An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.16, No.2, pp.555-565 (2026)
https://doi.org/10.36922/IJOCTA025100043

RESEARCH ARTICLE

Borzdyko's uniqueness theorems for fractal-fractional ordinary differential equations with power-law kernels and hysteresis

Abdon Atangana[1,2*] and Sonal Jain[3]

[1]Institute for Groundwater Studies (IGS) Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein, South Africa
[2]Department of Medical Research, China Medical University Hospital, Taichung, Taiwan
[3]Faculty of Data Science and Analytics, Sir Padampat Singhania University, Udaipur, Rajasthan, India
atanganaa@ufs.ac.za, sonaljainmaths@gmail.com

ARTICLE INFO

Article History:
Received: March 4, 2025
1st revised: July 4, 2025
2nd revised: November 5, 2025
Accepted: December 9, 2025
Published Online: January 19, 2026

Keywords:
Power law
Fractal-fractional operators
Uniqueness
Borzdyko's conditions

AMS Classification 2010:
90B23, 90B56

ABSTRACT

Fractal–fractional differential equations have emerged as a powerful mathematical framework for modeling complex systems exhibiting memory effects, nonlocality, and hysteresis phenomena. This study investigates a class of fractal-fractional ordinary differential equations characterized by a power-law memory kernel and influenced by hysteresis behavior. The continuity of the function $g(t, w(t))$ over closed subsets of $\mathbb{R}$, is used to establish the foundational results. A supporting lemma is introduced to facilitate the development of a uniqueness theorem. Drawing upon Borzdyko's framework, we derive existence results pertinent to the targeted family of equations.

1. Introduction

Fractal-fractional differential operators, introduced in earlier studies, extend the notion of fractional calculus to settings that model transport and dynamics in media with irregular or fractal-like structure. These operators and applications have been discussed widely in the literature. In particular, anomalous diffusion and related phenomena have been modeled using combinations of fractal and fractional operators and extensions thereof. A number of previous studies have reported representative developments and discussions on this topic.[1–4]

Fractal-fractional operators incorporate several memory kernels, including power-law, exponential-decay, and Mittag–Leffler types, which allowing flexible representation of nonlocal and history-dependent effects in constitutive relations and evolution equations. Although the term "fractal" is used, these operators do not attempt to reproduce classical fractals such as Julia or Mandelbrot sets; rather, they provide effective tools to account for dynamics that arise in media exhibiting fractal geometrical or scaling properties.[2,3]

These operators have motivated new formulations of both ordinary and partial differential equations that capture memory, hereditary effects, and anomalous transport. Such equations have been shown to offer improved modeling capacity for complex real-world phenomena that are not adequately described by integer-order models. Representative numerical and analytical developments include recent work on numerical approximations and applications of fractal-fractional differential equations.[5–7]

When exact analytical solutions are unavailable, the mathematical foundation of existence and uniqueness is essential to ensure the well-posedness of models and the meaningfulness of numerical approximations. Classical uniqueness criteria for ordinary differential equations have been augmented by several alternative conditions

*Corresponding Author

that relax Lipschitz or contraction requirements; among these, the criteria developed by Borzdyko (and discussed in standard references) have been influential.[8]

In this paper, we consider a general class of fractal-fractional ordinary differential equations with a power-law memory kernel augmented by a hysteresis term. Building on Borzdyko's framework and related techniques, we derive conditions under which uniqueness of solutions can be established for the considered family of equations. Finally, to place our discussion in historical context, we recall classical works on iteration theory and fractal geometry that helped motivate modern investigations of complex dynamical structures.[9,10]

2. Background

As mentioned in Section 1, there are three different fractal-fractional differential operators; however, in this work, we shall only consider the following fractal-fractional power law differential equation with an hysteresis component p:

$$\begin{aligned} &{}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}w(t) = f(t,w(t),p(t)) \quad \text{if} \quad t>\tau_0\neq 0\\ &w(\tau_0)=w_0 \quad \text{if} \quad t=\tau_0\\ &p(t)=\omega(\tau_0,w_0,p_0)w(t)\\ &w(\tau_0)=w_0,\ p(\tau_0)=p_0 \end{aligned} \tag{1}$$

where $\eta,\epsilon\in(0,1]$, $w(t)$ and $p(t)$ are continuous real functions in $[\tau_0+a]$. By fixing τ_0, w_0, and p_0, the defined operator, $\omega(\tau_0,w_0,p_0)$, is known as a single value on $[\tau_0,\tau_0+a]$ and maps each function $w(t)$ continuous in $[\tau_0,\tau_0+a]$ such that $w(\tau_0)=w_0$ into a function $p(t)\in C[\tau_0,\tau_0+a]$, for which $p(\tau_0)=p_0$. Additionally, the function $\omega(\tau_0,w_0,p_0)$ satisfies the following Lipschitz condition:

$$\left|\omega(\tau_0,w_0,p_0)w(t)-\omega(\tau_0,w_0,p_0)\bar{w(t)}\right|<L \max_{t\in[\tau_0,\tau_0+a]}\left|w(t)-\bar{w(t)}\right| \tag{2}$$

Add indentation note that if $w(t)$ is continuous in $[\tau_0,\tau_0+a]$,

$${}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}w(t)=\frac{1}{\Gamma(1-\eta)}\frac{d}{dt^{\epsilon}}\int_{\tau_0}^{t}(t-s)^{-\eta}w(s)ds \tag{3}$$

where

$$\frac{d}{dt^{\epsilon}}z(t)=\lim_{t_1\to t}\frac{z(t_1-z(t))}{t_1^{\epsilon}-t^{\epsilon}} \tag{4}$$

with $\eta,\epsilon\in(0,1]$. We note that

$$\frac{d}{dt^{\epsilon}}z(t)=\frac{z'(t)}{\epsilon t^{\epsilon-1}} \tag{5}$$

when $z(t)$ is differentiable. The integral associated τ_0, ${}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}$ is defined by

$${}^{FFP}_{\tau_0}J_t^{\eta,\epsilon}m(t)=\frac{\epsilon}{\Gamma(\eta)}\int_{\tau_0}^{t}s^{\epsilon-1}(t-s)^{\eta-)}m(s)ds \tag{6}$$

In fact,

$$\begin{aligned} {}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}w(t)&=\frac{1}{\Gamma(1-\eta)}\frac{d}{dt^{\epsilon}}\int_{\tau_0}^{t}(t-s)^{-\eta}w(s)ds\\ &=\frac{1}{\epsilon t^{\epsilon-1}\Gamma(1-\eta)}\frac{d}{dt}\int_{\tau_0}^{t}(t-s)^{-\eta}w(s)ds \end{aligned} \tag{7}$$

we define the following interval:

$$\begin{aligned} \Omega=\{(t,w,p)|\tau_0\le t\le\tau_0+a,|w-w_0|\le l,|p-p_0|\\ \le c,\quad a\le\left(\frac{l\Gamma(\eta+1)}{M\epsilon\tau_0^{\epsilon-1}}\right)^{\frac{1}{\eta}} \end{aligned} \tag{8}$$

3. Results

We shall recall that the space of continuous functions is larger than that of functions that satisfy the Lipschitz and contraction conditions, that is the reason why several authors have devoted their efforts in the last few decades to introducing new conditions that allow a larger class of functions. This is the case with the work by Borzdyko. In this section, owing to the larger applicability of fractal-fractional power law ordinary differential equations, we should use Borzdyko's conditions to establish the uniqueness of the solutions to these equations.

