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RESEARCH ARTICLE

Revisiting Krasnoselskii's fixed point theorem: Extensions and applications to operator equations

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ABSTRACT

Fixed point theory stands as a fundamental pillar in nonlinear functional analysis, being essential for proving the existence of solutions for nonlinear differential and integral equations. Krasnoselskii's hybrid fixed point theorem, which combines the Banach contraction principle with Schauder's theorem, is a pivotal contribution. Recent efforts have focused on refining and relaxing the conditions of this theorem. This study aims to extend the theoretical framework of Krasnoselskii-type fixed point theorems to address a broader and more general class of nonlinear operator equations within a Banach algebra setting. It also seeks to establish rigorous conditions for the existence (and uniqueness, where possible) of solutions. The approach involves developing local variants of the classic Krasnoselskii fixed point theorems. We performed a comparative analysis of previous studies, introduced modifications to the operator equations to relax restrictive assumptions, and theoretically generalized the theorems to accommodate a complex structure involving four operators. To validate the results, they were applied to a nonlinear functional integral equation within the Banach space C[0,1]. We successfully generalized existing results by incorporating the Hölder continuity condition, which is less restrictive than the standard Lipschitz condition. The unified theoretical framework led to the establishment of a comprehensive set of theorems and corollaries covering a wide class of operator equations such as : $Ax^{(\rho 2)}Bx^{\rho 1} + Cx^{(\rho 3)}Dx^{\rho 1} = x$. Our results provide less restrictive local existence conditions and wider applicability in the analysis of complex mathematical systems.

1. Introduction

The application of fixed point theorems remains a powerful tool in analyzing various complex systems, particularly in the realm of fractional calculus and control theory.

In the domain of control theory and system analysis, Ref.[1] utilized these theorems in their study of regional boundary observability for semilinear time-fractional systems with Caputo derivatives, demonstrating their utility. Similarly, Ref.[2] employed fixed point techniques to investigate approximate controllability for fractional stochastic integro-differential inclusion systems, a theme further explored by Ref.[3] for fractional order systems with state-dependent delay. The existence and uniqueness of solutions for various fractional differential equations are another significant area where fixed-point theorems

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are indispensable. This is consistent with findings in Ref.[4] for non-instantaneous impulsive fractional order differential equations and Ref.[5] for Hilfer fuzzy fractional inclusion with infinite delay. Fixed point theory is also central to proving existence and uniqueness for systems involving generalized derivatives, as shown by Ref.[6] for non-instantaneous impulsive fractional differential equations, and for coupled and tripled systems of fractional differential equations with nonlocal integro-multi-point boundary conditions by. Ref.[7]

Beyond specific fractional derivatives, a wide array of fixed-point theorems has been developed and applied in various metric and generalized metric spaces. For instance, new common fixed point results for generalized $\alpha - \psi$-contractive mappings in complete metric spaces were presented by, Ref.[8] while Ref.[9] focused on rational-type contractions. Generalized contractions in b-metric spaces and their applications were investigated by, Ref.[10] and extensions to complex- valued b-metric spaces were made by Ref.[11] and. Ref.[12] Further developments include fixed point theorems in generalized metric spaces and their applications to integral equations by. Ref.[13] Finally, Ref.[14] utilized fixed point principles in their work on the exponential stability of higher-order impulsive fractional neutral stochastic integro-delay differential equations with mixed Brownian motions and non-local conditions, highlighting the broad and enduring applicability of these theorems in analyzing complex dynamical systems. More results are presented in Ref.[15,16]

Fixed point theory is widely regarded as a foundational pillar of nonlinear functional analysis, owing to its central role in establishing existence results for both global and local solutions of nonlinear differential and integral equations. In this context, the pioneering contribution of Krasnoselskii marks a significant milestone: his hybrid fixed point theorem effectively merges Banach's contraction principle with Schauder's topological fixed point theorem within the framework of Banach spaces. It has played a crucial role in solving perturbed differential equations via nonlinear integral equations. Over the years, considerable effort has been devoted to refining and relaxing the conditions of Krasnoselskii's theorem, as noted in. Ref.[17–19] Notably, Dhage[20–22] has contributed extensively to clarifying and extending these developments in a series of works.

Recent advancements in fixed-point theory have expanded its applicability to broader classes of nonlinear problems. For instance, new fixed-point theorems for the sum of two operators—without requiring compactness or continuity–were introduced in, Ref.[23] extending and generalizing the classical Krasnoselskii theorem. These results have been effectively applied to prove the existence and uniqueness of solutions for various types of equations, including transport equations, Darboux problems, difference equations, and perturbed Volterra-type integral equations. A recent study Ref.[24] addresses Caputo-type fractional differential equations by converting them into integral equations and applying generalized fixed-point theorems. Using weighted norms, they avoid compactness conditions and obtain existence and growth properties of solutions through extensions of Schaefer's and Krasnoselskii's theorems. Investigates Ref.[25] linear maps between complex Banach algebras that preserve products equal to fixed elements, extending classical results involving zero or identity products. The study reveals that such maps often reduce to scalar multiples of algebra homomorphisms and provides a broader framework related to invertibility preservers and Kaplansky's problem.

Agarwal and collaborators[26,27] have advanced fixed point theory and its applications to nonlinear analysis. A new Leray–Schauder alternatives and generalized Krasnoselskii expansion–compression theorems were developed for multivalued maps in Fréchet spaces, using the framework of pseudo-open sets and projective limits of Banach spaces. Also, a Krasnoselskii-type fixed point theorem was applied to establish the existence and uniqueness of integrable solutions for general quadratic-Urysohn integral equations on bounded intervals, see.[26] More recently,[27] introduced new fixed point and coupled fixed point theorems—based on measures of noncompactness—to prove the existence of solutions for fractional evolution equations and coupled systems in Banach spaces. These findings not only generalize classical theorems but also extend recent developments in the study of nonlinear and fractional systems.

Within this framework, the present work aims to develop local variants of Krasnoselskii's classical fixed point theorems and to explore their practical applications to functional integral equations.

This work investigates specific hypotheses in fixed point theory and applies them to a novel class of operator equations. Additionally, the study explores the application of recently developed local fixed point theorems to the analysis of these equations.

Our key contributions include:

(i) Generalization of existing results: We extended existing results by analyzing a complex structure involving four operators and introducing Hölder continuity, which is less restrictive than the traditional Lipschitz condition.
(ii) Establishment of less restrictive conditions: We developed local existence conditions that are less restrictive and more practically applicable than those previously available.
(iii) Unified theoretical framework: We unified the analysis of a broad class of operator equations (Equations [4–11]) under a single theoretical framework, demonstrating its effectiveness through the analysis of a nonlinear functional integral equation.

The research was conducted through the following steps:

(i) Comparative analysis: We began by reviewing and comparing previous studies involving operator equations where local fixed point theorems had been applied, identifying limitations and gaps in existing frameworks.
(ii) Formulation and modification: We introduced new conditions to a set of operator equations, modifying them to relax restrictive assumptions from earlier works and broaden their scope of applicability.
(iii) Theoretical generalization: We extended existing local fixed point theorems to accommodate these newly formulated operator equations, establishing new existence results under weaker and more practical assumptions.

