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RESEARCH ARTICLE

Fixed point results for generalized rectangular fuzzy b-metric-like spaces and applications

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ABSTRACT

This paper introduces a new extended rectangular fuzzy b-metric-like space that generalizes the existing frameworks of the extended rectangular and fuzzy b-metric spaces. Within this setting, we establish several fixed-point theorems for Ciric-type and Banach-type contractions, accompanied by a series of corollaries, propositions, and conditions that further illustrate the proposed concept. These results unify and extend many known theorems in fuzzy metric theory. Moreover, we provided several non-trivial examples to validate of the main results. A flow diagram was provided to demonstrate the generalized structure. Additionally, we applied the fuzzy integral equation to establish the uniqueness and existence of our main result.

1. Introduction

Fréchet[1] introduced the concept metric space which paved the path for distance-based scientific disciplines such as mathematics and physics. Subsequently, Zadeh[2] introduced a novel concept in the sets that extends beyond the traditional approach of classical set. In 1960, Schweizer and Sklar[3] introduced the concept of a continuous t-norm (CTN), a crucial operation in fuzzy set theory. Recently, Gul and Sarfraz[4] enhanced artificial intelligence models with interval-valued picture fuzzy sets and Sugeno–Weber triangular norms. In 1975, Karamata and Michalek[5] introduced the concept of a fuzzy metric space (FMS), opening new avenues for research in mathematics. George and Veeramani[6] improved the FMS by proving a fixed-point theorem (FPT) in an FMS. This extension opened up new avenues for research in fuzzy mathematics and its applications in a range of fields, including decision-making and control theory. Subsequent studies that build on these key notions investigate the relationship between fuzziness and continuity in more complex mathematical settings. A study of modules of

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fractions under fuzzy sets and soft sets was conducted by Ayub *et al.*[7]. In 1988, Grabiec[8] extended this framework by establishing a fuzzy counterpart of the Banach contraction principle, demonstrating its significance in FPT and mathematical analysis.

Vasuki and Veeramani[9] compared two types of Cauchy sequences (CS) utilized to identify FPT in the FMS. They also provided literature-related applications for such theorems. Heilpern[10] investigated FPT in FMS and established several FPT in complete FMS. Kaleva 11 developed the concept of fuzzy differential equations, providing a way for understanding dynamical systems in a fuzzy setting. Buckley and Feuring[12] used fuzzy partial differential equations to investigate complex systems with several variables. Puri and Ralescu[13] developed fuzzy function differentials, which influenced fuzzy calculus and its applications.

Czerwik[14,15] introduced the concept of a b-metric space (BMS), which extends standard metric spaces by relaxing the triangle inequality with a positive constant. He also developed the FPT for single-valued and multi-valued mappings in these domains. Kamran *et al.*[16] introduced the concept of extended BMS (EBMS), which generalizes traditional BMS by reducing the distance requirements. Aydil *et al.*[17] proposed a FPT for nonlinear contractive mappings and expanded the concept of functional inequalities. Roshan *et al.*[18] defined b-rectangular metric spaces and developed FPT for mappings with rational-type contraction criteria and almost generalized weak contractiveness. They introduced a new concept into FPT, and scholars work on it to advance their discipline.

Nădăban[19] further developed the concept of metric space, providing a new method for FMS researchers to obtain fixed point (FP). Mehmood *et al.*[20] introduced the notion of fuzzy BMS (FBMS), which extends the classic fuzzy b-metric framework by relaxing constraints. Kanwal *et al.*[21] established various FP results under the context of complete fuzzy strong b-metric spaces. These spaces have essential properties, such as the openness of open balls, that are not typically found in traditional b-metric and FBMS spaces. The concept of an extended FBMS (EFBMS) was introduced by Mehmood *et al.*[22], which generalizes FMS by incorporating a broader set of distance conditions. They developed some FPTs on this framework. Again,[23], Mehmood *et al.*[22] introduced the concept of rectangular FBMS (RFBMS) and established FPT in this generalized setting, thereby advancing FPT and its applications in fuzzy mathematical structures.

Saleem *et al.*[24] introduced the notion of extended RFBMS (ERFBMS), expanding the scope of RFBMS and rectangular fuzzy metric space (RFMS). They provided some FP thresholds for this framework. Asif *et al.*[25] introduced fractional-order mathematical modeling of toxoplasmosis transmission dynamics with a harmonic mean-type incidence rate. They proved several FP results for fuzzy contractive mappings using this expanded approach. Zoto *et al.*[26] extended Wardowski-type contractions for b-metric-like spaces "(BMLS)". They created generalized forms (s, ϕ, F) and (s, q, ϕ, F) contractions. Their work provided a broader view of contraction principles, which contributed to the development of FPT in generalized metric structures. Younis *et al.*[27] have recently explored FP method to connect with Chua's attractor model, incorporating the Atangana–Baleanu derivative using a two-step Lagrange polynomial. Similarly, the FP result explored by Younis and Bahuguna[28] was utilized in graph-based metric spaces to model physical processes, such as rocket ascent. Younis *et al.*[29] established some novel results concerning graph contractions in a more generalized setting.

Shereen *et al.*[30] found FP findings for $(\alpha - F)$ contraction in newly regulated S-metric spaces. Ud-din *et al.*[31] developed FP solutions for complex-valued neutrosophic metric spaces and used them to solve an integral equation. Ishtiaq *et al.*[32] discovered some fixed solutions for bipolar metric spaces and applied them to solve a boundary value problem in chemical science. Malik *et al.*[33] demonstrated novel types of higher-order chaotic polynomial maps. Alfaqih *et al.*[34] obtained FP findings and discovered a fractional differential equation. Panda *et al.*[35] derived FP results in a graphical extended S-supra-metric space and applied them to a fractal-fractional-order system. Ahmad *et al.*[36] demonstrated the contractive technique in generalized suprametric spaces, using it for fractional boundary-value and epidemiological problems. Tudorache and Luca [37] found answers to a set of fractional q-difference boundary value problems. Panda *et al.*[38] demonstrated the stability of fractional-order complex-valued neural networks through numerical simulations and application in game theory. Khan *et al.*[39] developed FP findings in fuzzy S-metric space and demonstrated their applicability to the Fractals and Satellite Web Coupling Problem.