Theorem 1. *Assuming that the function $g(t,w,p)$ is continuous, then $\forall t\in[\tau_0,\tau_0+a]$, we have that $(t,w,p)\in\Omega$.*

Proof. If $t\in[\tau_0,\tau_0+a]$, we shall have that

$$w(t)=\frac{\epsilon}{\Gamma(\eta)}\int_{\tau_0}^{t}s^{\epsilon-1}(t-s)^{\eta-1}g(s,w,p)ds \tag{9}$$

$$p(t)=\omega(\tau_0,w_0,p_0)w(t),\quad p(\tau_0)=p_0,\quad w(\tau_0)=w_0 \tag{10}$$

$$\begin{aligned} |w-w_0|&=\left|\frac{\epsilon}{\Gamma(\eta)}\int_{\tau_0}^{t}s^{\epsilon-1}(t-s)^{\eta-1}g(s,w,p)ds-w_0\right|\\ &\le|w_0|+\frac{\epsilon}{\Gamma(\eta)}\int_{\tau_0}^{t}s^{\epsilon-1}(t-s)^{\eta-1}\left|g(s,w,p)\right|ds \end{aligned} \tag{11}$$

since the function $g(t,w,p)$ is continuous, we have that $\exists t\in[\tau_0,\tau_0+a]$, such that $\forall t\in[\tau_0,\tau_0+a],|g(\bar{t}),w(\bar{t}),p(\bar{t})|>|g(t,w(t),p(t))|$. We put $M=|(g(\bar{t}),w(\bar{t}),p(\bar{t}))|$, then

$$|w-w_0|\le|w_0|+\frac{\epsilon M}{\Gamma(\eta)}\int_{\tau_0}^{t}s^{\epsilon-1}(t-s)^{\eta-1}ds \tag{12}$$

we shall note that $M[\tau_0, \tau_0 + a]$, the function $t^{\epsilon-1}$ is non-increasing, therefore

$$\begin{aligned} |w - w_0| &\le |w_0| + \frac{\epsilon M}{\Gamma(\eta)}\tau_0^{\epsilon-1}\frac{(t-\tau_0)^\eta}{\eta} \\ &\le |w_0| + \frac{\epsilon M \tau_0^{\epsilon-1}}{\Gamma(\eta+1)}a^\eta \end{aligned} \tag{13}$$

$$|y| \le \frac{\epsilon M \tau_0^{\epsilon-1}}{\Gamma(\eta+1)}a^\eta \le l$$

$$a \le \left(\frac{l\Gamma(\eta+1)}{\epsilon M \tau_0^{\epsilon-1}}\right)^{\frac{1}{\eta}} \tag{14}$$

$$\implies |y| \le l$$

$$\begin{aligned} |p(t) - p_0| &= |\Omega(\tau_0, w_0, p_0)w(t) - \omega(\tau_0, w_0, p_0)w_0| \\ &\le L \max_{t\in[\tau_0,\tau_0+a]} |w(t) - w_0| \\ &= L(l + |w_0|) \end{aligned} \tag{15}$$

we shall need $L \le \frac{c}{l+|w_0|}$ then $|p(t) - p_0| \le c$. Thus $(t, w(t), p(t)) \in \Omega$ note the condition that $(l + |w_0|)L \le c$ and $a \le \left|\frac{l\Gamma(\eta+1)}{\epsilon M \tau_0^{\epsilon-1}}\right|^{\frac{1}{\epsilon}}$.

Under the same condition, we want to show that $w(t)$ is continuously differentiable and that $p(t)$ is continuous in $[\tau_0, \tau_0 + a]$.

$$w'(t) = \frac{d}{dt}\left[\int_{\tau_0}^t \frac{\epsilon s^{\epsilon-1}}{\Gamma(\eta)}(t-s)^{\eta-1}g(s, w(s), p(s))ds\right] \tag{16}$$

$$F(s, w(s), p(s)) = \frac{\epsilon s^{\epsilon-1}}{\Gamma(\eta)}g(s, w(s), p(s))$$

Indeed the function $g(t, w(t))$ is continuous, $\eta, \epsilon \in (0, 1), \tau_0 \ne 0$. For simplicity we put $F(t, w(t)) = \frac{\epsilon t^{\epsilon-1}}{\Gamma(\eta)}g(t, w(t))$, we then have

$$w'(t) = \frac{d}{dt}\int_{\tau_0}^t F(s, w(s), p(s))(t-s)^{\eta-1}ds \tag{17}$$

Using the derivative of a convolution, we obtain

$$\begin{aligned} w'(t) =& (\eta - 1)\int_{\tau_0}^t F(s, w(s), p(s))(t-s)^{\eta-2}ds \\ =& \frac{(\eta-1)\epsilon}{\Gamma(\eta)}\int_{\tau_0}^t s^{\epsilon-1}(t-s)^{\eta-2}g(s, w(s), p(s))ds \end{aligned} \tag{18}$$

$w'(t)$ is continuous in $[\tau_0, \tau_0 + a]$. Therefore $w(t)$ is continuously differentiable in $[\tau_0, \tau_0 + a]$.
Since $w(t)$ is continuous and, by definition, $p(t)$ atisfies a Lipschitz condition. Therefore $p(t)$ is continuous in $[\tau_0, \tau_0 + a]$.

We shall show the first inequality.

Lemma 1. *Let $\phi(t)$ be absolutely continuous in $[\tau_0, \tau_0 + a]$, such that $\phi(\tau_0) = 0$ and non-decreasing. Let $\psi(\bar{t}) = \max_{\tau_0 \le t \le \bar{t}} |\phi(t)|$, which is non-decreasing and absolutely continuous. In addition, Let $\phi'(t)$ and $\psi'(t)$ exist in any point $t \in [\tau_0, \tau_0 + a]$ then*

$$0 \le_{\tau_0}^{FFP} D_t^{\eta,\epsilon}\psi(t) \le_{\tau_0}^{FFP} D_t^{\eta,\epsilon}|\phi(t)| \tag{19}$$

Proof. since $\psi(t)$ is non-decreasing, we have that $\forall t \in [\tau_0, \tau_0 + a]$

$$\begin{aligned} &0 \le \psi'(t) \\ &0 \le \int_{\tau_0}^t \frac{\psi'(s)(t-s)^{-\eta}}{\Gamma(1-\eta)}ds, \quad \tau_0 \le s < t, \eta \in (0, 1] \\ &\le \frac{d}{dt}\int_{\tau_0}^t \frac{\psi(s)(t-s)^{-\eta}}{\Gamma(1-\eta)}ds - \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)}\psi(\tau_0) \\ &\le \frac{1}{\epsilon t^{\epsilon-1}}\frac{d}{dt}\int_{\tau_0}^t \frac{\psi(s)(t-s)^{-\eta}}{\Gamma(1-\eta)}ds - \frac{\psi(\tau_0)}{\epsilon t^{\epsilon-1}}\frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} \\ &\le \frac{d}{dt^\epsilon}\int_{\tau_0}^t \frac{\psi(s)(t-s)^{-\eta}}{\Gamma(1-\eta)}ds \end{aligned} \tag{20}$$

since $\psi(\tau_0) = 0$

$$0 \le_{\tau_0}^{FFP} D_t^{\eta,\epsilon}\psi(t), \quad \forall t \in [\tau_0, \tau_0 + a]$$

For the second part, we have two cases:
Case 1: There exists $t_1 \in [\tau_0, t]$, where $|\psi(t_1)| = \max_{\tau_0 \le t \le \bar{t}} |\phi(t)|$.
Based on definition, we have that

$$\psi(t_1) = \psi(\bar{t}), \quad \tau_0 \le t \le \bar{t}$$

and hence $\psi'(\bar{t}) = 0$. With this, we get $\psi'(t)(t - s)^{-\eta} = 0, \quad \forall t \in [\tau_0, \tau_0 + a]$.