The primary objectives of this study are:

(i) Broadening applicability: To enable the application of local fixed point theorems to a wider class of operator equations, particularly those not previously covered.
(ii) Establishing existence and uniqueness: To develop rigorous conditions for the existence—and where possible, uniqueness—of solutions to these operator equations.
(iii) Theoretical advancement: To generalize existing results in fixed point theory, thereby extending their relevance and applicability to a wider range of mathematical models and functional equations.

In this study, we have successfully extended the framework of Krasnoselskii-type fixed point theorems to address a broad class of nonlinear operator equations within a Banach algebra setting. Our primary objective was to establish local existence results for solutions by refining and clarifying the hypotheses for theorems involving multiple operators.
Our main contributions regarding the proofs include:

(i) Generalization of applicability: The core contribution of our proofs lies in relaxing restrictive assumptions, allowing fixed point theory to be applied to a broader and more general class of operator equations. This is clearly illustrated in the proof of Theorem 2 (see Equation [4]), which extends applicability beyond traditional frameworks.
(ii) Demonstration through applications: To validate our theoretical results, we applied them to a nonlinear functional integral equation, providing a concrete example in the Banach space $C[0,1]$. This demonstrates how abstract results can be effectively used in practical settings, bridging the gap between theory and application.

It is important to note that the proposed technique is primarily theoretical, focusing on proving the existence (and sometimes uniqueness) of solutions to a broad class of operator equations using fixed-point theorems. As such, it does not involve algorithmic procedures or numerical methods that require traditional computational complexity analysis. However, the theoretical framework has been systematically developed and applied to derive a unified set of results—including several theorems and corollaries—that cover general equations such as $Ax^{\rho 2}Bx^{\rho 1}+Cx^{\rho 3}Dx^{\rho 1}=x$, as well as several important special cases. Compared to previous studies, our approach is more general and flexible, offering broader applicability rather than computational efficiency in the algorithmic sense.

Throughout this study, let Y be a Banach space with a norm $\|.\|$. Let $s \in Y$ and let e be a positive real number. Then by $B_e(s)$ and $\overline{B_e}(s)$, we denote, respectively, an open and a closed ball in Y centered at the point $s \in Y$ of radius e. The local version of the well-known Banach fixed-point theorem is stated as follows:

Theorem 1 ([18]). *Let $Q : \overline{B_e}(s) \longrightarrow Y$ be a contraction with contraction constant β. If Q satisfies*

$$\|s - Qs\| \leq (1-\beta)e, \tag{1}$$

for some $s \in Y$ and $e > 0$, then Q has a unique fixed point in $\overline{B_e}(s)$.

Definition 1. *A mapping $Q : Y \longrightarrow Y$ is called* ***Hölderian*** *if there exist constants $\mu > 0$ and*

$\rho \in (0,1]$ such that

$$\|Q(t_1) - Q(t_2)\| \leq \mu \|t_1 - t_2\|^{\rho} \quad \text{for all } t_1, t_2 \in Y. \tag{2}$$

The constant μ is the Hölder constant.

Definition 2. *A mapping $Q : Y \longrightarrow Y$ is called **Lipschitzian** if it is Hölderian with $\rho = 1$. That is, there exists a constant $\mu > 0$ such that*

$$\|Qt_1 - Qt_2\| \leq \mu \|t_1 - t_2\| \quad \text{for all } t_1, t_2 \in Y. \tag{3}$$

The constant μ is the Lipschitz constant.

This work aims to extend existing results by establishing existence theorems for the following operator equations in a Banach algebra Y:

$$Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3}Dx^{q} = x \tag{4}$$

$$AxBx + CxDx = x \tag{5}$$

$$Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3} = x \tag{6}$$

$$Ax^{\rho_2}Bx^{\rho_3} + C = x \tag{7}$$

$$Ax^{\rho_2}Bx^{\rho_3} = x \tag{8}$$

$$Ax^{\rho_2} + Bx^{\rho_3} = x \tag{9}$$

$$Ax^{\rho_2} + B = x \tag{10}$$

$$Ax^{\rho_2} = x \tag{11}$$

This study is structured as follows: Section 2 introduces the core contributions of this work, presenting our main results and a newly developed fixed point theorem. We then demonstrate its applicability through novel examples involving nonlinear integral equations. Finally, Section 3 provides a summary of our findings, along with conclusions and an outlook on future research directions.

2. Results

2.1. Fixed point theorem

Fixed point theorems were applied to a class of operator equations in a Banach algebra, building upon a series of works by Dhage.[22,28]

Theorem 2. *Let Y be a Banach algebra, $s \in Y$, e a positive real number, and $\rho_1, \rho_2, \rho_3 \in (0,1]$ with $\rho_2 \geq \rho_3$. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(s) \longrightarrow Y$ be four operators such that:*

(i) A and C are Hölderian with Hölder constants μ and ν and exponents ρ_2 and ρ_3, respectively.

(ii) B and D are completely continuous, with $U = \sup_{r\in\overline{B_e}(s)} \|Br^{\rho_1}\|$ and $V = \sup_{r\in\overline{B_e}(s)} \|Dr^{\rho_1}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu V < 1$:

$$\|s - (As^{\rho_2}Br^{\rho_1} + Cs^{\rho_3}Dr^{\rho_1})\| \leq (1-(\mu U+\nu V))e. \tag{12}$$

Then, the operator equation $Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3}Dx^{\rho_1} = x$ has a solution in $\overline{B_e}(s)$.

Proof. Let $r \in \overline{B_e}(s)$ be fixed and define a mapping A_r on $\overline{B_e}(s)$ by

$$A_r(x) = Ax^{\rho_2}Br^{\rho_1} + Cx^{\rho_3}Dr^{\rho_1}.$$

We show that A_r is a contraction on $\overline{B_e}(s)$. Let x_1, $x_2 \in \overline{B_e}(s)$ be arbitrary. Then we have

$$\begin{aligned}\|A_r(x_1) - A_r(x_2)\| &\leq \|Ax_1^{\rho_2} - Ax_2^{\rho_2}\| \, \|Br^{\rho_1}\| \\ &\quad + \|Cx_1^{\rho_3} - Cx_2^{\rho_3}\| \, \|Dr^{\rho_1}\| \\ &\leq \mu \|x_1 - x_2\|^{\rho_2} \|Br^{\rho_1}\| \\ &\quad + \nu \|x_1 - x_2\|^{\rho_3} \|Dr^{\rho_1}\| \\ &\leq (\mu U + \nu V) \|x_1 - x_2\|^{\rho_3},\end{aligned}$$

where $0 < \mu U + \nu V < 1$. Since $\rho_3 \in (0,1]$, A_r is a contraction on $\overline{B_e}(s)$. By Hypothesis (3),

$$\begin{aligned}\|s - A_r(s)\| &= \|s - (As^{\rho_2}Br^{\rho_1} + Cs^{\rho_3}Dr^{\rho_1})\| \\ &\leq (1 - (\mu U + \nu V))e.\end{aligned}$$

Hence, by Theorem 1 , there is a unique point $\widetilde{x}_r$ in $\overline{B_e}(s)$ such that $A_r(\widetilde{x}_r) = \widetilde{x}_r$, i.e.,

$$A\widetilde{x}_r^{\rho_2}Br^{\rho_1} + C\widetilde{x}_r^{\rho_3}Dr^{\rho_1} = \widetilde{x}_r.$$

Define an operator $K : \overline{B_e}(s) \longrightarrow \overline{B_e}(s)$ by $Kr = \widetilde{x}_r$. We show that K is a continuous operator on $\overline{B_e}(s)$. Let $\{r_n\}$ be a sequence in $\overline{B_e}(s)$ converging to a point $r \in \overline{B_e}(s)$.