Ishtiaq *et al.*[40] demonstrated FP results for controlled rectangular modular metric spaces with fractional differential equation solutions.

Building on the previous discussion, we present numerous FP results in the context of new extended rectangular fuzzy BMLS (NERFBMLS). The proposed NERFBMLS differs from extended rectangular BMLS and extended rectangular fuzzy BMLS (ERFBMLS) in terms of structure and scope. While ERFBMLS employs a two-variable control function $\phi(\sigma, x)$, NERFBMLS introduces a triple-variable function $\phi(\sigma, x, \rho)$, allowing the fuzzy distance to depend on three interacting points concurrently. In the first section, we review fundamental principles and significant findings from the existing literature. The second section introduces an enlarged version of rectangular fuzzy BMLS (RFBMLS), then formulates and proves FPTs in this context, with illustrative examples to validate our conclusions. In the third section, we use our primary findings to demonstrate the existence and uniqueness of a solution to the Volterra integral equation of the second sort, along with a relevant example. We also present corollaries and remarks that connect our results to previously known findings in the literature. The fourth section summarizes our contributions.

Next, we present a comparative overview of various fuzzy metric frameworks, highlighting how the proposed NERFBMLS extends existing structures such as ERFBMS and RFBMS by introducing a triple-variable control function and relaxed self-distance conditions.

In this work, we enhance prior research by applying a triple-variable control function, $\phi(\sigma, x)$. As a result, RFBMS and ERFBMS are unique examples where $\phi(\sigma, x, \rho)$ is reduced to b or $\phi(\sigma, x)$. This generalization increases modeling flexibility for higher-order dependence and non-uniform fuzzy relationships. Furthermore, unlike NERFBMLS, the self-distance in ERFBMS is not always equal to one, allowing it to contain a more complex fuzzy metric structure.

1.1. Preliminaries

We begin by introducing fundamental concepts that serve as the foundation for our further analysis.

Definition 1. [17] *Let $\mathcal{X} \neq \Phi$, and $\phi : \mathcal{X}\times\mathcal{X}\times\mathcal{X} \to [1, \infty)$ be a given function. A new extended b-metric (NEBM) is a function $D : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ that satisfies the following assertions for all $\sigma, x, \rho \in \mathcal{X}$:*

: (E1) $D(\sigma, x) \geq 0$;

: (E2) $D(\sigma, x) = 0 \Leftrightarrow \sigma = x$;

: (E3) $D(\sigma, x) = D(x, \sigma)$;

: (E4) $D(\sigma, \rho) \leq \phi(\sigma, x, \rho)[D(\sigma, x) + D(x, \rho)]$.

The pair $(\mathcal{X}, D)$ is called an NEBM space (NEBMS).

Remark 1. [17] *AN NEBM is not a continuous function.*

Definition 2. [23] *Let $\mathcal{X} \neq \Phi$ and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ in $\mathcal{X} \times \mathcal{X} \times [0, \infty)$ is called a rectangular fuzzy metric if, for all $\sigma, y, \rho \in \mathcal{X}$, it satisfies the following assertions:*

: ($1bM_\phi$): $\mathcal{M}_\phi(\sigma, x, 0) = 0$;

: ($2bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = 1, \quad \forall\, \zeta > 0 \Leftrightarrow \sigma = x$;

: ($3bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = \mathcal{M}_\phi(x, \sigma, t)$;

: ($4bM_\phi$): $\mathcal{M}_\phi(\sigma, \rho, \zeta + \mu + \nu)) \geq \mathcal{M}_\phi(\sigma, x, \zeta) \star \mathcal{M}_\phi(x, u, \mu) \star \mathcal{M}_\phi(u, \rho, \nu), \ \forall\, \zeta, \mu, \nu \geq 0$;

: ($5bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma, x, t) = 1$.

The triplet $(\mathcal{X}, \mathcal{M}_\phi, \star)$ is called a RFMS.

Definition 3. [23] *Let $\mathcal{X} \neq \Phi$, $b \geq 1$ be a real number, and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ defined on $\mathcal{X} \times \mathcal{X} \times [0, \infty)$ is called an rectangular fuzzy b-metric (RFBM) if for all $\sigma, x, \rho \in \mathcal{X}$, it satisfies the following assertions:*

: ($1bM_\phi$): $\mathcal{M}_\phi(\sigma, x, 0) = 0$;

: ($2bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = 1, \quad \forall\, \zeta > 0 \Leftrightarrow \sigma = x$;

: ($3bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = \mathcal{M}_\phi(x, \sigma, \zeta)$;

: ($4bM_\phi$): $\mathcal{M}_\phi(\sigma, \rho, b(\zeta + \mu + \nu)) \geq \mathcal{M}_\phi(\sigma, x, \zeta) \star \mathcal{M}_\phi(x, u, \mu) \star \mathcal{M}_\phi(u, \rho, \nu), \ \forall\, \zeta, \mu, \nu \geq 0$;

: ($5bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma, x, \zeta) = 1$.

The quadruple $(\mathcal{X}, \mathcal{M}_\phi, \star, b)$ is named an RFBMS.