$$\begin{aligned} &\frac{1}{\Gamma(1-\eta)}\int_{\tau_0}^t \frac{\psi'(s)(t-s)^{-\eta}}{\Gamma(1-\eta)} \\ &ds = \frac{1}{\Gamma(1-\eta)}\frac{d}{dt}\int_{\tau_0}^t \frac{\psi(s)(t-s)^{-\eta}}{\Gamma(1-\eta)} \\ &ds - \frac{\psi(\tau_0)(t-\tau_0)^{-\eta}}{\Gamma(1-\eta)} \\ &\frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^t \frac{d\psi(s)}{dt}\frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} \\ &ds = \frac{1}{\epsilon t^{\epsilon-1}\Gamma(1-\eta)}\frac{d}{dt}\int_{\tau_0}^t (t-s)^{-\eta}\psi(s) \\ &ds - \frac{\psi(\tau_0)(t-\tau_0)^{-\eta}}{\epsilon t^{\epsilon-1}\Gamma(1-\eta)} \end{aligned} \tag{21}$$

However, $\psi(\tau_0) = 0$

$$\begin{aligned} &\frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^t \frac{\psi' s(t-s)^{-\eta}}{\Gamma(1-\eta)ds} = 0 \\ &= \frac{1}{\Gamma(\eta-1)}\frac{d}{dt^\epsilon}\int_{\tau_0}^t (t-s)^{-\eta}\psi(s)ds =_{\tau_0}^{FFP} D_t^{\eta,\epsilon}\psi(t) \end{aligned} \tag{22}$$

For the second case it was in Ref.[8] that $\forall t \in [\tau_0, t_a]$ under the conditions presented above

$$\psi'(t) \le |\phi'(t)|$$

Now, from the above result, since $\phi(t)$ is non-decreasing based on hypothesis, we will have that

$$\psi'(t) \leq \phi'(t)$$

This implies using the fact that $\forall t \in [\tau_0, \tau_0 + \eta], \forall\ t - s > 0$

$$(t-s)^{-\eta}\psi'(t) \leq (t-s)^{-\eta}\phi'(t) \tag{23}$$

$$\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}\psi'(s)}{\Gamma(1-\eta)} ds \leq \int_{\tau_0}^{t} \frac{(t-s)^{-\eta}\phi'(s)}{\Gamma(1-\eta)} ds \tag{24}$$

From the above and using the relation between Caputo and Riemann–Liouville derivatives, and the fact at the initial condition the function is 0, we have the following

$$\frac{1}{\epsilon t^{\epsilon-1}}\frac{d}{dt}\left[\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}\psi(s)}{\Gamma(1-\eta)} ds - \frac{\psi(\tau_0)(t-\tau_0)^{-\eta}}{\Gamma(1-\eta)}\right]$$
$$\leq \frac{1}{\epsilon t^{\epsilon-1}}\frac{d}{dt}\left[\int_{\tau_0}^{t} (t-s)^{-\eta}\frac{\phi(s)|}{\Gamma(1-\eta)}\right.$$
$$\left. ds - \frac{(t-s)^{-\eta}|\phi(\tau_0)|}{\Gamma(1-\eta)}\right] \tag{25}$$

$$\psi(\tau_0) = \phi(\tau_0) = 0$$

implies that

$$\frac{1}{\epsilon t^{\epsilon-1}}\frac{d}{dt}\left[\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}\psi(s)}{\Gamma(1-\eta)} ds\right] \leq \frac{1}{\epsilon t^{\epsilon-1}}\frac{d}{dt}\int_{\tau_0}^{t} |\phi(s)|\frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} ds \tag{26}$$

Therefore,

$${}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}\psi(t) \leq {}^{FFP}_{\tau_0} D_t^{\eta,\epsilon}|\phi(t)|$$

the proof is completed.

Theorem 2. *From the Borzdyko's Uniqueness Theorem for Fractal-Fractional differential equations with power law kernel, let $\omega(\tau_0, w_0, p_0)$ in $C[\tau_0, \tau_0 + a]$ satisfy the Lipschitz condition, as $\omega(\tau_0, w_0, p_0)w(t) - \omega(\tau_0, w_0, p_0)\bar{w}(t) \leq \max_{\tau_0 \leq t \leq \bar{t}} |w(t) - \bar{w}(t)|$ and the function $g(t, w, p)$ is continuous and $\forall(t, w, p), (t, \bar{w}, \bar{p}) \in \Omega$ satisfies*

$$|g(t, w, p) - g(t, \bar{w}, \bar{p})| < g(|w - \bar{w}| + |p - \bar{p}|) \tag{27}$$

where $g(t) > 0$ continuous for $0 < t \leq \eta \leq t_0 + a$ now derivative for $0 < t < \delta \leq \eta$, $g(0) = 0$ and g satisfying the following condition.[8]

$$\lim_{\epsilon \to 0^+} \int_{\epsilon} \frac{1}{g(s)} dz = \infty$$

then, the fractal fractional equation considered here has at most one solution in a small interval that contains τ_0.

Proof. Let (t, w, p) and $(\tau_0, \bar{w}, \bar{p}) \in \Omega$ such that if $w(t) \neq \bar{w}(t), p(t) \neq \bar{p}(t)$. Set $\phi(t) = w(t) - \bar{w}(t)$ and hence $\psi(\bar{t}) = \max_{\tau_0 \leq t \leq \bar{t}} |\phi(t)|$. Indeed that exist $t_1 \in [\tau_0, \tau_0 + a]$ such that $\psi(t) = 0$,
$\forall t \in [\tau_0, t_1]$, and $\psi(t) > 0$ for $t_1 < t \leq \tau_0 + a$. From Lemma 3.1, if we assume that $\phi(t)$ is increasing, we get

$$FFP_{\tau_0}D_t^{\eta,\epsilon}\psi(t) \leq^{FFP}_{\tau_0} D_t^{\eta,\epsilon}|\phi(t)|$$

$${}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}|\phi(t)| = \frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)}\phi'(s) ds \tag{28}$$

since $\psi(\tau_0) = 0$, we shall have

$$\frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)}\phi'(s) ds =$$
$$\frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} (t-s)^{-\eta}w' ds - \frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} (t-s)^{-\eta}\bar{w}'(s) ds \tag{29}$$

$$={}^{FFP}_{\tau_0} D_t^{\eta,\epsilon}w(t) - {}^{FFP}_{\tau_0} D_t^{\eta,\epsilon}\bar{w}(t)$$
$$= g(t, w, p) - g(t, \bar{w}, \bar{p}) \leq g((L+1)\psi(t))$$

in some interval $[t_1, t_1 + \delta_1], \delta_1 > 0$, but the continuously differentiable functions $w(t)$ and $\bar{w}(t)$ are absolutely continuous, non-decreasing, and have a finite derivative almost everywhere, and then

$$0 \leq^{FFP}_{\tau_0} D_t^{\eta,\epsilon}\psi(t) \leq^{FFP}_{\tau_0} D_t^{\eta,\epsilon}|\phi(t)|$$

thus

$$0 \leq^{FFP}_{\tau_0} D_t^{\eta,\epsilon}\psi(t) \leq^{FFP}_{\tau_0} D_t^{\eta,\epsilon}|\phi(t)| \leq g((L+1)\psi(t))$$

The above is almost everywhere in $[t_1, t_1 + \delta_1]$. Then let $\{t_n\}$ be a decreasing sequence such that $t_1 < t_n < t_1 + \delta_1$ and $t_n \to t_1$ as $n \to \infty$.

