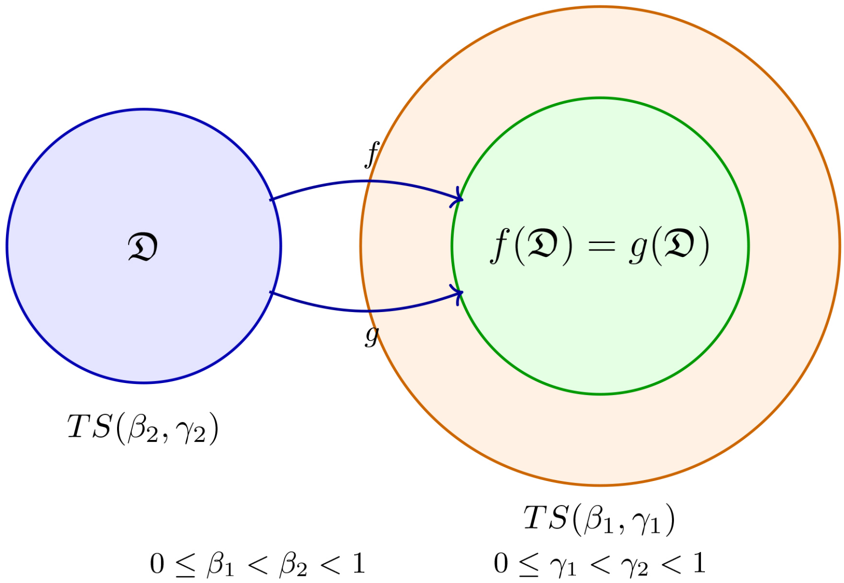

Figure 5. Inclusion of image regions under $f \in TS(\beta_2, \gamma_2)$

Utilizing tools from fractional calculus, we derived various analytical characteristics for the $TS(\beta, \gamma)$ class. Notably, we established precise bounds for Taylor–Maclaurin coefficients and determined necessary and sufficient conditions that functions must satisfy to belong to this subclass. Furthermore, we investigated key geometric aspects, including growth behavior, distortion estimates, and radii connected to properties like starlikeness, convexity, and close-to-convexity. These findings contribute to a deeper understanding of the function's mapping properties under fractional integral transformations.

An important finding of this work is that each function within the $TS(\beta, \gamma)$ class can be expressed as a convex linear combination of specific extremal functions. This structural representation enhances our understanding of the subclass and provides the analysis of extremal problems, coefficient bounds, and geometric optimization in the broader scope of univalent function theory. A central contribution of this work lies in the development and proof of the inclusion criterion for the subclass $TS(\beta, \gamma)$. Specifically, we demonstrated that for any $0 \leq \beta_1 < \beta_2 < 1$ and $0 \leq \gamma_1 < \gamma_2 < 1$, the inclusion

$$TS(\beta_2, \gamma_2) \subset TS(\beta_1, \gamma_1) \tag{85}$$

holds. This result introduces a natural hierarchy among subclass, whereby increasing either the weighting parameter β or the starlikeness threshold γ imposes stricter geometric constraints, thereby reducing the admissible function space. The theoretical insight gained from this inclusion property enables broader applicability in analytic estimations: for instance, any geometric or analytic bound proven for a stricter subclass remains valid for all larger, encompassing subclasses.

This inclusion framework also has practical significance in applications governed by fractional differential equations. For example, in viscoelastic models or anomalous diffusion systems, solutions expressed via fractional operators like $\mathcal{K}_{\delta}^{\vartheta}$ often need to satisfy certain geometric conditions for stability and physical realism. Suppose a strain function $\varepsilon(t)$ is known to lie in a more restricted class $TS(\beta_2, \gamma_2)$, then, the inclusion result assures that this function also belongs to $TS(\beta_1, \gamma_1)$ for any $\beta_1 < \beta_2$ and $\gamma_1 < \gamma_2$, then, widening the modeling flexibility without compromising analytic structure. This is particularly beneficial in numerical simulations, conformal mappings, and inverse problems where adjusting parameters within broader classes can improve computational efficiency and solution robustness. Additionally, our methodological approach highlights the growing interaction between fractional calculus and geometric function theory. The Erdélyi–Kober operators used herein, recognized for their nonlocal and memory-retaining properties, reflect the increasing relevance of such tools in modeling real-world systems with hereditary characteristics. As such, the theoretical results presented hold potential implications across areas like conformal mappings, viscoelastic material modeling, anomalous transport, and control systems with fractional-order dynamics.

In summary, the $TS(\beta, \gamma)$ subclass significantly contributes to the development of operator-defined analytic function classes. The introduction of an inclusion principle enriches the structural understanding of the class, expands its applicability, and opens avenues for future work, including higher-dimensional generalizations, advanced operator theory, and computational modeling frameworks based on fractional geometric constraints.

8. Conclusion

This work presents a unified operator-theoretic framework for a newly defined subclass $TS(\beta, \gamma)$ of analytic and univalent functions in the open unit disk, constructed via the convolution with the normalized Erdélyi–Kober fractional integral operator. Through rigorous analysis, we established sharp bounds for initial coefficients, derived distortion and growth estimates, and identified precise radii related to starlikeness, convexity, and close-to-convexity. A central contribution is the formulation and proof of an inclusion criterion based on the parameters β and γ, which introduces a hierarchy of function classes and ensures that stronger geometric properties in narrower subclasses persist in broader ones. This inclusion property, alongside the representation of class members as convex combinations of extremal functions, enhances both theoretical insight and practical modeling flexibility. Moreover, the findings highlight the versatility of the Erdélyi–Kober operator in capturing memory effects and geometric constraints, with direct implications for fractional differential systems in viscoelasticity, signal processing, and conformal mapping. The results pave the way for further investigations into multidimensional generalizations, subordination principles, and numerical methods informed by operator-defined geometric function theory.

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Conflict of interest

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