Remark 2. *The class of RFBMS is larger than the class of RFMS because a RFBMS becomes a FRMS when $b = 1$.*

Remark 3. *An RFBM on a space $\mathcal{X}$ may not be a fuzzy rectangular metric on $\mathcal{X}$.*

Example 1. [23] *Let $\mathcal{M}_\phi(\sigma, x, t) = e^{\frac{-|\sigma - x|^p}{t}}$, where $p > 1$ is a real number. Then $\mathcal{M}_\phi$ is an RFBM b-metric with $b = 2^{p-1}$. But, it is not a fuzzy rectangular metric for $p = 2$.*

Definition 4. [24] *Let $\mathcal{X} \neq \Phi$ and $\phi : \mathcal{X} \times \mathcal{X} \to [1, \infty)$ ϕ be a given function and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ defined in $\mathcal{X}\times\mathcal{X}\times[0, \infty)$ is called an extended RFBM (ERFBM) if for all $\sigma, x, z \in \mathcal{X}$, it satisfies the following assertions:*

: ($1bM_\phi$): $\mathcal{M}_\phi(\sigma, x, 0) = 0$;

: ($2bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = 1, \forall\zeta > 0 \Leftrightarrow \sigma = x$;

: ($3bM_\phi$): $\mathcal{M}_\phi(\sigma, x, \zeta) = \mathcal{M}_\phi(x, \sigma, \zeta)$;

Table 1. Comparison of different fuzzy metric frameworks

Framework	Defining function	Self-distance	Triangle inequality type	Generalizes
RFBMS	$b \geq 1$	$\mathcal{M}(\sigma,\sigma,\zeta)=1$	$\mathcal{M}(\sigma,\rho,b(\zeta+\mu+\nu))$	RFMS
ERFBMS	$\phi(\sigma,x)$	$\mathcal{M}(\sigma,\sigma,\zeta)=1$	$\mathcal{M}(\sigma,\rho,\phi(\sigma,\rho)(\zeta+\mu+\nu))$	RFBMS
NERFBMLS (Proposed)	$\phi(\sigma,x,\rho)$	$\mathcal{M}(\sigma,\sigma,\zeta)\neq 1$	$\mathcal{M}(\sigma,\rho,\phi(\sigma,x,\rho)(\zeta+\mu+\nu))$	RFMS, ERFBMS

Abbreviations: ERFBMS: Extended rectangular fuzzy b-metric space; NERFBMLS: New extended rectangular fuzzy b-metric-like space; RFBMS: Rectangular fuzzy b-metric space; RFMS: Rectangular fuzzy metric space.

: (4bM$_\phi$): $\mathcal{M}_\phi(\sigma,\rho,\phi(\sigma,\rho)(\zeta+\mu+\nu)\geq \mathcal{M}_\phi(\sigma,x,\zeta)\star\mathcal{M}_\phi(x,u,\mu)\star\mathcal{M}_\phi(u,\rho,\nu)$, $\forall\ \zeta,\mu,\nu\geq 0$;
: (5bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\cdot):[0,\infty)\to[0,1]$ is left continuous and $\lim_{n\to\infty} M_\phi(\sigma,x,\zeta)=1$.

The quadruple $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is named as an ERFBMS.

Remark 4. *An ERFBMS reduces to an RFBMS when $\phi(\sigma,\rho)=b$.*

Definition 5. [24] *Let $\mathcal{X}\neq\Phi$ and $\phi:\mathcal{X}\times\mathcal{X}\to[1,\infty)$ be a given function, and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ defined on $\mathcal{X}\times\mathcal{X}\times[0,\infty)$ is called an ERFBM-like (ERFBML) if for all $\sigma,x,\rho\in\mathcal{X}$, it satisfies the following assertions:*

: (1bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,0)=0$;
: (2bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=1,\forall\ \zeta>0\Rightarrow\sigma=x$;
: (3bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=\mathcal{M}_\phi(x,\sigma,\zeta)$;
: (4bM$_\phi$): $\mathcal{M}_\phi(\sigma,\rho,\phi(\sigma,\rho)(\zeta+\mu+\nu))\geq\mathcal{M}_\phi(\sigma,x,\zeta)\star\mathcal{M}_\phi(x,u,\mu)\star\mathcal{M}_\phi(u,\rho,\nu)$, $\forall\ \zeta,\mu,\nu\geq 0$;
: (5bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\cdot):[0,\infty)\to[0,1]$ is left continuous and $\lim_{n\to\infty} M_\phi(\sigma,x,\zeta)=1$.

The quadruple $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is called an ERFBMLS.

2. Main results

The concept of a new ERFBMS (NERFBMS) is defined as follows:

Definition 6. *Let $\mathcal{X}\neq\Phi$, $\phi:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[1,\infty)$ be a given function and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ defined on $\mathcal{X}\times\mathcal{X}\times[0,\infty)$ is called a new ERFBM (NERFBM) if for all $u,\sigma,x,\rho\in\mathcal{X}$ and $\zeta,\mu,\nu>0$, it satisfies following assertions:*

: (1bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,0)=0$;
: (2bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=1$, iff $\sigma=x \quad \forall\ \zeta>0$;
: (3bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=\mathcal{M}_\phi(x,\sigma,\zeta)$;
: (4bM$_\phi$): $\mathcal{M}_\phi(\sigma,\rho,\phi(\sigma,x,\rho)(\zeta+\mu+\nu))\geq\mathcal{M}_\phi(\sigma,x,\zeta)\star\mathcal{M}_\phi(x,w,\mu)\star\mathcal{M}_\phi(w,\rho,\nu),\forall\zeta,\mu,\nu\geq 0$;
: (5bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\cdot):[0,\infty)\to[0,1]$ is left continuous and $\lim_{n\to\infty}\mathcal{M}_\phi(\sigma,x,\zeta)=1$.

The quadruple $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is a NERFBMS.

Remark 5. *A NERFBM generalizes ERFBMS, RFBMS, and RFMS.*

Example 2. *Let $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ be an NEBMS. Consider $\mathcal{M}_\phi:\mathcal{X}\times\mathcal{X}\times[0,\infty)\to[0,1]$, such that:*

$$\mathcal{M}_\phi(\sigma,x,\zeta)=\begin{cases}\frac{\zeta}{\zeta+D(\sigma,x)}, & \textit{if } \zeta>0;\\ 0, & \textit{if } \zeta=0,\end{cases}$$

where $\phi(\sigma,x,z)=\sigma^2+x^2+\rho^2+1$ and $D_\phi(\sigma,x)=(\sigma-x)^2$, then $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is an NERFBM. $\mathcal{M}_\phi$ is known as the standard NERFBM.