The above inequality leads us to

$$0 \leq \frac{{}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}\psi(t)}{g\left((L+1)\psi(t)\right)} \leq 1 \tag{30}$$

Since $\psi(t)$ has a finite derivative, we can minimize

$$\begin{aligned}
{}^{FFP}_{\tau_0}D_t^{\eta,\epsilon}\psi(t) &= \frac{1}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} \psi'(s)\frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} ds \\
&\geq \frac{\inf_{t\in[\tau_0,t]} |\psi'(t)|}{\epsilon t^{\epsilon-1}}\int_{\tau_0}^{t} \frac{(t-s)^{-\eta}}{\Gamma(1-\eta)} ds \\
&\geq \frac{\inf_{t\in[\tau_0,t]} |\psi'(t)|}{\epsilon t^{\epsilon-1}}\frac{(t-s)^{-\eta}}{\Gamma(2-\eta)} \\
&\geq \frac{\delta_2}{\epsilon t^{\epsilon-1}}\frac{(t_1-\tau_0)^{1-\eta}}{\Gamma(2-\eta)} \\
&\geq \frac{\delta_1}{\epsilon}t^{1-\epsilon}\frac{(t_1-\tau_0)^{1-\eta}}{\Gamma(2-\eta)} \geq \frac{\delta_2 t_1^{1-\epsilon}(t_1-\tau_0)^{1-\eta}}{\epsilon\Gamma(2-\eta)}
\end{aligned} \tag{31}$$

$$0 \leq \frac{\frac{\delta_2 t_1^{1-\epsilon}(t_1-\tau_0)^{1-\eta}}{\epsilon\Gamma(2-\eta)}}{g(L+1)\psi(t)} \tag{32}$$

Therefore, integrating over t_n to $t_1+\delta_1$, we get

$$0 \leq \frac{\omega t_1^{1-\epsilon}(t_1-\tau_0)^{1-\eta}}{\epsilon\Gamma(2-\eta)} \int_{t_n}^{t_1+\delta_1} \frac{ds}{g((L+1)\psi(s))} \leq \int_{t_n}^{t_1+\delta_1} dt \tag{33}$$

Let $z=(L+1)\psi(t)$, then

$$0 \leq \frac{\delta t_1^{1-\epsilon}(t_1-\tau_0)^{1-\eta}}{(L+1)\epsilon\Gamma(2-\eta)} \int_{(L+1)\psi(t_n)}^{(L+1)\psi(t_1+\delta_1)} \frac{dz}{g(z)} \leq t_1+\delta_1-t_n \tag{34}$$

$\lim_{n\to\infty}$ of the above process

$$\lim_{n\to\infty} \frac{\delta t_1^{1-\epsilon}(t_1-\tau_0)^{1-\eta}}{(L+1)\epsilon\Gamma(2-\eta)} \int_{(L+1)\psi(t_n)}^{(L+1)\psi(t_1+\delta_1)} \frac{dz}{g(z)} \leq t_1+\delta_1-t_n < \delta_1 \tag{35}$$

which is a contration. Therefore $\psi(t)=0$, $\forall t \in [\tau_0, \tau_0+a]$, and $\phi(t)=0$. This implies $w(t)=\bar{w}(t)$, which completes the proof.

We shall now present different techniques of uniqueness for the considered fractal-differential equation. The foundation of our techniques is based on Borzdyko's Uniqueness, and this is achieved in different cases.

Case (1) We assume that $\eta=1$, then our equation is reduced to

$$\begin{cases} {}^{F}_{\tau_0}D_t^{\epsilon} w(t) = g(t,w(t),p(t)) \\ w(\tau_0)=w_0 \\ p(\tau_0)=p_0 \\ p(t)=w(\tau_0,w_0,p_0)w(t) \end{cases} \tag{36}$$

Theorem 3. *We assume that the defined operator $w(\tau_0,w_0,p_0)$ in $C[\tau_0,\tau_0+a]$ satisfies the presented Lipschitz condition, and it is possible to find two different functions, h_1 and h_2*

$$g(t,w,p)=h_1(t,w)h_2(t,w,p) \tag{37}$$

where $h_1(t,w)$ is defined and continuous in Ω_+ and $\frac{\partial h_1}{\partial t}(t,w)$ exists, non-negative and dominated by an (Lebesgue) integral function of w. The function $h_2(t,w,p)$ is defined in Ω_+ and $\forall (t,w,p)$ and $(t,\bar{w},\bar{p}) \in \Omega_+$ satisfies the following condition

$$(i)|h_2(t,w,p)-h_2(t,\bar{w},\bar{p})| \leq \lambda(t)[|w-\bar{w}|+|p-\bar{p}|] \tag{38}$$

$$g(t,w,p)=h_1(t,p)h_2(t,w,p) \tag{39}$$

$$(ii)|h_2(t,w,p)-h_2(t,\bar{w},\bar{p})| \leq \lambda(t)[|w-\bar{w}|+|p-\bar{p}|] \tag{40}$$

$$w^{\epsilon} g(t,w,p)=h_1(t,p)h_2(t,w,p) \tag{41}$$

$\lambda(t)$ is an integrable function in $[\tau_0,\tau_0+a]$.

Proof. Let assume that the equation under consideration has two solution w,p and $\bar{w},\bar{p}$ in $[\tau_0,\tau_0+a]$, $w \neq \bar{w}$,

(i) If we consider the following function by Agarwal and Lakshmikantham[8]

$$F(t)=\frac{1}{2}\left[\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)}\right]^2 \tag{42}$$

we have, based on hypothesis, that h_1 exists and (Lebesgue) integrable function $g(w)$ in $|w-w_0| \leq l$, such that

$$0 \leq \frac{\partial h_1(t,w)}{\partial t} \leq g(w) \quad \forall (t,w) \in \Omega^+ \tag{43}$$

we have that $F(t)$ is differentiable, thus

$$\begin{aligned} {}^{F}_{\tau_0}D_t^{\epsilon} H(t) &= \frac{1}{2} {}^{F}_{\tau_0}D_t^{\epsilon} \left(\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)} ds\right)^2 \\ &= \frac{1}{2\epsilon t^{\epsilon-1}} \frac{d}{dt}\left(\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)}\right)^2 \end{aligned} \tag{44}$$

using the Lipschitz rule for integral and following the deviation of in Theorem 3.10.1 by Agarwal and Lakshmikantham[8], we get

$$\begin{aligned} {}^{F}_{\tau_0}D_t^{\epsilon} H(t) =& \frac{1}{\epsilon t^{\epsilon-1}}\left[\frac{\bar{w}'(t)}{h_1(t,\bar{w}(t))} - \frac{w'(t)}{h_1(t,\bar{w}(t))} + \int_{w(t)}^{\bar{w}(t)} \frac{\partial}{\partial t}\left(\frac{1}{h_1(t,s)}\right) ds \left[\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)}\right]\right. \\ &\leq \frac{1}{\epsilon t^{\epsilon-1}}\left[\frac{\bar{w}'(t)}{h_1(t,\bar{w}(t))} - \frac{w'(t)}{h_1(t,\bar{w}(t))}\right]\left(\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)}\right) \\ &\leq \left[\frac{\frac{\bar{w}'(t)}{\epsilon t^{\epsilon-1}}}{h_1(t,\bar{w}(t))} - \frac{\frac{\bar{w}'(t)}{\epsilon t^{\epsilon-1}}}{h_1(t,w(t))}\right]\left(\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)}\right) \\ &\leq \left[\frac{{}^{F}_{\tau_0}D_t^{\epsilon}\bar{w}(t)}{h_1(t,\bar{w}(t))} - \frac{{}^{F}_{\tau_0}D_t^{\epsilon} w(t)}{h_1(t,w(t))}\right]\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)} \\ &\leq \frac{1}{2}\left[\frac{g(t,\bar{w},\bar{p})}{h_1(t,\bar{w}(t))} - \frac{g(t,w,p)}{h_1(t,w)}\right]\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)} \\ &\leq \frac{1}{2}[h_2(t,\bar{w},\bar{p})-h_2(t,w,p)]\int_{w(t)}^{\bar{w}(t)} \frac{ds}{h_1(t,s)} \end{aligned} \tag{45}$$