Let $x_n = Kr_n$ and $x = Kr$. Then we have

$$\begin{aligned}\|x_n - x\| &\leq \|A(x_n^{\rho_2})B(r_n^{\rho_1}) - A(x^{\rho_2})B(r_n^{\rho_1})\| \\ &\quad + \|A(x^{\rho_2})B(r_n^{\rho_1}) - A(x^{\rho_2})B(r^{\rho_1})\| \\ &\quad + \|C(x_n^{\rho_3})D(r_n^{\rho_1}) - C(x^{\rho_3})D(r_n^{\rho_1})\| \\ &\quad + \|C(x^{\rho_3})D(r_n^{\rho_1}) - C(x^{\rho_3})D(r^{\rho_1})\| \\ &\leq (\mu U + \nu V) \|x_n - x\|^{\rho_3} \\ &\quad + \|A(x^{\rho_2})\| \, \|B(r_n^{\rho_1}) - B(r^{\rho_1})\| \\ &\quad + \|C(x^{\rho_3})\| \, \|D(r_n^{\rho_1}) - D(r^{\rho_1})\|.\end{aligned}$$

This yields $(1 - (\mu U + \nu V)) \|x_n - x\| \leq \|A(x^{\rho_2})\| \, \|B(r_n^{\rho_1}) - B(r^{\rho_1})\| + \|C(x^{\rho_3})\| \, \|D(r_n^{\rho_1}) - D(r^{\rho_1})\|$.

Since B and D are continuous, the right-hand side tends to zero as $n \to \infty$. Thus, $\lim_{n\to\infty} \|Kr_n - Kr\| = 0$. This proves that K is continuous.

Next, we show that K is a compact operator on $\overline{B_e}(s)$. Let $\{r_n\}$ be a sequence in $\overline{B_e}(s)$. Since B and D are completely continuous, the sets $\{Br_n^{\rho_1}\}$ and $\{Dr_n^{\rho_1}\}$ are relatively compact. Thus, there exist subsequences, which we also denote by $\{r_n\}$, such that $Br_n^{\rho_1} \to u$ and $Dr_n^{\rho_1} \to v$

for some $u, v \in Y$. Let $x_n = Kr_n$. Then x_n satisfies $x_n = Ax_n^{\rho_2} Br_n^{\rho_1} + Cx_n^{\rho_3} Dr_n^{\rho_1}$. The sequence $\{x_n\}$ is bounded in $\overline{B_e}(s)$. The right-hand side involves continuous operators and compact sequences, implying that the image sequence $\{x_n\}$ lies in a compact set. Therefore, there exists a convergent subsequence $\{x_{n_k}\}$. This shows that $K(\overline{B_e}(s))$ is relatively compact.

Since K is continuous and compact, an application of Schauder's fixed point theorem yields that K has a fixed point in $\overline{B_e}(s)$. Consequently, the operator equation has a solution in $\overline{B_e}(s)$. □

By taking $\rho_2 = \rho_3 = \rho_1$ in the previous theorem, we obtain the following result.

Corollary 1. *Let $s \in Y$ and e be a positive real number and $\rho \in (0, 1]$. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(s) \longrightarrow Y$ be four operators such that*

(i) A and C are Hölderian with Hölder constants μ and ν and exponent ρ;

(ii) B and D are completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br^\rho\|$ and $V = \sup_{r\in\overline{B_e}(s)} \|Dr^\rho\|$;

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu V < 1$:

$$\|s - (As^\rho Br^\rho + Cs^\rho Dr^\rho)\| \le (1 - (\mu U + \nu V))e. \tag{13}$$

Then the operator equation $Ax^\rho Bx^\rho + Cx^\rho Dx^\rho = x$ has a solution in $\overline{B_e}(s)$.

By taking $s = 0$ in the main theorem, we obtain the following corollary.

Corollary 2. *Let e be a positive real number and $\rho_1, \rho_2, \rho_3 \in (0, 1]$ with $\rho_2 \ge \rho_3$. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(0) \longrightarrow Y$ be four operators such that*

(i) A and C are Hölderian with Hölder constants μ and ν and exponents ρ_2 and ρ_3, respectively;

(ii) B and D are completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^{\rho_1}\|$ and $V = \sup_{r\in\overline{B_e}(0)} \|Dr^{\rho_1}\|$;

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U + \nu V < 1$:

$$\|A(0)Br^{\rho_1} + C(0)Dr^{\rho_1}\| \le (1 - (\mu U + \nu V))e. \tag{14}$$

Then the operator equation $Ax^{\rho_2} Bx^{\rho_1} + Cx^{\rho_3} Dx^{\rho_1} = x$ has a solution in $\overline{B_e}(0)$.

By setting $s = 0$ and taking $\rho_1 = \rho_2 = \rho_3 = \rho$ in the main theorem, we obtain the following.

Corollary 3. *Let e be a positive real number and $\rho \in (0, 1]$. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(0) \longrightarrow Y$ be four operators such that:*

(i) A and C are Hölderian with Hölder constants μ and ν and exponent ρ,

(ii) B and D are completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^\rho\|$ and $V = \sup_{r\in\overline{B_e}(0)} \|Dr^\rho\|$;

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U + \nu V < 1$:

$$\|A(0)Br^\rho + C(0)Dr^\rho\| \le (1-(\mu U+\nu V))e. \tag{15}$$

Then the operator equation $Ax^\rho Bx^\rho + Cx^\rho Dx^\rho = x$ has a solution in $\overline{B_e}(0)$.

By setting $\rho_1 = \rho_2 = \rho_3 = 1$ in Theorem 2, we obtain the following result for Lipschitzian operators.

Proposition 1. *Let $s \in Y$ and e a positive real number. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(s) \longrightarrow Y$ be four operators such that:*

(i) A and C are Lipschitzian with Lipschitz constants μ and ν respectively.

(ii) B and D are completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br\|$ and $V = \sup_{r\in\overline{B_e}(s)} \|Dr\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu V < 1$:

$$\|s - (AsBr + CsDr)\| \le (1-(\mu U+\nu V))e. \tag{16}$$

Then the operator Equation (5) has a solution in $\overline{B_e}(s)$.