Remark 6. *In general, an NERFBMS is not continuous.*

Example 3. *Let $\mathcal{X}=[0,\infty)$, $\mathcal{M}_\phi(\sigma,x,\zeta)=e^{\frac{-D(\sigma,x)}{\zeta}}$, $\star$ be a product t-norm and:*

$$D(\sigma,x)=\begin{cases}0, & \textit{if } \sigma=x\\ 3|\sigma-x|, & \textit{if } x,x\in[0,1)\\ \frac{1}{3}|\sigma-x| & \textit{otherwise.}\end{cases}$$

Then $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is an NERFBMS with $\phi=6$. Note that $\mathcal{M}$ is not continuous. Since:

$$\lim_{n\to\infty}\mathcal{M}_\phi(1,1-\frac{1}{n},\zeta)=\lim_{n\to\infty}e^{-3/n\zeta}=1=\mathcal{M}(1,1,\zeta),$$

we have $1-\frac{1}{n}\to 1$ as $n\to\infty$. Also:

$$\lim_{n\to\infty}\mathcal{M}_\phi(0,1-\frac{1}{n},\zeta)=\lim_{n\to\infty}e^{\frac{-3|1-\frac{1}{n}|}{\zeta}}=e^{\frac{-3}{\zeta}}\neq e^{\frac{-1}{3\zeta}}=\mathcal{M}(0,1,\zeta).$$

Therefore, $\mathcal{M}_\phi$ is not continuous.

Definition 7. *Let $\mathcal{X}\neq\Phi$, $\phi:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[1,\infty)$ be a given function and $\star$ be a CTN. A fuzzy set $\mathcal{M}_\phi$ in $\mathcal{X}\times\mathcal{X}\times[0,\infty)$ is called an NERFBM-like (NERFBML), if for all distinct $u,\sigma,x,\rho\in\mathcal{X}$ and $\zeta,\mu,\nu>0$, it satisfies the following assertions:*

: *(1bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,0)=0$;*
: *(2bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=1,\quad\forall\ \zeta>0$ then $\sigma=x$;*
: *(3bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\zeta)=\mathcal{M}_\phi(x,\sigma,\zeta)$;*
: *(4bM$_\phi$): $\mathcal{M}_\phi(\sigma,z,\phi(\sigma,x,z)(\zeta+\mu+\nu))\geq\mathcal{M}_\phi(\sigma,x,\zeta)\star\mathcal{M}_\phi(x,u,\mu)\star\mathcal{M}_\phi(u,z,w=\nu),\forall\zeta,\mu,\nu\geq 0$;*
: *(5bM$_\phi$): $\mathcal{M}_\phi(\sigma,x,\cdot):[0,\infty)\to[0,1]$ is left continuous and $\lim_{n\to\infty}\mathcal{M}_\phi(\sigma,x,\zeta)=1$.*

The quadruple $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ is an NERFBMLS

By using the following proposition, several examples of fuzzy metric-like spaces can be obtained.

Remark 7. *An ERFBMLS generalizes an ERFBMS, as every ERFBMLS is an ERFBMS, but the converse may not hold.*

Proposition 1. *Let (X, d_ϕ) be any new extended rectangular BMLS (NERBMLS). Then $(X, \mathcal{M}_\phi, \star, \phi)$ is a NERFBMLS, where $\star$ is an product t-norm given by:*

$$\mathcal{M}_\phi(\sigma, x, \zeta) = \frac{k\zeta^n}{k\zeta^n + D_\phi(\sigma, x)},$$

where $D_\phi(\sigma, x)$ is a rectangular BMLS, $k \in \mathbb{R}$, and $n \geq 1$.

Proof. The proof of properties $(1bM_\phi) - (3bM_\phi)$ and $(5bM_\phi)$ are obvious. For $(4bM_\phi)$, $u, \sigma, x, \rho \in \mathcal{X}$ and $\zeta, \mu, \nu > 0$, and $a = D_\phi(\sigma, x)$, $b = D_\phi(x, u)$, $c = D_\phi(u, \rho)$, $d = D_\phi(\sigma, \rho)$, then for all $t, s, w \geq 0$; we know that:

$$\begin{aligned} d &\leq a + b + c \\ k\zeta^n\mu^n\nu^n d &\leq k(\zeta + \mu + \nu)^n\nu^n a + k(\zeta + \mu + \nu)^n\zeta^n b + \\ &\quad k(\zeta + \mu + \nu)^n\nu^n c \\ k\zeta^n\mu^n\nu^n d &\leq (\zeta + \mu + \nu)^n \left[k\mu a + k\zeta^n b + k\nu^n c + abc\right] \\ k\zeta^n\mu^n\nu^n \left[k(\mu + \zeta + \nu)^n + d\right] &\leq (\zeta + \mu + \nu)^n \left[(k\zeta^n + a)(k\mu^n + b)(k\nu^n + c)\right] \\ \frac{k\zeta^n}{k\zeta^n + a} \cdot \frac{k\mu^n}{k\mu^n + b} \cdot \frac{k\nu^n}{k\nu^n + c} &\leq \frac{k(\zeta + \mu + \nu)^n}{k(\zeta + \mu + \nu)^n + d} \\ \mathcal{M}_\phi(\sigma, x, \zeta) \star \mathcal{M}_\phi(x, u, \mu) \star \mathcal{M}_\phi(u, \rho, \nu) &\leq \mathcal{M}_\phi(\sigma, \rho, \phi(\sigma, x, \rho)(\zeta + \mu + \nu)). \end{aligned}$$

Therefore, $(4bm_\phi)$ is also satisfied and $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ is an NERFBMLS.