Now from Ref.[8] and the properties of $h_1(t,w)$, we get

$$ {}^{F}_{\tau_0}D_t^{\epsilon} F(t) \leq \frac{(L+1)}{m}\lambda(t)\left[\max_{\tau_0 \leq \bar{t} \leq t} |\bar{w}(\bar{t})-w(\bar{t})|\right]^2 \tag{46}$$

also from Ref.[8], we have

$$\left[\max_{\tau_0 \leq \bar{t} \leq t} |\bar{w}(\bar{t})-w(\bar{t})|\right]^2 \leq 2\bar{M}^2 \max_{\tau_0 \leq \bar{t} \leq t} F(\bar{t}) \tag{47}$$

letting

$$\psi(t)=\max_{\tau_0 \leq \bar{t} \leq t} F(\bar{t}) \tag{48}$$

and the inequality of the lemma presented above, we get

$$ {}_{\tau_0}^{F}D_t^{\epsilon}\psi(t) \leq \frac{2(L+1)}{m}\lambda(t)\bar{m}^2\psi(t) \tag{49} $$

$$ \begin{aligned} \psi'(t) &\leq \frac{2(L+1)}{m}\lambda(t)\bar{m}^2\psi(t)\epsilon t^{\epsilon-1} \\ &\leq \frac{2(L+1)}{m}\lambda(t)\bar{m}^2\psi(t)\epsilon(\bar{t}_0)^{\epsilon-1} \end{aligned} \tag{50} $$

following the routine in Ref.[8], we yield

$$ w(t) = \bar{w}(t) \quad \forall t \in [\tau_0, \tau_0 + a] \tag{51} $$

(ii) For the second case, we select the following function

$$ F(t) = \frac{1}{2}\left[\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds\right]^2 \tag{52} $$

we get $F(\tau_0) = 0$ with the properties of $h_1(t,w)$, we then have that $F(t)$ is differentiable, thus fractionally differentiable

$$ \begin{aligned} {}_{\tau_0}^{F}D_t^{\epsilon}F(t) &= \frac{1}{2\epsilon t^{\epsilon-1}}\left(\left[\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds\right]^2\right) \\ &= \frac{1}{\epsilon t^{\epsilon-1}}\left[\frac{\bar{w}^{\epsilon}(t)\bar{w}'(t)}{h_1(t,\bar{w}(t))} - \frac{w^{\epsilon}(t)w'(t)}{h_1(t,w(t))} + \right. \\ &\left.\int_{w(t)}^{\bar{w}(t)} \frac{\partial}{\partial t}\left(\frac{s^{\epsilon}}{h_1(t,s)}ds\right)\right]\left(\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds\right) \end{aligned} \tag{53} $$

we have that

$$ \frac{\partial}{\partial t}\left(\frac{s^{\epsilon}}{h_1(t,s)}ds\right) \leq 0 $$

and $F(t) \geq 0$, therefore

$$ \begin{aligned} {}_{\tau_0}^{F}D_t^{\epsilon}F(t) &\leq \frac{1}{\epsilon t^{\epsilon-1}}\left[\frac{\bar{w}^{\epsilon}(t)\bar{w}'(t)}{h_1(t,\bar{w}(t))} - \frac{w^{\epsilon}(t)w'(t)}{h_1(t,w(t))}\right] \\ &\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \\ &\leq \left[\frac{\frac{\bar{w}^{\epsilon}(t)\bar{w}'(t)}{\epsilon t^{\epsilon-1}}}{h_1(t,\bar{w}(t))} - \frac{\frac{w^{\epsilon}(t)w'(t)}{\epsilon t^{\epsilon-1}}}{h_1(t,w(t))}\right] \\ &\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \\ &\leq \left[\frac{\bar{w}^{\epsilon}(t){}_{\tau_0}^{F}D_t^{\epsilon}\bar{w}(t)}{h_1(t,\bar{w}(t))} - \frac{w^{\epsilon}(t){}_{\tau_0}^{F}D_t^{\epsilon}w(t)}{h_1(t,w(t))}\right] \\ &\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \\ &\leq \left[\frac{\bar{w}^{\epsilon}(t)g(t,w,p)}{h_1(t,\bar{w}(t))} - \frac{w^{\epsilon}(t)g(t,\bar{w},\bar{p})}{h_1(t,w(t))}\right] \\ &\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \\ &\leq [h_2(t,w,p) - h_2(t,\bar{w},\bar{p})]\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \\ &\leq \lambda(t)\left[|w-\bar{w}| + |p-\bar{p}|\right]\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds \end{aligned} \tag{54} $$

to find a single way to proceed, we maximize s^{ϵ} in $[\tau_0, \tau_0 + a]$, then

$$ \begin{aligned} {}_{\tau_0}^{F}D_t^{\epsilon}F(t) &\leq \lambda(t)(\tau_0+a)^{\epsilon}\frac{(L+1)}{m}\max_{\tau_0 \leq \bar{t} \leq t}|\bar{w}(\bar{t}) - w(\bar{t})|^2 \\ &\leq 2\bar{M}^2\frac{(L+1)}{m}\lambda(t)\max_{\tau_0 \leq \bar{t} \leq t}[F(\bar{t})](\tau_0+a)^{3\epsilon} \end{aligned} \tag{55} $$

since

$$ \begin{aligned} F(t) &= \frac{1}{2}\left[\int_{w(t)}^{\bar{w}(t)} \frac{s^{\epsilon}}{h_1(t,s)}ds\right]^2 \\ &\leq \frac{(\tau_0+a)^{2\epsilon}}{2m^2}[\bar{w} - w]^2 \end{aligned} \tag{56} $$

therefore,

$$ {}_{\tau_0}^{F}D_t^{\epsilon}F(t) \leq \frac{2\bar{M}^2(L+1)\lambda(t)(\tau_0+a)^{3\epsilon}}{m}\max_{\tau_0 \leq \bar{t} \leq t}F(t) \tag{57} $$

Based on the lemma result, if

$$ \psi(t) = \max_{\tau_0 \leq \bar{t} \leq t} F(\bar{t}) \tag{58} $$

the following

$$ {}_{\tau_0}^{F}D_t^{\epsilon}F(t) \leq \frac{2\bar{M}^2(L+1)\lambda(t)(\tau_0+a)^{3\epsilon}}{m}\psi(t) \tag{59} $$

$$ {}_{\tau_0}^{F}D_t^{\epsilon}\psi(t) = \frac{2\bar{M}^2(L+1)\lambda(t)(\tau_0+a)^{3\epsilon}}{m}\psi(t) \tag{60} $$

$$ \begin{aligned} \psi'(t) &\leq \frac{2\epsilon t^{\epsilon-1}\bar{M}^2(L+1)\lambda(t)(\tau_0+a)^{3\epsilon}}{m}\psi(t) \\ &\leq \frac{2\epsilon(\bar{t}_0)^{\epsilon-1}\bar{M}^2(L+1)\lambda(t)(\tau_0+a)^{3\epsilon}}{m}\psi(t) \end{aligned} \tag{61} $$

is almost everywhere in $[\tau_0, \tau_0 + a]$. By letting

$$ \Omega(t) = -\psi(t)\exp\left[-\frac{2\epsilon(\bar{t}_0)^{\epsilon-1}\bar{M}^2(L+1)(\tau_0+a)^{3\epsilon}}{m}\right. \\ \left.\int_{\tau_0}^{t}\lambda(s)ds\right] \tag{62} $$

then

$$ \Omega(t)' \geq 0 \tag{63} $$

is almost everywhere in $[\tau_0, \tau_0 + a]$. Following the routine in Ref.[8], we conclude that

$$ \psi = 0 \to w(t) = \bar{w}(t) \;\; \forall t \in [\tau_0, \tau_0 + a] $$

which completes the proof.