Proof. Let $r \in \overline{B_e}(s)$ be fixed and define a mapping A_r on $\overline{B_e}(s)$ by $A_r(x) = AxBr + CxDr$. We show that A_r is a contraction on $\overline{B_e}(s)$. Let $x_1, x_2 \in \overline{B_e}(s)$ be arbitrary. Then we have

$$\begin{aligned}\|A_r(x_1) - A_r(x_2)\| &= \|(Ax_1 - Ax_2)Br \\ &\quad + (Cx_1 - Cx_2)Dr \\ &\le \|Ax_1 - Ax_2\| \, \|Br\| \\ &\quad + \|Cx_1 - Cx_2\| \, \|Dr\| \\ &\le \mu \|x_1 - x_2\| \, \|Br\| \\ &\quad + \nu \|x_1 - x_2\| \, \|Dr\| \\ &\le (\mu U + \nu V) \|x_1 - x_2\|,\end{aligned}$$

where $0 < \mu U + \nu V < 1$, and so A_r is a contraction. By Hypothesis (3), $\|s - A_r(s)\| \le (1-(\mu U+\nu V))e$. By Theorem 1, there is a unique point $\widetilde{x}_r \in \overline{B_e}(s)$ such that $A\widetilde{x}_r Br + C\widetilde{x}_r Dr = \widetilde{x}_r$. The remainder of the proof follows the same argument as in Theorem 2: define an operator $K(r) = \widetilde{x}_r$, show it is continuous and compact, and apply Schauder's fixed point theorem. □

Taking $s = 0$ in the previous proposition yields the following result.

Corollary 4. *Let e be a positive real number. Let $A, C : Y \longrightarrow Y$ and $B, D : \overline{B_e}(0) \longrightarrow Y$ be four operators such that:*

(i) A and C are Lipschitzian with Lipschitz constants μ and ν, respectively.

(ii) B and D are completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br\|$ and $V = \sup_{r\in\overline{B_e}(0)} \|Dr\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U + \nu V < 1$:

$$\|A(0)Br + C(0)Dr\| \leq (1 - (\mu U + \nu V))e. \quad (17)$$

Then the operator Equation (5) has a solution in $\overline{B_e}(0)$.

Now we consider the operator Equation (6).

Theorem 3. *Let $s \in Y$, $e > 0$, and $\rho_1, \rho_2, \rho_3 \in (0,1]$ with $\rho_2 \geq \rho_3$. Let $A, C : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be three operators such that:*

(i) A and C are Hölderian with constants μ, ν and exponents ρ_2, ρ_3 respectively.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br^{\rho_1}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu < 1$:

$$\|s - (As^{\rho_2}Br^{\rho_1} + Cs^{\rho_3})\| \leq (1-(\mu U+\nu))e. \quad (18)$$

Then the operator equation $Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3} = x$ has a solution in $\overline{B_e}(s)$.

Proof. Let $r \in \overline{B_e}(s)$ be fixed and define $A_r(x) = Ax^{\rho_2}Br^{\rho_1} + Cx^{\rho_3}$. We show that A_r is a contraction on $\overline{B_e}(s)$. Let $x_1, x_2 \in \overline{B_e}(s)$.

$$\begin{aligned}\|A_r(x_1) - A_r(x_2)\| &\leq \|(Ax_1^{\rho_2} - Ax_2^{\rho_2})Br^{\rho_1}\| \\ &\quad + \|(Cx_1^{\rho_3} - Cx_2^{\rho_3})\| \\ &\leq \|Ax_1^{\rho_2} - Ax_2^{\rho_2}\| \, \|Br^{\rho_1}\| \\ &\quad + \|Cx_1^{\rho_3} - Cx_2^{\rho_3}\| \\ &\leq \mu \|x_1 - x_2\|^{\rho_2} \|Br^{\rho_1}\| \\ &\quad + \nu \|x_1 - x_2\|^{\rho_3} \\ &\leq (\mu U + \nu) \|x_1 - x_2\|^{\rho_3},\end{aligned}$$

where $0 < \mu U + \nu < 1$. So A_r is a contraction. By Hypothesis (3), $\|s - A_r(s)\| \leq (1-(\mu U+\nu))e$. By Theorem 1, there is a unique point $\widetilde{x}_r \in \overline{B_e}(s)$ such that $A\widetilde{x}_r^{\rho_2}Br^{\rho_1} + C\widetilde{x}_r^{\rho_3} = \widetilde{x}_r$. The proof is completed by defining $K(r) = \widetilde{x}_r$ and applying Schauder's theorem as in Theorem 2. □

The following are corollaries of the preceding theorem under various special conditions.

Corollary 5. *Let $s \in Y$, $e > 0$, and $\rho \in (0,1]$. Let $A, C : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be three operators such that:*

(i) A and C are Hölderian with constants μ, ν and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu < 1$:

$$\|s - (As^{\rho}Br^{\rho} + Cs^{\rho})\| \leq (1 - (\mu U + \nu))e. \quad (19)$$

Then the operator equation $Ax^{\rho}Bx^{\rho} + Cx^{\rho} = x$ has a solution in $\overline{B_e}(s)$.

Corollary 6. *Let $e > 0$ and $\rho_1, \rho_2, \rho_3 \in (0,1]$ with $\rho_2 \geq \rho_3$. Let $A, C : Y \longrightarrow Y$ and $B : \overline{B_e}(0) \longrightarrow Y$ be three operators such that:*

(i) A and C are Hölderian with constants μ, ν and exponents ρ_2, ρ_3, respectively.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^{\rho_1}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U + \nu < 1$:

$$\|A(0)Br^{\rho_1} + C(0)\| \leq (1 - (\mu U + \nu))e. \quad (20)$$

Then the operator equation $Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3} = x$ has a solution in $\overline{B_e}(0)$.

Corollary 7. *Let $e > 0$ and $\rho \in (0,1]$. Let $A, C : Y \longrightarrow Y$ and $B : \overline{B_e}(0) \longrightarrow Y$ be three operators such that:*

(i) A and C are Hölderian with constants μ, ν and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U + \nu < 1$:

$$\|A(0)Br^{\rho} + C(0)\| \leq (1 - (\mu U + \nu))e. \quad (21)$$

Then the operator equation $Ax^{\rho}Bx^{\rho} + Cx^{\rho} = x$ has a solution in $\overline{B_e}(0)$.

Corollary 8. *Let $s \in Y$ and $e > 0$. Let $A, C : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be three operators such that:*

(i) A and C are Lipschitzian with constants μ and ν.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U + \nu < 1$:

$$\|s - (AsBr + Cs)\| \leq (1 - (\mu U + \nu))e. \quad (22)$$

Then the operator equation $AxBx + Cx = x$ has a solution in $\overline{B_e}(s)$.

Now, we focus on Equation (7).

Proposition 2. *Let $s \in Y$, $e > 0$, and $\rho_2, \rho_3 \in (0,1]$. Let $A : Y \longrightarrow Y$, $B : \overline{B_e}(s) \longrightarrow Y$, and $C \in Y$ be an element (a constant operator) such that:*

(i) A is Hölderian with constant μ and exponent ρ_2.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br^{\rho_3}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - (As^{\rho_2} Br^{\rho_3} + C)\| \le (1 - \mu U)e. \tag{23}$$

Then the operator equation $Ax^{\rho_2} Bx^{\rho_3} + C = x$ has a solution in $\overline{B_e}(s)$.