Remark 8. *It is important to mention that the previous proposition remains valid even when using the minimum t-norm.*

Example 4. *Let $\mathcal{X} = \mathbb{R}^+$. Consider $\mathcal{M}_\phi : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0, 1]$, such that:*

$$\mathcal{M}_\phi(\sigma, x, t) = \frac{t}{t + D(\sigma, x)}$$

where $\phi(\sigma, x, \rho) = \sigma^2 + x^2 + \rho^2 + 1$ and $D_\phi(\sigma, x) = (\sigma + x)^2$. Then $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ is a NERFBMLS. It is easy to see that self-distance is not necessarily equal to 1; hence, it is neither an EFBMS nor an new EFBMS (NEFBMS), as:

$$\mathcal{M}_\phi(\sigma, \sigma, \zeta) = \frac{\zeta}{\zeta + D_\phi(\sigma, \sigma)} \neq 1 \forall \sigma > 0 \text{ and } \zeta > 0.$$

Example 5. *Let $\mathcal{X} = \mathbb{N}$. Consider $\mathcal{M}_\phi : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0, 1]$, such that:*

$$\mathcal{M}_\phi(\sigma, x, \zeta) = \frac{\zeta}{\zeta + \max\{\sigma, x\}}$$

Also, consider $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, \infty)$, as:

$$\phi(\sigma, x, \rho) = \begin{cases} 1, & \text{if } \sigma = \rho \text{ and } x \text{ is even or odd} \\ \frac{\sigma\rho}{\sigma + \rho}, & \text{if } \sigma \neq \rho,\ \sigma \text{ and } \rho \text{ are even and } x \text{ is odd}, \\ \frac{x}{2}, & \text{if } \sigma \neq \rho,\ \sigma \text{ and } \rho \text{ are odd and } x \text{ is even}, \\ \frac{3}{2}, & \text{if } \sigma \neq \rho,\ \sigma, x \text{ and } \rho \text{ are all even or all odd}, \\ \frac{\sigma + x(1+\sigma)}{\sigma(1+x)}, & \text{if } \sigma \neq \rho,\ \sigma \text{and } x \text{ are even and } \rho \text{ is odd}, \\ \frac{\rho + x(\rho+1)}{\rho(x+1)}, & \text{if } \sigma \neq \rho,\ \sigma \text{ is odd and } x \text{ and } \rho \text{ are even}, \\ \frac{2+\rho}{1+\rho}, & \text{if } \sigma \neq \rho,\ \sigma \text{ and } x \text{ are odd and } \rho \text{ is even}, \\ \frac{\sigma+1}{\sigma}, & \text{if } \sigma \neq \rho,\ \sigma \text{ is even and } x \text{ and } \rho \text{ are odd}. \end{cases}$$

Then $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ is a NERFBMLS with $\star$ product t-norm. It is easy to see that self-distance is not necessarily equal to 1; hence, it is not an EFBMS, nor an NEFBMS.

Remark 9. *Every NERFBMLS is an ERFBMLS, but the converse is not necessarily true.*

Remark 10.
- *Taking* $\phi(\sigma, x, \rho) = \phi(\sigma, \rho)$, *an NERFBMS similar to that of the ratchet reduces to an ERFBMS similar to that of the ratchet.*
- *Taking* $\phi(\sigma, z) = b \geq 1$, *an ERFBMLS reduces to an RFBMLS.*

Now we define convergence, CS, and completeness in the NERFBMLS.

Definition 8. *A sequence in* $\{\sigma_n\}$ *in NERFBMLS* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *is said to be convergent to* $\sigma \in \mathcal{X}$ *if*

$$\lim_{n\to\infty} \mathcal{M}_\phi(\sigma_n, \sigma, \zeta) = \mathcal{M}_\phi(\sigma, \sigma, \zeta) \quad \forall\ \zeta > 0.$$

Definition 9. *A sequence in* $\{\sigma_n\}$ *in NERFBMLS* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *is said to be CS, if:* $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma_{n+p}, \sigma_n, \zeta)$ *for all* $\zeta > 0, p \geq 1$ *exists and finite.*

Definition 10. *In NERFBMLS,* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *is said to be convergent to* $\{x_n\} \in \mathcal{X}$ *converges to some* $\sigma \in \mathcal{X}$ *such that*

$$\lim_{n\to\infty} \mathcal{M}_\phi(\sigma_n, \sigma, \zeta) = \mathcal{M}_\phi(\sigma, \sigma, \zeta) = \lim_{n\to\infty} \mathcal{M}_\phi(\sigma_{n+p}, \sigma_n, \zeta) \forall \zeta > 0, p \geq 1.$$

Definition 11. *Let* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *be a NERFBMLS. A mapping* $\Gamma : \mathcal{X} \to \mathcal{X}$ *is called a Ciri'c type contraction if it satisfies:*

$$\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq C(\sigma, x, \zeta)\ \forall\ \sigma, x \in \mathcal{X}, \tag{1}$$

where:

$$C(\sigma, x, \zeta) = \min\left\{ \frac{\mathcal{M}_\phi(x, \Gamma x, 3\zeta)\big[1 + \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta)\big]}{1 + \mathcal{M}_\phi(\sigma, x, \zeta)}, \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta), \right.$$
$$\left. \mathcal{M}_\phi(x, \Gamma x, 3\zeta), \mathcal{M}_\phi(x, \Gamma\sigma, \zeta), \mathcal{M}_\phi(\sigma, x, \zeta) \right\}.$$

In the framework of an NERFBMLS, we prove the following result by using the definition of a CS given by George and Veeramani.[6]

Theorem 1. *Let* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *be a complete NERFBMLS,* $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, 1/k)$ *be a continuous function with* $k \in (0, 1)$*, such that:*

$$\lim_{n\to\infty} \mathcal{M}_\phi(\sigma, x, \zeta) = 1,\ \forall\ \sigma, x \in \mathcal{X}.$$

Let $\Gamma : \mathcal{X} \to \mathcal{X}$ *be an Ciric type fuzzy contraction then there is a unique FP (UFP) of* Γ*.*