For the second part, we shall assume that $p = 1$, such that the considered equation is reduced to

$$ \begin{aligned} &{}_{\tau_0}^{RL}D_t^{\eta}w(t) = g(t,w,p) \\ &w(\tau_0) = w_0 \\ &p(t) = w(\tau_0, w_0, p_0)w(t) \end{aligned} \tag{64} $$

We shall now show the conditions under which the above system has at most one solution.

Theorem 4. *Let h_1 and h_2 be two functions satisfying the condition presented in Theorem 3.3, and that*
(i)

$$\forall(t,w,p)\in\Omega$$

(ii)

$$\forall(t,\bar{w},\bar{p}),(t,w,p)\in\Omega$$

$|h_2(t,\bar{w},\bar{p})-h_2(t,w,p)|\leq\lambda(t)[|\bar{w}-w|+|\bar{p}-p|]$ *where $\lambda(t)$ is positive and non-null in $[\tau_0,\tau_0+a]$ such that*

$$\frac{1}{\Gamma(\eta)}\int_{\tau_0}^{t}\lambda(s)(t-s)^{\eta-1}ds$$

exists.
(iii) We have $0<m\leq h_1(t,w)\leq M$. Then Equation (64) has at most one solution in τ_0,τ_0+a.

Proof. Let h_2,h_1 be two functions that satisfy the conditions of the theorem.
we set

$$F(t)=\frac{1}{2}\left(\int_{w(t)}^{\bar{w}(t)}\frac{ds}{h_1(t,s)}\right)^2 \tag{65}$$

Indeed at $t=\tau_0$, we have that $F(\tau_0)=0$, hence

$${}_{\tau_0}^{C}D_t^{\eta}F(t)={}_{\tau_0}^{RL}D_t^{\eta}F(t)$$

$$\begin{aligned}{}_{\tau_0}^{C}D_t^{\eta}F(t)&=\frac{1}{\Gamma(1-\eta)}\int_{\tau_0}^{t}(t-s)^{-\eta}\left[\frac{1}{2}\right.\\&\left.\left(\int_{w(s)}^{\bar{w}(s)}\frac{dl}{h_1(s,l)}\right)^2 ds\right.\\&\leq\frac{1}{\Gamma(1-\eta)}\int_{\tau_0}^{t}(t-s)^{-\eta}\\&\left[\left[\frac{\bar{w}'(s)}{h_1(s,\bar{w}(s))}-\frac{w'(s)}{h_1(s,w(s))}\right]\right.\\&\int_{w(s)}^{\bar{w}(s)}\frac{dl}{h_1(s,l)}ds\end{aligned} \tag{66}$$

The calculation is then divided into two cases. If

$$\frac{\bar{w}'(s)}{h_1(s,\bar{w}(s))}-\frac{w'(s)}{h_1(s,w(s))}<0 \tag{67}$$

$$\int_{w(s)}^{\bar{w}(s)}\frac{ds}{g(t,s)}\geq\frac{\bar{w}(s)-w(s)}{M} \tag{68}$$

thus

$$\begin{aligned}{}_{\tau_0}^{C}D_t^{\eta}F(t)&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M\Gamma(1-\eta)}\int_{\tau_0}^{t}(t-s)^{-\eta}\\&\left[\frac{\bar{w}'(t)}{h_1(\bar{w},l)}-\frac{w'(t)}{h_1(y,l)}\right]ds\\&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2\Gamma(1-\eta)}\int_{\tau_0}^{t}(t-s)^{-\eta}\\&(\bar{w}'-w')ds\\&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2}({}_{\tau_0}^{RL}D_t^{\eta}\bar{w}-{}_{\tau_0}^{RL}D_t^{\eta}w)\\&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2}(g(t,\bar{w},\bar{p})-g(t,w,p))\\&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2}(h_1(t,\bar{w})h_2(t,\bar{w},\bar{p})\\&-h_1(t,w)h_2(t,w,p))\\&\leq\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2}m[h_2(t,\bar{w},\bar{p})\\&-h_2(t,w,p)]\end{aligned} \tag{69}$$

Using the hypothesis, we obtain

$${}_{\tau_0}^{RL}D_t^{\eta}F(t)\min_{\tau_0\leq t\leq\bar{t}}\frac{|\bar{w}(\bar{t})-w(t)|}{M^2}m\lambda(t)[|\bar{w}-w|+|\bar{p}-p|] \tag{70}$$

$${}_{\tau_0}^{RL}D_t^{\eta}\left(\min_{\tau_0\leq\bar{t}\leq t}F(\bar{t})\right)\leq\frac{(L+1)}{M^2}\lambda(t)m\min_{\tau_0\leq\bar{t}\leq t}|\bar{w}(t)-w(\bar{t})| \tag{71}$$

we note that

$$F(t)=\frac{1}{2}\left(\int_{w(t)}^{\bar{w}(t)}\frac{dt}{h_1(t,s)}\right)^2\geq\frac{1}{2}\frac{(\bar{w}(t)-w(\bar{t}))^2}{m} \tag{72}$$

therefore

$$2F(t)M^2\geq\bar{w}(t)-w(\bar{t}) \tag{73}$$

replacing yields

$${}_{\tau_0}^{RL}D_t^{\eta}\left(\min_{\tau_0\leq\bar{t}\leq t}F(\bar{t})\right)\leq 2\frac{(L+1)}{M^2}m^2\lambda(t)\min_{\tau_0\leq\bar{t}\leq t}F(\bar{t}) \tag{74}$$

we let

$$\psi(t)=\min_{\tau_0\leq\bar{t}\leq t}F(\bar{t}) \tag{75}$$

we get

$${}_{\tau_0}^{RL}D_t^{\eta}\psi(t)\leq 2\frac{(L+1)}{M^2}m^2\lambda(t)\psi(t) \tag{76}$$

we can now apply the Riemann–Liouville integral to get

$$\psi(t)\leq 2\frac{(L+1)m^2}{M^2\Gamma(\eta)}\int_{\tau_0}^{t}\lambda(s)\psi(t)(t-s)^{\eta-1}ds \tag{77}$$