Proof. Let $r \in \overline{B_e}(s)$ be fixed and define $A_r(x) = Ax^{\rho_2} Br^{\rho_3} + C$. For $x_1, x_2 \in \overline{B_e}(s)$,

$$\begin{aligned} \|A_r(x_1) - A_r(x_2)\| &= \|Ax_1^{\rho_2} Br^{\rho_3} - Ax_2^{\rho_2} Br^{\rho_3}\| \\ &\le \|Ax_1^{\rho_2} - Ax_2^{\rho_2}\| \, \|Br^{\rho_3}\| \\ &\le \mu \, \|x_1 - x_2\|^{\rho_2} \, U. \end{aligned}$$

The operator A_r is a contraction since $\mu U < 1$. By Hypothesis (3) and Theorem 1, there is a unique solution $\widetilde{x}_r = A_r(\widetilde{x}_r)$. Define the operator $K(r) = \widetilde{x}_r$. The operator K is continuous and compact because B is completely continuous and A is continuous. Schauder's theorem then implies the existence of a fixed point for K. □

The following corollaries are direct consequences of Proposition 2 .

Corollary 9. *Let $s \in Y$, $e > 0$, and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$, $B : \overline{B_e}(s) \longrightarrow Y$, and $C \in Y$ be such that:*

(i) A is Hölderian with constant μ and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(s)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - (As^{\rho} Br^{\rho} + C)\| \le (1 - \mu U)e. \tag{24}$$

Then the operator equation $Ax^{\rho} Bx^{\rho} + C = x$ has a solution in $\overline{B_e}(s)$.

Corollary 10. *Let $e > 0$ and $\rho_2, \rho_3 \in (0, 1]$. Let $A : Y \longrightarrow Y$, $B : \overline{B_e}(0) \longrightarrow Y$, and $C \in Y$ be such that:*

(i) A is Hölderian with constant μ and exponent ρ_2.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(0)} \|Br^{\rho_3}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U < 1$:

$$\|-A(0) Br^{\rho_3} - C\| \le (1 - \mu U)e. \tag{25}$$

Then the operator equation $Ax^{\rho_2} Bx^{\rho_3} + C = x$ has a solution in $\overline{B_e}(0)$.

Corollary 11. *Let $e > 0$ and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$, $B : \overline{B_e}(0) \longrightarrow Y$, and $C \in Y$ be such that:*

(i) A is Hölderian with constant μ and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(0)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U < 1$:

$$\|-A(0) Br^{\rho} - C\| \le (1 - \mu U)e. \tag{26}$$

Then the operator equation $Ax^{\rho} Bx^{\rho} + C = x$ has a solution in $\overline{B_e}(0)$.

Corollary 12. *Let $s \in Y$ and $e > 0$. Let $A : Y \longrightarrow Y$, $B : \overline{B_e}(s) \longrightarrow Y$, and $C \in Y$ be such that:*

(i) A is Lipschitzian with constant μ.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(s)} \|Br\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - (AsBr + C)\| \le (1 - \mu U)e. \tag{27}$$

Then the operator equation $AxBx + C = x$ has a solution in $\overline{B_e}(s)$.

The next theorem addresses Equation (8).

Theorem 4. *Let $s \in Y$, $e > 0$, and $\rho_2, \rho_3 \in (0, 1]$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be two operators such that:*

(i) A is Hölderian with constant μ and exponent ρ_2.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(s)} \|Br^{\rho_3}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - (As^{\rho_2} Br^{\rho_3})\| \le (1 - \mu U)e. \tag{28}$$

Then the operator equation $Ax^{\rho_2} Bx^{\rho_3} = x$ has a solution in $\overline{B_e}(s)$.

Proof. This is a special case of Proposition 2 with $C = 0$. The proof is identical in structure. □

The following corollaries specialize this result.

Corollary 13. *Let $s \in Y$, $e > 0$, and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be two operators such that:*

(i) A is Hölderian with constant μ and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r \in \overline{B_e}(s)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - As^{\rho} Br^{\rho}\| \le (1 - \mu U)e. \tag{29}$$

Then the operator equation $Ax^{\rho} Bx^{\rho} = x$ has a solution in $\overline{B_e}(s)$.

Corollary 14. *Let $e > 0$ and $\rho_2, \rho_3 \in (0, 1]$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(0) \longrightarrow Y$ be two operators such that:*

(i) A is Hölderian with constant μ and exponent ρ_2.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^{\rho_3}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U < 1$:

$$\|-A(0)Br^{\rho_3}\| \leq (1-\mu U)e. \tag{30}$$

Then the operator equation $Ax^{\rho_2}Bx^{\rho_3} = x$ has a solution in $\overline{B_e}(0)$.

Corollary 15. *Let $e > 0$ and $\rho \in (0,1]$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(0) \longrightarrow Y$ be two operators such that:*

(i) A is Hölderian with constant μ and exponent ρ.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(0)} \|Br^{\rho}\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(0)$ with $\mu U < 1$:

$$\|-A(0)Br^{\rho}\| \leq (1-\mu U)e. \tag{31}$$

Then the operator equation $Ax^{\rho}Bx^{\rho} = x$ has a solution in $\overline{B_e}(0)$.

Corollary 16. *Let $s \in Y$ and $e > 0$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be two operators such that:*

(i) A is Lipschitzian with constant μ.

(ii) B is completely continuous with $U = \sup_{r\in\overline{B_e}(s)} \|Br\|$.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$ with $\mu U < 1$:

$$\|s - AsBr\| \leq (1-\mu U)e. \tag{32}$$

Then the operator equation $AxBx = x$ has a solution in $\overline{B_e}(s)$.

Remark 1. *Taking $s = 0$ in the preceding corollary recovers a result similar to Theorem 3.4 in Ref.[20]] Taking $A \equiv 0$ or $B \equiv 0$ leads to results related to Theorem 2.1 in Ref.[22] and Theorem 3.3 in Ref.[20]*

The next theorem addresses Equation (9).

Theorem 5. *Let $s \in Y$, $e > 0$, and $\rho_2, \rho_3 \in (0,1]$. Let $A : Y \longrightarrow Y$ be an operator and $B : \overline{B_e}(s) \longrightarrow Y$ be a completely continuous operator, such that:*

(i) A is Hölderian with constant μ and exponent ρ_2, with $\mu < 1$.

(ii) The following inequality holds for each $r \in \overline{B_e}(s)$:

$$\|s - (As^{\rho_2} + Br^{\rho_3})\| \leq (1-\mu)e. \tag{33}$$

Then the operator equation $Ax^{\rho_2} + Bx^{\rho_3} = x$ has a solution in $\overline{B_e}(s)$.