Proof. Fix $\sigma_0 \in \mathcal{X}$. Let $\{\sigma_n\}$ be a sequence in $\mathcal{X}$ so that $\sigma_n = \Gamma^n\sigma_0 = \Gamma\sigma_{n-1}$, $(n \in \mathbb{N})$. Now,

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) &= \mathcal{M}_\phi(\Gamma^n\sigma_0, \Gamma^{n+1}\sigma_0, k\zeta) \\
&= \mathcal{M}_\phi(\Gamma\sigma_{n-1}, \Gamma(\Gamma\sigma_n), k\zeta) \\
&\geq C(\sigma_{n-1}, \sigma_n, \zeta) \\
&= \min\left\{ \frac{\mathcal{M}_\phi(\sigma_n, \Gamma\sigma_n, 3\zeta)\big[1 + \mathcal{M}_\phi(\sigma_{n-1}, \Gamma\sigma_{n-1}, \zeta)\big]}{1 + \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)}, \right.
\end{aligned}$$
$$\left. \mathcal{M}_\phi(\sigma_{n-1}, \Gamma\sigma_{n-1}, \zeta), \mathcal{M}_\phi(\sigma_n, \Gamma\sigma_n, 3\zeta), \mathcal{M}_\phi(\sigma_n, \Gamma\sigma_{n-1}, \zeta), \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta) \right\},$$
$$= \min\left\{ \frac{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta)\big[1 + \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)\big]}{1 + \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)}, \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta) \right.$$
$$\left. \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta), \mathcal{M}_\phi(\sigma_n, \sigma_n, \zeta), \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta) \right\},$$
$$= \min\Big\{ \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta), \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, t) \Big\}.$$

If $\min\{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta), \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta), 1\} = \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta)$ or 1, then

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) \geq \mathcal{M}(\sigma_n, \sigma_{n+1}, 3\zeta) \text{ or } \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) \geq 1.$$

This implies that $\sigma_n = \sigma_{n+1}$. If $\min\{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, 3\zeta), \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta), 1\} = \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)$, then we have $\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) \geq \mathcal{M}(\sigma_{n-1}, \sigma_n, \zeta)$, which implies that:

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta) \geq \mathcal{M}_\phi(\sigma_0, \sigma_1, \zeta/k^n), \quad \forall\, \zeta > 0$$

Case(1): When p is odd, say $p = 2m+1$, for $m \geq 1$, by using $(4bM_\phi)$, we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) &\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+2m+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+2m+1}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
&\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta/3^3}{\prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{\zeta/3^3}{\prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+6}, \sigma_{n+2m+1}, \frac{\zeta/3^3}{\prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})}\right)
\end{aligned}$$

$$\vdots$$

$$\star\, \mathcal{M}_\phi \left(\sigma_{n+2m}, \sigma_{n+2m+1}, \frac{\zeta/3^m}{\prod_{j=0}^{m-1} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})} \right).$$

Now, using Equation (), we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq\ & \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})k^n} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})k^{n+1}} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})k^{n+2}} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})k^{n+3}} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^3 \prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})k^{n+4}} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^3 \prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})k^{n+5}} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^3 \prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})k^{n+6}} \right) \\
& \vdots \\
& \star\, \mathcal{M}_\phi \left(\sigma_0, \sigma_1, \frac{\zeta}{3^m \prod_{j=0}^{m-1} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})k^{n+2m}} \right).
\end{aligned}$$

Case(2): When p is even, say $p = 2m$, for $m \geq 1$, by using $(4bM_\phi)$, we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq\ & \mathcal{M}_\phi \left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})} \right) \\
& \star\, \mathcal{M}_\phi \left(\sigma_{n+2}, \sigma_{n+2m}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})} \right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{t/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+2m}, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right).
\end{aligned}
$$

$$
\begin{aligned}
&\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi\left(\sigma_n, \sigma_{n+1}, \sigma_{n+2m}\right)}\right) \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi\left(\sigma_n, \sigma_{n+1}, \sigma_{n+2m}\right)}\right) \\
&\star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta}{3^2\phi\left(\sigma_n, \sigma_{n+1}, \sigma_{n+2m}\right)\phi\left(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m}\right)}\right) \\
&\star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta}{3^2\phi\left(\sigma_n, \sigma_{n+1}, \sigma_{n+2m}\right)\phi\left(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m}\right)}\right) \\
&\star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi\left(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m}\right)}\right) \\
&\star \mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi\left(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m}\right)}\right) \\
&\star \mathcal{M}_\phi\left(\sigma_{n+6}, \sigma_{n+2m}, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi\left(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m}\right)}\right) \\
&\quad \vdots \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2m-2}, \sigma_{n+2m}, \frac{\zeta}{3^{m-1}\prod_{j=0}^{m-2}\phi\left(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m}\right)}\right).
\end{aligned}
$$

Again, by using Equation (), we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n,\sigma_{n+2m},\zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta/3}{\phi\left(\sigma_n,\sigma_{n+1},\sigma_{n+2m}\right)k^n}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta/3}{\phi\left(\sigma_n,\sigma_{n+1},\sigma_{n+2m}\right)k^{n+1}}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^2\phi\left(\sigma_n,\sigma_{n+1},\sigma_{n+2m}\right)\phi\left(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m}\right)k^{n+2}}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^2\phi\left(\sigma_n,\sigma_{n+1},\sigma_{n+2m}\right)\phi\left(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m}\right)k^{n+3}}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi\left(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m}\right)k^{n+4}}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi\left(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m}\right)k^{n+5}}\right)\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi\left(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m}\right)k^{n+6}}\right)\\
&\vdots\\
&\star\,\mathcal{M}_\phi\left(\sigma_0,\sigma_1,\frac{\zeta}{3^{m-1}\prod\limits_{j=0}^{m-2}\phi\left(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m}\right)k^{n+2m-2}}\right).
\end{aligned}$$