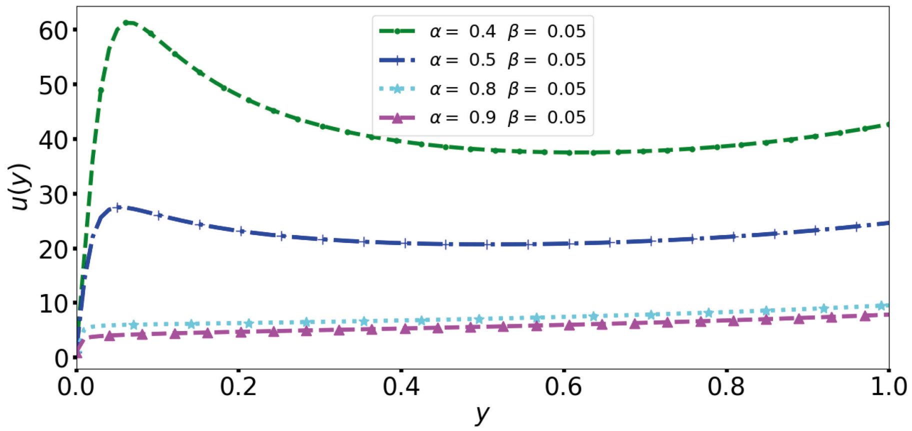

Figure 1. Numerical simulation of example 1 for different values of η and ϵ

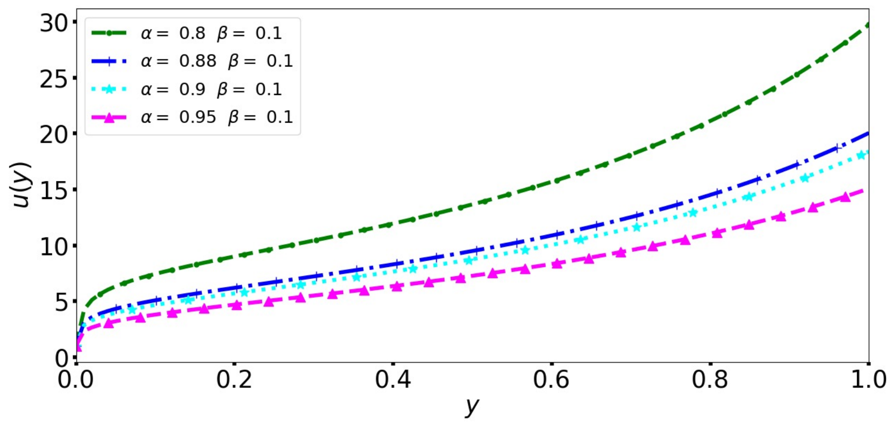

Figure 2. Numerical simulation of example 2 for different values of η and ϵ

Based on the Gronwall inequality, we yield

$$\psi(t) \leq 0 \exp\left[\frac{1}{\Gamma(\eta)}\int_{\tau_0}^{t} \lambda(s)(t-s)^{\eta-1}ds\right] \leq 0 \tag{78}$$

which is a contradiction, therefore

$$\psi(t) = 0, \quad \bar{w}(t) = w(t) \tag{79}$$

if

$$\frac{\bar{w}'(t)}{h_1(t,\bar{w})} - \frac{w'(t)}{h_1(t,y} > 0 \tag{80}$$

then we shall maximize

$$\int_{w(s)}^{\bar{w}(s)} \frac{dl}{g(t,l)} \leq \frac{\bar{w}(s)-w(s)}{m} \tag{81}$$

then

$$\begin{aligned}
{}^{RL}_{\tau_0}D_t^{\eta}F(t) &\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m\Gamma(1-\eta)}\int_{\tau_0}^{t}(t-s)^{-\eta} \\
&\left[\frac{\bar{w}'(t)}{h_1(\bar{w},l)} - \frac{w'(t)}{h_1(y,l)}\right]ds \\
&\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m^2\Gamma(1-\eta)} \\
&\int_{\tau_0}^{t}(t-s)^{-\eta}(\bar{w}'-w')ds \\
&\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m^2}({}^{RL}_{\tau_0}D_t^{\eta}\bar{w} - {}^{RL}_{\tau_0}D_t^{\eta}w) \\
&\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m^2}(g(t,\bar{w},\bar{p}) - g(t,w,p)) \\
&\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m^2}(h_1(t,\bar{w})h_2(t,\bar{w},\bar{p}) \\
&- h_1(t,w)h_2(t,w,p)) \\
&\leq \max_{\tau_0\leq\bar{t}\leq t} \frac{|\bar{w}(\bar{t})-w(t)|}{m^2}[h_2(t,\bar{w},\bar{p}) \\
&- h_2(t,w,p)]\rho(t)
\end{aligned} \tag{82}$$

where

$$\rho(t) = \frac{h_1(t,\bar{w})h_2(t,\bar{w},\bar{p}) - h_1(t,w)h_2(t,w,p)}{h_1(t,\bar{w}(t)) - h_1(t,w(t))} \tag{83}$$

$$\wedge = \max_{t\in[\tau_0,\tau_0+a]}(\rho(t)) \tag{84}$$

we obtain

$$\begin{aligned}{}^{RL}_{\tau_0}D^{\eta}_{t}F(t) \leq& \max_{\tau_0\leq\bar{t}\leq t}\frac{|\bar{w}(t)-w(t)|\wedge}{m^2}|h_2(t,\bar{w},\bar{p})\\ &- h_2(t,w,p)|\\ \leq& \max_{\tau_0\leq\bar{t}\leq t}\frac{|\bar{w}(t)-w(t)|\wedge}{m^2}[|\bar{w}-w|+|\bar{p}-p|]\lambda(t)\\ \leq& \max_{\tau_0\leq\bar{t}\leq t}\frac{|\bar{w}(t)-w(t)|^2\wedge}{m^2}(L+1)\lambda(t)\end{aligned} \tag{85}$$

$$ {}^{RL}_{\tau_0}D^{\eta}_{t}\max_{\tau_0\leq\bar{t}\leq t}F(\bar{t}) \leq 2\max_{\tau_0\leq\bar{t}\leq t}F(\bar{t})(L+1)\lambda(t)\wedge \tag{86}$$

$$ {}^{RL}_{\tau_0}D^{\eta}_{t}\psi(t) \leq 2(L+1)\wedge -\psi(t)\lambda(t) \tag{87}$$

$$\psi(t) \leq \frac{1}{\Gamma(\eta)}\int_{\tau_0}^{t} 2(L+1)\wedge\lambda(t)(t-s)^{\eta-1}\psi(s)ds \tag{88}$$

Based on the Gronwall inequality, we get

$$\begin{aligned}\psi(t) =&0 \quad \forall t\in[\tau_0,\tau_0+a] \implies\\ &\bar{w}(t)=w(t).\end{aligned}$$

which concludes the proof.

In the next section, we provide two practical examples with their numerical solution.

4. Examples and numerical solutions

The aim of this section is not to provide any theoretical analysis of the method that used to provide a numerical solution. Rather, we aim to present several examples and their numerical solutions; the existence and uniqueness of these solutions are guaranteed.

4.1. Example 1

We consider the following function

$$g(t,w,p) = (1+t)\left(w(t)e^{\lambda t}+ue^{\lambda t}\right) \tag{89}$$

we have

$$h_1(t,w) = 1+t, \quad \frac{\partial h_1}{\partial t}(t,w) => 0 \tag{90}$$

$$h_2(t,w,p) = w(t)e^{\lambda t}+ue^{\lambda t} \tag{91}$$

$$\begin{aligned}|h_2(t,\bar{w},\bar{p})-h_2(t,w,p)| &\leq |\bar{w}(t)\\ &- w(t)|e^{\lambda t}+|\bar{p}(t)-p(t)|e^{\lambda t}\\ &\leq e^{\lambda t}[|\bar{w}(t)\\ &- w(t)|-|\bar{p}(t)-p(t)|]\end{aligned} \tag{92}$$

Taking $\lambda(t)=e^{\lambda t}$, we have

$$\begin{aligned}|h_2(t,\bar{w},\bar{p})-h_2(t,w,p)| &\leq \lambda t[|\bar{w}(t)\\ &- w(t)|-|\bar{p}(t)-p(t)|]\end{aligned} \tag{93}$$

we have that $p(t) = Lw(t)$, with $L>0, t\in[\tau_0,\tau_0+a]$. By choosing $w(\tau_0)=w(0)=1,\quad \lambda = 1,\quad L=2$, we obtain the following solution for $(\eta,\epsilon)\in(0,1]$.