Proof. Let $r \in \overline{B_e}(s)$ be fixed and define $A_r(x) = Ax^{\rho_2} + Br^{\rho_3}$. We show A_r is a contraction on $\overline{B_e}(s)$. Let $x_1, x_2 \in \overline{B_e}(s)$.

$$\begin{aligned}\|A_r(x_1) - A_r(x_2)\| &= \|Ax_1^{\rho_2} - Ax_2^{\rho_2}\| \\ &\leq \mu \|x_1 - x_2\|^{\rho_2}.\end{aligned}$$

Since $0 < \mu < 1$, A_r is a contraction. By Hypothesis (2), $\|s - A_r(s)\| \leq (1-\mu)e$. By Theorem 1, there is a unique point $\widetilde{x}_r \in \overline{B_e}(s)$ such that $A\widetilde{x}_r^{\rho_2} + Br^{\rho_3} = \widetilde{x}_r$. Define an operator $K : \overline{B_e}(s) \longrightarrow \overline{B_e}(s)$ by $K(r) = \widetilde{x}_r$. To show K is continuous, let $r_n \to r$. Let $x_n = Kr_n$ and $x = Kr$.

$$\begin{aligned}\|x_n - x\| &= \|(Ax_n^{\rho_2} + Br_n^{\rho_3}) - (Ax^{\rho_2} + Br^{\rho_3})\| \\ &\leq \|Ax_n^{\rho_2} - Ax^{\rho_2}\| + \|Br_n^{\rho_3} - Br^{\rho_3}\| \\ &\leq \mu \|x_n - x\|^{\rho_2} + \|Br_n^{\rho_3} - Br^{\rho_3}\|.\end{aligned}$$

Thus, $(1-\mu)\|x_n - x\| \leq \|Br_n^{\rho_3} - Br^{\rho_3}\|$. Since B is continuous, $\|x_n - x\| \to 0$. Thus, K is continuous. To show K is compact, let $\{r_n\}$ be a sequence in $\overline{B_e}(s)$. Since B is completely continuous, there is a subsequence, also denoted $\{r_n\}$, such that $\{Br_n^{\rho_3}\}$ converges. The relation $x_n = Ax_n^{\rho_2} + Br_n^{\rho_3}$ shows that $\{x_n\}$ must lie in a compact set, so a subsequence of $\{x_n = Kr_n\}$ converges. Hence, K is a compact operator. By Schauder's fixed point theorem, K has a fixed point. ☐

Corollary 17. *Let $s \in Y$, $e > 0$, and $\rho \in (0,1]$. Let $A : Y \longrightarrow Y$ and $B : \overline{B_e}(s) \longrightarrow Y$ be operators such that:*

(i) A is Hölderian with constant $\mu < 1$ and exponent ρ.

(ii) B is completely continuous.

(iii) The following inequality holds for each $r \in \overline{B_e}(s)$:

$$\|s - (As^{\rho} + Br^{\rho})\| \leq (1-\mu)e. \tag{34}$$

Then the operator equation $Ax^{\rho} + Bx^{\rho} = x$ has a solution in $\overline{B_e}(s)$.

Corollary 18. *Let $e > 0$ and $\rho_2, \rho_3 \in (0,1]$. Let $A : Y \longrightarrow Y$ be Hölderian with constant $\mu < 1$ and exponent ρ_2, and let $B : \overline{B_e}(0) \longrightarrow Y$ be completely continuous. If the following inequality holds for each $r \in \overline{B_e}(0)$:*

$$\|-A(0) - Br^{\rho_3}\| \leq (1-\mu)e, \tag{35}$$

then the operator equation $Ax^{\rho_2} + Bx^{\rho_3} = x$ has a solution in $\overline{B_e}(0)$.

Corollary 19. *Let $e > 0$ and $\rho \in (0,1]$. Let $A : Y \longrightarrow Y$ be Hölderian with constant $\mu < 1$ and exponent ρ, and let $B : \overline{B_e}(0) \longrightarrow Y$ be completely continuous. If the following inequality holds for each $r \in \overline{B_e}(0)$:*

$$\|-A(0) - Br^{\rho}\| \leq (1-\mu)e, \tag{36}$$

then the operator equation $Ax^\rho + Bx^\rho = x$ has a solution in $\overline{B_e}(0)$.

Corollary 20. *Let $s \in Y$ and $e > 0$. Let $A : Y \longrightarrow Y$ be Lipschitzian with constant $\mu < 1$, and let $B : \overline{B_e}(s) \longrightarrow Y$ be completely continuous. If the following inequality holds for each $r \in \overline{B_e}(s)$:*

$$\|s - (As + Br)\| \leq (1 - \mu)e, \tag{37}$$

then the operator equation $Ax + Bx = x$ has a solution in $\overline{B_e}(s)$.

Remark 2. *Setting $s = 0$ in the preceding corollary leads to a theorem similar to one in.[21]*

Proposition 3. *Let $s \in Y$, $e > 0$, and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$ be an operator and $B \in Y$ be an element such that:*

(i) A is Hölderian with constant $\mu \in (0, 1)$ and exponent ρ.

(ii) The following inequality holds:

$$\|s - (As^\rho + B)\| \leq (1 - \mu)e. \tag{38}$$

Then the operator Equation (10), $Ax^\rho + B = x$, has a solution in $\overline{B_e}(s)$.

Proof. Define the operator $T(x) = Ax^\rho + B$. This operator does not depend on a parameter r. For $x_1, x_2 \in \overline{B_e}(s)$, we have

$$\|T(x_1) - T(x_2)\| = \|Ax_1^\rho - Ax_2^\rho\| \leq \mu \|x_1 - x_2\|^\rho .$$

Since $\mu < 1$, T is a contraction mapping from $\overline{B_e}(s)$ to Y. Condition (b) is $\|s - T(s)\| \leq (1 - \mu)e$. By Theorem 1, which does not require a self-mapping, T has a unique fixed point in $\overline{B_e}(s)$. ☐

Corollary 21. *Let $e > 0$ and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$ be an operator and $B \in Y$ such that:*

(i) A is Hölderian with constant $\mu \in (0, 1)$ and exponent ρ.

(ii) The following inequality holds:

$$\|-A(0) - B\| \leq (1 - \mu)e. \tag{39}$$

Then the operator Equation (10) has a solution in $\overline{B_e}(0)$.

Corollary 22. *Let $s \in Y$ and $e > 0$. Let $A : Y \longrightarrow Y$ be a Lipschitzian operator with constant $\mu \in (0, 1)$ and let $B \in Y$. If*

$$\|s - (As + B)\| \leq (1 - \mu)e, \tag{40}$$

then the operator equation $Ax + B = x$ has a solution in $\overline{B_e}(s)$.

Corollary 23. *Let $e > 0$. Let $A : Y \longrightarrow Y$ be a Lipschitzian operator with constant $\mu \in (0, 1)$ and let $B \in Y$. If*

$$\|-A(0) - B\| \leq (1 - \mu)e, \tag{41}$$

then the operator equation $Ax + B = x$ has a solution in $\overline{B_e}(0)$.

Setting $B = 0$ in Proposition 2.25 gives the following theorem for Equation (11).

Theorem 6. *Let $s \in Y$, $e > 0$, and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$ be an operator such that:*

(i) A is Hölderian with Hölder constant $\mu \in (0, 1)$ and exponent ρ.

(ii) The following inequality holds:

$$\|s - As^\rho\| \leq (1 - \mu)e. \tag{42}$$

Then the operator Equation (11), $Ax^\rho = x$, has a solution in $\overline{B_e}(s)$.

Proof. The operator $T(x) = Ax^\rho$ is a contraction on $\overline{B_e}(s)$ since $\|T(x_1) - T(x_2)\| \leq \mu\|x_1 - x_2\|^\rho$. The result follows directly from Theorem 1. ☐

Corollary 24. *Let $e > 0$ and $\rho \in (0, 1]$. Let $A : Y \longrightarrow Y$ be an operator such that A is Hölderian with constant $\mu \in (0, 1)$ and*

$$\|-A(0)\| \leq (1 - \mu)e. \tag{43}$$

Then, the operator equation $Ax^\rho = x$ has a solution in $\overline{B_e}(0)$.