Now, since $0<k<1$ and as $n\to\infty$, we have $k^n\to 0$ as $t\to\infty$. So we have $\lim\limits_{n\to\infty}\mathcal{M}_\phi(\sigma_n,\sigma_{n+p},t)\geq 1\star 1\star\cdots\star 1=1$. This implies that $\lim\limits_{n\to\infty}\mathcal{M}_\phi(\sigma_n,\sigma_{n+p},\zeta)\geq 1$. It shows that $\{\sigma_n\}$ is a CS in $\mathcal{X}$. Since $\mathcal{X}$ is complete, there exists a point $\sigma\in\mathcal{X}$ such that $\lim_{n\to\infty}\sigma_n=\sigma$, or $\lim_{n\to\infty}\mathcal{M}_\phi(\sigma_n,\sigma,\zeta)=1$. Now, to prove σ is the UFP. Let us consider:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma,\Gamma\sigma,\zeta) \geq\ & \mathcal{M}_\phi\left(\sigma,\sigma_n,\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\star\mathcal{M}_\phi\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\\
&\star\mathcal{M}_\phi\left(\sigma_{n+1},\Gamma\sigma,\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\\
\geq\ & \mathcal{M}_\phi\left(\sigma,\sigma_n,\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\star\mathcal{M}_\phi\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\\
&\star\mathcal{M}_\phi\left(\Gamma\sigma_n,\Gamma\sigma,\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\\
\geq\ & \mathcal{M}_\phi\left(\sigma,\sigma_n,\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\star\mathcal{M}_\phi\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\\
&\star C\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)
\end{aligned}$$

where:

$$\begin{aligned}
C\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right) &= \min\Bigg\{\frac{\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\left[1+\mathcal{M}_\phi\left(\sigma_n,\Gamma\sigma_n,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\right]}{\left[1+\mathcal{M}_\phi\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\right]},\\
&\quad \mathcal{M}_\phi\left(\sigma_n,\Gamma\sigma_n,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\\
&\quad \mathcal{M}_\phi\left(\sigma,\Gamma\sigma_n,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\mathcal{M}_\phi\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\Bigg\}\\
&= \min\Bigg\{\frac{\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\left[1+\mathcal{M}_\phi\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\right]}{\left[1+\mathcal{M}_\phi\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\right]},\\
&\quad \mathcal{M}_\phi\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\\
&\quad \mathcal{M}_\phi\left(\sigma,\sigma_{n+1},\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right),\mathcal{M}_\phi\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)\Bigg\}.
\end{aligned}$$

Taking $n\to\infty$, we have:

$$\begin{aligned}
\lim_{n\to\infty} C\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right) &= \min\Bigg\{\frac{\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)\left[1+\mathcal{M}_\phi\left(\sigma,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)\right]}{\left[1+\mathcal{M}_\phi\left(\sigma,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)\right]},\\
&\quad \mathcal{M}_\phi\left(\sigma,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right),\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right),\\
&\quad \mathcal{M}_\phi\left(\sigma,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right),\mathcal{M}_\phi\left(\sigma,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)\Bigg\}\\
&= \min\left\{\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right),1\right\}.
\end{aligned}$$

If $\lim_{n\to\infty} C\left(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\right)=1$ then:

$$\mathcal{M}_\phi(\sigma,\Gamma\sigma,\zeta)\geq 1\star 1\star 1$$

that is, $\Gamma\sigma=\sigma$. If:

$$\lim_{n\to\infty} C\Big(\sigma_n,\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma_n)}\Big)=\mathcal{M}_\phi\Big(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\Big) \tag{2}$$

then:

$$\mathcal{M}_\phi(\sigma,\Gamma\sigma,\zeta)\geq 1\star 1\star \mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)=\mathcal{M}_\phi\left(\sigma,\Gamma\sigma,\frac{\zeta/3}{k\phi(\sigma,\Gamma\sigma,\sigma)}\right)$$

that is, $\Gamma\sigma=\sigma$.

Uniqueness: Let σ and σ^* be two FP of the mapping Γ. Now:

$$\begin{aligned}\mathcal{M}_\phi(\sigma,\sigma^*,\zeta) &= \mathcal{M}_\phi(\Gamma\sigma,\Gamma\sigma^*,\zeta)\\ &\geq C(\sigma,\sigma^*,\zeta/k)\\ &= min\{\frac{\mathcal{M}(\sigma^*,\Gamma\sigma^*,3\zeta/k)[1+\mathcal{M}(\sigma,\Gamma\sigma,\zeta/k)]}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)},\mathcal{M}(\sigma,\Gamma\sigma,\zeta/k),\mathcal{M}(\sigma^*,\Gamma\sigma^*,3\zeta/k),\\ &\quad \mathcal{M}(\sigma^*,\Gamma\sigma,\zeta/k),\mathcal{M}(\sigma,\sigma*,\zeta/k)\}\\ &= min\{\frac{\mathcal{M}(\sigma^*,\sigma^*,3\zeta/k)[1+\mathcal{M}(\sigma,\sigma,\zeta/k)]}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)},\mathcal{M}(\sigma,\sigma,\zeta/k),\mathcal{M}(\sigma^*,\sigma^*,3\zeta/k),\\ &\quad \mathcal{M}(\sigma^*,\sigma,\zeta/k),\mathcal{M}(\sigma,\sigma^*,\zeta/k)\}\\ &= min\{\frac{2}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)},1,\mathcal{M}(\sigma,\sigma*,\zeta/k)\},\end{aligned}$$

that is:

$$\mathcal{M}_\phi(\sigma,\sigma^*,t) \quad \geq \quad C(\sigma,\sigma^*,\zeta/k) = \min\left\{\frac{2}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)},1,\mathcal{M}(\sigma,\sigma^*,\zeta/k)\right\}$$