This dynamic equation describes systems where growth factors change over time, where there are memory effects, where external inputs affect the overall system response, and where previous states interact in such a way that their influence increases or decreases depending on how the system has previously evolved. The dynamic equation can be applied in material science as a descriptor of stress–strain paths during cyclic loading; climate modeling as a plan for feedback mechanisms (i.e., ice-albedo effect and green house gas effects); ecology as a basis for population dynamics influenced by environmental memory; neuroscience as a model of neural activity and synaptic plasticity; economic field by explaining unemployment rates and market behavior; chemical kinetics where it describes the concentration of reactant in pathways dependent on their history; and behavioral sciences as a model for decision-making processes and adaptive mechanisms. Its versatility provides insights into systems that rely on memory in the presence of changing external conditions in diverse domains, aiding the comprehension and decision-making on complex, dynamic phenomena.

4.2. Example 2

We shall consider the following function

$$\begin{cases} g(t,w,p) = t(w(t)t^{\epsilon}+p(t)t^{\epsilon})\\ p(t) = Lw(t), \quad t\in[0,T], \ \ L>0\end{cases} \tag{94}$$

here,

$$h_1(t,w)=t, \quad \frac{\partial h_1}{\partial t}(t,w(t)) = 1>0 \tag{95}$$

$$\begin{aligned}h_2(t,w,p) = t^{\epsilon}w(t)+t^{\epsilon}p(t)\\ \forall(t,\bar{w},\bar{p}),(t,w,p),\end{aligned} \tag{96}$$

we have

$$\begin{aligned}|h_2(t,\bar{w},\bar{p})-h_2(t,w,p)| &\leq t^{\epsilon}[|\bar{w}(t)\\ &- w(t)|-|\bar{p}(t)-p(t)|]\\ &\leq \lambda t[|\bar{w}(t)-w(t)|-|\bar{p}(t)-p(t)|]\end{aligned} \tag{97}$$

therefore,

$$\begin{aligned}|h_2(t,\bar{w},\bar{p})-h_2(t,w,p)| &\leq \lambda t[|\bar{w}(t)\\ &- w(t)|-|\bar{p}(t)-p(t)|]\end{aligned} \tag{98}$$

By choosing $L=2$, we get the following numerical solution for different values of $(\eta,\epsilon=\beta)\in(0,1]$.

5. Conclusion

Borzdyko's conditions of uniqueness have attracted the attention of researchers working within the field of existence and uniqueness in the last decades. This approach has been applied mostly to differential equations with classical derivatives, and some extensions have been suggested. Fractal-fractional power law differential equations with hysteresis are a special class of differential equations that have not received adequate attention. In this paper, we have established conditions under which these equations yield at most one solution.

Acknowledgments

The first author would like to thank the National Research Foundation of South Africa (NRF).

Funding

None.

Conflict of interest

The authors declare they have no competing interests.

Author contributions

Conceptualization: All authors
Investigation: All authors
Methodology: All authors
Formal analysis: All authors
Writing–original draft: All authors
Writing–review & editing: All authors

Availability of data

The study presented here is purely theoretical. Therefore, there is no data available.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

References

1. Chen W, Sun H, Zhang X, Korošak D. Anomalous diffusion modeling by fractal and fractional derivatives. *Comput Math Appl.* 2010;59(5):1754–1758. https://doi.org/10.1016/j.camwa.2009.08.019
2. Chen W. Time–space fabric underlying anomalous diffusion. *Chaos Solitons Fractals.* 2006;28(4):923–929. https://doi.org/10.1016/j.chaos.2005.08.199
3. Sun HG, Meerschaert MM, Zhang Y, Zhu J, Chen W. A fractal Richards' equation to capture the non-Boltzmann scaling of water transport in unsaturated media. *Adv Water Resour.* 2013;52:292–295. https://doi.org/10.1016/j.advwatres.2012.11.005
4. Atangana A. Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex systems. *Chaos Solitons Fractals.* 2017;102:396–406.https://doi.org/10.1016/j.chaos.2017.04.027
5. Atangana A, Jain S. A new numerical approximation of the fractal ordinary differential equation. *Eur Phys J Plus.* 2018;133:37. https://doi.org/10.1140/epjp/i2018-11895-1
6. Jain S, El-Khatib Y. Modelling chaotic dynamical attractors with fractal-fractional differential operators. *AIMS Math.* 2021;6(12):13689–13725. https://doi: 10.3934/math.2021795
7. Jain S. Numerical analysis for the fractional diffusion and fractional Buckmaster equation by the two-step Laplace Adam–Bashforth method. *Eur Phys J Plus.* 2018;133(1):19. https://doi.org/10.1140/EPJP/I2018-11854-X
8. Agarwal RP, Lakshmikantham V. Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations. Singapore: World Scientific; 1993. https://doi.org/10.1142/1988
9. Julia G. Mémoire sur l'itération des fonctions rationnelles. *J Math Pures Appl.* 1918;8:47–245. http://eudml.org/doc/234994
10. Tan L. Similarity between the Mandelbrot set and Julia sets. *Commun Math Phys.* 1990;134:587–617. https://doi.org/10.1007/BF02098448

Abdon Atangana *is a Professor at the University of the Free State, Bloemfontein, South Africa. He is the founder of fractal–fractional calculus and fractional differential calculus with the Mittag–Leffler kernel, and a co-founder of piecewise differential and integral calculus. He is the author of more than ten books and has published over 400 research articles in top-tier international journals.*
https://orcid.org/0000-0002-1886-3125

Sonal Jain *is a faculty member in the Department of Data Science and Mathematics at Sir Padampat Singhania University, Udaipur. She specializes in Mathematical Modeling, Computational Mathematics, Epidemiology, Fractional and Fractal Calculus, and the application of Artificial Intelligence and Machine Learning in applied sciences. She holds a Ph.D. in Mathematics from MNIT Jaipur and an EPGP in Data Science Artificial Intelligence from IIT Guwahati. Dr. Jain has an extensive research record with 45+ publications in reputed SCI and Scopus-indexed journals. According to Google Scholar, she has an h-index of 15, an i10-index of 18, and over 685 citations, reflecting the impact and visibility of her scholarly contributions. Her work spans fractional-order differential equations, reaction–diffusion systems, epidemiological modeling, and numerical methods. She has served in academic and research positions across*

India, UAE, and Turkey, including a Postdoctoral Fellowship at UAE University and a Visiting Associate Professorship at Usak University. She has received notable recognitions such as the Best Research Award (Woxsen University, 2023) and the NBHM International Travel Grant, and has been actively involved in organizing international conferences, curriculum development, and interdisciplinary collaborations. Her current research focuses on integrating advanced computational techniques with AI/ML frameworks to study complex biological, chemical, and ecological systems. She continues to contribute to the global research community through publications, mentorship, and international academic engagement.

https://orcid.org/0000-0002-4094-2575

An International Journal of Optimization and Control: Theories & Applications
(https://accscience.com/journal/ijocta)