Corollary 25. *Let $s \in Y$ and $e > 0$. Let $A : Y \longrightarrow Y$ be a Lipschitzian operator with constant $\mu \in (0, 1)$ such that*

$$\|s - As\| \leq (1 - \mu)e. \tag{44}$$

Then the operator equation $Ax = x$ has a solution in $\overline{B_e}(s)$.

Corollary 26. *Let $e > 0$. Let $A : Y \longrightarrow Y$ be a Lipschitzian operator with constant $\mu \in (0, 1)$ such that*

$$\|-A(0)\| \leq (1 - \mu)e. \tag{45}$$

Then the operator equation $Ax = x$ has a solution in $\overline{B_e}(0)$.

2.2. Nonlinear integral equations

Integral equations play a pivotal role in both theoretical and applied mathematics, serving as a foundation for modeling phenomena across physics, engineering, and differential systems. For instance, Mennouni[29] proposed has contributed significantly to this field through various analytical and numerical approaches. He developed piecewise constant Galerkin methods to solve Cauchy singular integral equations, ensuring convergence and stability. In, he proposed a new application of projection methods by introducing two projection techniques for solving skew-Hermitian operator equations with singular kernels. These contributions highlight the depth and practical relevance of integral equation theory in solving complex mathematical problems. This section illustrates the theory by applying it to a nonlinear functional integral equation, establishing existence results following the approach in the studies by Xu S and Radenovi S[30] and Arab R, et al.[31]

Let $I = [0, 1] \subset \mathbb{R}$. We seek a solution to the nonlinear functional integral equation

$$x(t) = g(t, x(\mu(t))) \left(q(t) + \int_0^{\sigma(t)} h(\tau, x(\eta(\tau)))d\tau \right) \tag{46}$$

$$+ k(t, x(\theta(t))) \left(p(t) + \int_0^{\sigma(t)} h(\tau, x(\eta(\tau)))d\tau \right) \tag{47}$$

for $t \in I$, where $p, q : I \longrightarrow \mathbb{R}$, $\mu, \theta, \sigma, \eta : I \longrightarrow I$ and $g, k, h : I \times \mathbb{R} \longrightarrow \mathbb{R}$.

We seek a solution in the space $C(I, \mathbb{R})$ of continuous real-valued functions on I, which is a Banach algebra with the supremum norm $\|x\|_C = \max_{t \in I} |x(t)|$.

Definition 3. *A function $\Psi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is called L^1-Carathéodory if:*

(i) $t \longmapsto \Psi(t, x)$ is measurable for each $x \in \mathbb{R}$.
(ii) $x \longmapsto \Psi(t, x)$ is continuous for almost every $t \in I$.
(iii) For each real number $e > 0$, there exists a function $\phi_e \in L^1(I, \mathbb{R}^+)$ such that $|\Psi(t, x)| \le \phi_e(t)$ for a.e. $t \in I$ and for all $x \in \mathbb{R}$ with $|x| \le e$.

We consider the following set of assumptions:

(i) (Hypothesis 1) The functions $\mu, \theta, \sigma, \eta : I \longrightarrow I$ are continuous.
(ii) (Hypothesis 2) The functions $p, q : I \longrightarrow \mathbb{R}$ are continuous.
(iii) (Hypothesis 3) The function $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and there exists a function $\mu_1 \in C(I, \mathbb{R}^+)$ with bound $\|\mu_1\|_C$ such that for all $x, y \in \mathbb{R}$ and $t \in I$, $|g(t, x) - g(t, y)| \le \mu_1(t)|x - y|$.
(iv) (Hypothesis 4) The function $k : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and there exists a function $\nu_1 \in C(I, \mathbb{R}^+)$ with bound $\|\nu_1\|_C$ such that for all $x, y \in \mathbb{R}$ and $t \in I$, $|k(t, x) - k(t, y)| \le \nu_1(t)|x - y|$.
(v) (Hypothesis 5) The function h is L^1-Carathéodory.

Theorem 7. *Assume that Hypothesis (1)–(5) hold. Further, if there exists a real number $e > 0$ and a function $\phi_e \in L^1(I, \mathbb{R}^+)$ from Hypothesis (5) such that*

$$\|\mu_1\|_C \left(\|q\|_C + \|\phi_e\|_{L^1}\right) + \|\nu_1\|_C \left(\|p\|_C + \|\phi_e\|_{L^1}\right) < 1 \tag{48}$$

and

$$\frac{G\left(\|q\|_C + \|\phi_e\|_{L^1}\right) + K\left(\|p\|_C + \|\phi_e\|_{L^1}\right)}{1 - \left[\|\mu_1\|_C\left(\|q\|_C + \|\phi_e\|_{L^1}\right) + \|\nu_1\|_C\left(\|p\|_C + \|\phi_e\|_{L^1}\right)\right]} \le e, \tag{49}$$

where $G = \sup\{|g(t, 0)| : t \in I\}$ and $K = \sup\{|k(t, 0)| : t \in I\}$. Then Equation (47) has a solution $x \in C(I, \mathbb{R})$ with $\|x\|_C \le e$.

Proof. Consider the closed ball $\overline{B}_e(0)$ in $C(I, \mathbb{R})$. Define four operators A, B, C, D on $C(I, \mathbb{R})$ by:

$$(Ax)(t) = g(t, x(\mu(t))),$$
$$(Bx)(t) = q(t) + \int_0^{\sigma(t)} h(\tau, x(\eta(\tau)))d\tau,$$
$$(Cx)(t) = k(t, x(\theta(t))),$$
$$(Dx)(t) = p(t) + \int_0^{\sigma(t)} h(\tau, x(\eta(\tau)))d\tau.$$

The integral Equation (47) is equivalent to the operator equation $AxBx + CxDx = x$. We shall show that the operators satisfy the conditions of Corollary 4.

First, we show that A is Lipschitzian. For any $x, y \in C(I, \mathbb{R})$:

$$\begin{aligned} |Ax(t) - Ay(t)| &= |g(t, x(\mu(t))) - g(t, y(\mu(t)))| \\ &\le \mu_1(t)|x(\mu(t)) - y(\mu(t))| \\ &\le \|\mu_1\|_C \|x - y\|_C. \end{aligned}$$

Taking the supremum over t, we get $\|Ax - Ay\|_C \le \|\mu_1\|_C \|x - y\|_C$. Thus, A is Lipschitzian with constant $\|\mu_1\|_C$. Similarly, C is Lipschitzian with constant $\|\nu_1\|_C$.

Next, we show that B and D are completely continuous on $\overline{B}_e(0)$. For any $x \in \overline{B_e}(0)$, the operators map to $C(I, \mathbb{R})$ due to the continuity of the terms and the absolute continuity of the integral. For $x \in \overline{B_e}(0)$, we have $|(Bx)(t)| \le \|q\|_C + \int_0^1 |h(\tau, x(\eta(\tau)))|d\tau \le \|q\|_C + \|\phi_e\|_{L^1}$. This shows that $B(\overline{B_e}(0))$ is uniformly bounded. For equicontinuity, consider $t_1, t_2 \in I$:

$$\begin{aligned} |(Bx)(t_1) - (Bx)(t_2)| &\le |q(t_1) - q(t_2)| \\ &\quad + \left| \int_{\sigma(t_2)}^{\sigma(t_1)} \phi_e(\tau)d\tau \right|. \end{aligned}$$

Since q is uniformly continuous and $\int_0^t \phi_e(\tau)d\tau$ is absolutely continuous, the set $B(\overline{B_e}(0))$ is equicontinuous. By the Arzelà-Ascoli theorem, $B(\overline{B_e}(0))$ is relatively compact. Since B is also continuous (by the dominated convergence theorem), it is completely continuous. A similar argument holds for D.

Let $U = \sup_{r \in \overline{B_e}(0)} \|Br\|_C \le \|q\|_C + \|\phi_e\|_{L^1}$ and $V = \sup_{r \in \overline{B_e}(0)} \|Dr\|_C \le \|p\|_C + \|\phi_e\|_{L^1}$. Condition (48) ensures that $\|\mu_1\|_C U + \|\nu_1\|_C V < 1$.

Finally, for any $r \in \overline{B}_e(0)$, we check the condition for Corollary 4:

$$\begin{aligned}\|A(0)Br + C(0)Dr\|_C &\le \|A(0)\|_C \|Br\|_C \\ &+ \|C(0)\|_C \|Dr\|_C \\ &\le G \cdot U + K \cdot V \\ &\le G(\|q\|_C + \|\phi_e\|_{L^1}) \\ &+ K(\|p\|_C + \|\phi_e\|_{L^1}).\end{aligned}$$

By Inequality (49), this is less than or equal to $(1 - (\|\mu_1\|_C U + \|\nu_1\|_C V))e$. All hypotheses of Corollary 4 are satisfied. Thus, Equation (47) has a solution in $\overline{B}_e(0)$. □

Example 1. *Let $Y = C[0,1]$ be the Banach algebra of continuous functions on $[0,1]$ with the supremum norm. We consider the operator equation*

$$A(f)B(f) + C(f)D(f) = f,$$

where the operators $A, B, C, D : C[0,1] \longrightarrow C[0,1]$ are defined as:

$$Af(t) = \frac{1}{4}f(t) + \frac{1}{8}$$

which is Lipschitz with constant $\mu = \dfrac{1}{4}$,

$$Cf(t) = \frac{1}{5}f(t) - \frac{1}{10}$$

which is Lipschitz with constant $\nu = \dfrac{1}{5}$,

$$Bf(t) = \int_0^1 \frac{t+s}{2} f(s)ds$$

which is a compact integral operator,

$$Df(t) = \int_0^1 \frac{\sin(\pi ts)}{3} f(s)ds$$

which is a compact integral operator.

We verify the conditions of Proposition 1(or its corollary, Case 4, for $s = 0$). The operators A and C are affine and thus Lipschitz. The operators B and D are integral operators with continuous kernels on a compact domain, so they are compact operators on $C[0,1]$.

Let's find bounds for the operators on a ball $\overline{B_e}(0)$, i.e., for $\|f\| \le e$. For B, we have

$$\begin{aligned}|Bf(t)| &\le \int_0^1 \frac{|t+s|}{2}|f(s)|ds \\ &\le \int_0^1 \frac{1+1}{2}\|f\|ds = \|f\|.\end{aligned}$$

Thus, $\|Bf\| \le \|f\|$. On the ball $\overline{B_e}(0)$, the bound is $U = \sup_{\|f\| \le e} \|Bf\| \le e$. For D, we have

$$\begin{aligned}|Df(t)| &\le \int_0^1 \frac{|\sin(\pi ts)|}{3}|f(s)|ds \\ &\le \int_0^1 \frac{1}{3}\|f\|ds = \frac{1}{3}\|f\|.\end{aligned}$$

Thus, $\|Df\| \le \frac{1}{3}\|f\|$. On the ball $\overline{B_e}(0)$, the bound is $V = \sup_{\|f\| \le e} \|Df\| \le \frac{e}{3}$.

The contraction condition is $\mu U + \nu V < 1$. Using our bounds, we need $\frac{1}{4}e + \frac{1}{5}\frac{e}{3} < 1$, which implies $\frac{e}{4} + \frac{e}{15} < 1$, so $\frac{19e}{60} < 1$. This is true for $e < 60/19 \approx 3.15$.

Let's choose $e = 1$. Then we can take $U = 1$ and $V = 1/3$. The condition is $\frac{1}{4}(1) + \frac{1}{5}(\frac{1}{3}) = \frac{1}{4} + \frac{1}{15} = \frac{19}{60} < 1$, which is satisfied. Now we check the third condition of Corollary 4 for $s = 0$ and any $r \in \overline{B_1}(0)$:

$$\begin{aligned}\|A(0)Br + C(0)Dr\| &\le \|A(0)\|\|Br\| + \|C(0)\|\|Dr\| \\ &= \left|\frac{1}{8}\right| \|Br\| + \left|-\frac{1}{10}\right| \|Dr\| \\ &\le \frac{1}{8}U + \frac{1}{10}V \le \frac{1}{8} + \frac{1}{10}\left(\frac{1}{3}\right) \\ &= \frac{1}{8} + \frac{1}{30} = \frac{19}{120}.\end{aligned}$$

We need to satisfy

$\|A(0)Br + C(0)Dr\| \le (1 - (\mu U + \nu V))e$. *With $e = 1$, this is*

$$\frac{19}{120} \le 1 - \left(\frac{1}{4}(1) + \frac{1}{5}\left(\frac{1}{3}\right)\right) = 1 - \frac{19}{60} = \frac{41}{60}.$$

Since $\frac{19}{120} \le \frac{82}{120} (= \frac{41}{60})$, the condition holds. All hypotheses of Corollary 4 are satisfied for $e = 1$. Therefore, the operator equation has a solution in the unit ball of $C[0,1]$.

3. Conclusion

In this study, we have successfully extended the framework of Krasnoselskii-type fixed point theorems to address a broad class of nonlinear operator equations within a Banach algebra setting. Our primary objective was to establish local existence results for solutions by refining and clarifying the hypotheses for theorems involving multiple operators.

The core of our methodology involved a two-step hybrid fixed-point argument. We first established the existence of a unique solution for a parameterized family of contraction mappings within a specified closed ball. Subsequently, by demonstrating the continuity and compactness of the operator mapping the parameter to this unique solution, we applied Schauder's fixed-point theorem to guarantee the existence of a solution to the original operator equation. This approach was systematically applied to derive a comprehensive suite of theorems and corollaries for various operator equations, including general forms such as $Ax^{\rho_2}Bx^{\rho_1} + Cx^{\rho_3}Dx^{\rho_1} = x$ and several important special cases.

This study contributes in three key aspects. First, we have generalized existing results by considering a more complex structure with up to four operators and incorporating Hcontinuity, which provides greater flexibility than standard Lipschitz conditions. Second, our results are fundamentally local, providing conditions for existence within a specific domain, which can be less restrictive and more practical for applications. Third, we have unified the analysis of a wide range of operator equations (Equations [4]–[11]) under a single theoretical framework. The applicability of our findings was demonstrated through a detailed analysis of a nonlinear functional integral equation and a concrete example in $C[0,1]$, bridging the gap between theory and practice. The results presented in this work can also be applied to the problems considered in.[32,33]

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Conflict of interest

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