If $C(\sigma,\sigma^*,\zeta/k)=1$ then $\sigma=\sigma^*$. If $C(\sigma,\sigma^*,\zeta/k)=\mathcal{M}(\sigma,\sigma^*,\zeta/k)$ then:

$$\mathcal{M}_\phi(\sigma,\sigma^*,\zeta) \quad \geq \quad \mathcal{M}(\sigma,\sigma*,\zeta/k)$$

which implies $\sigma=\sigma^*$. If $C(\sigma,\sigma^*,\zeta/k)=\frac{2}{1+\mathcal{M}(\sigma,\sigma^*,t/k)}$, then:

$$\mathcal{M}_\phi(\sigma,\sigma^*,\zeta) \quad \geq \quad \frac{2}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)}.$$

As, $\mathcal{M}(\sigma,\sigma^*,\zeta/k)\leq 1$, $\frac{2}{1+\mathcal{M}(\sigma,\sigma^*,\zeta/k)}\geq 1$ and $\mathcal{M}_\phi(\sigma,\sigma^*,t)\geq 1$. Now, $1\leq\mathcal{M}_\phi(\sigma,\sigma^*,\zeta)\leq 1$, that is, $\mathcal{M}_\phi(\sigma,\sigma^*,\zeta)=1$, which implies that $\sigma=\sigma^*$.

Example 6. *Let $\mathcal{X}=\mathbb{N}$. Consider $\mathcal{M}_\phi:\mathcal{X}\times\mathcal{X}\times[0,\infty)\to[0,1]$ such that*

$$\mathcal{M}_\phi(\sigma,x,\zeta)=e^{-\dfrac{(\sigma+x)^2}{\zeta}}$$

Also, consider $\phi:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[1,\infty)$ as

$$\phi(\sigma,x,\rho)=\begin{cases}1, & \text{if } \sigma=\rho \text{ and } x \text{ is even or odd}\\ \frac{\sigma\rho}{\sigma+\rho}, & \text{if } \sigma\neq\rho,\ \sigma \text{ and } \rho \text{ are even and } x \text{ is odd,}\\ \frac{x}{2}, & \text{if } \sigma\neq\rho,\ \sigma \text{ and } \rho \text{ are odd and } x \text{ is even,}\\ \frac{3}{2}, & \text{if } \sigma\neq\rho,\ \sigma,x \text{ and } \rho \text{ are all even or all odd },\\ \frac{\sigma+x(1+\sigma)}{\sigma(1+x)}, & \text{if } \sigma\neq\rho,\ \sigma \text{and } x \text{ are even and } \rho \text{ is odd,}\\ \frac{\rho+x(\rho+1)}{\rho(x+1)}, & \text{if } \sigma\neq\rho,\ \sigma \text{ is odd and } x \text{ and } \rho \text{ are even,}\\ \frac{2+\rho}{1+\rho}, & \text{if } \sigma\neq\rho,\ \sigma \text{ and } x \text{ are odd and } \rho \text{ is even,}\\ \frac{\sigma+1}{\sigma}, & \text{if } \sigma\neq\rho,\ \sigma \text{ is even and } x \text{ and } \rho \text{ are odd.}\end{cases}$$

Then $(\mathcal{X},\mathcal{M}_{d_\phi},\star,\phi)$ is a NERFBMLS with $\star$ product t-norm. Let $\Gamma:\mathcal{X}\to\mathcal{X}$ be a mapping defined as $\Gamma\sigma=\frac{\sigma}{2}$, for all $\sigma\in\mathcal{X}$. Observe that all the conditions of Theorem 1 hold with a UFP of 0.

Remark 11. *If we take $C(\sigma,x,\zeta)=\mathcal{M}(\sigma,x,\zeta)$ in the Ciric-type contraction in Equation (1), then we have the following Banach contraction Theorem.*

Theorem 2. *Let $(\mathcal{X},\mathcal{M}_\phi,\star,\phi)$ be a complete NERFBMLS, $\star$ is a CTN and: $\mathcal{M}_\phi$ is strictly increasing in the variable ζ and*

$$\lim_{n\to\infty}\mathcal{M}_\phi(\sigma,x,\zeta)=1,\ \forall\ \sigma,x\in\mathcal{X}.$$

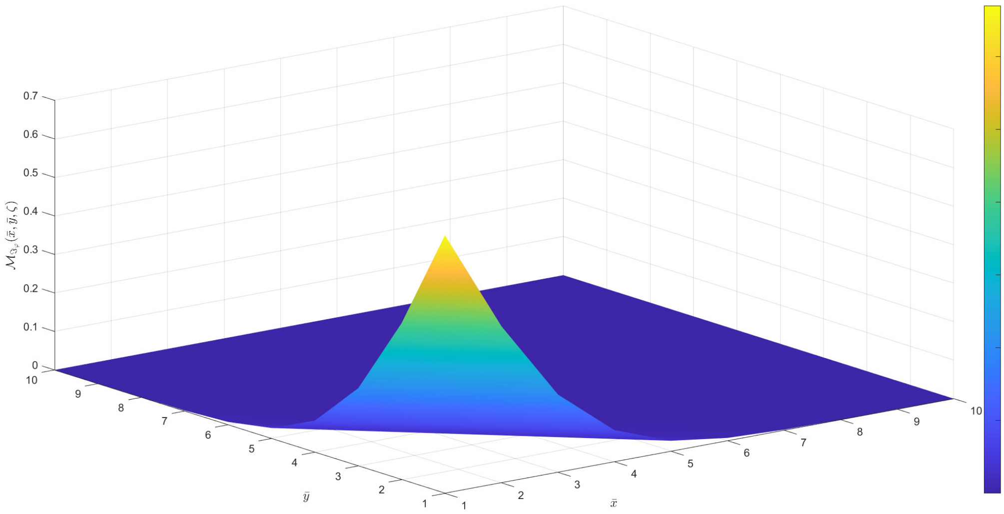

Figure 1. The graphical behavior of the fuzzy metric $\mathcal{M}_\phi(\sigma, x, \zeta)$ for a fixed $\zeta > 0$, showing its variation with respect to σ and $x \in \mathbb{N}$

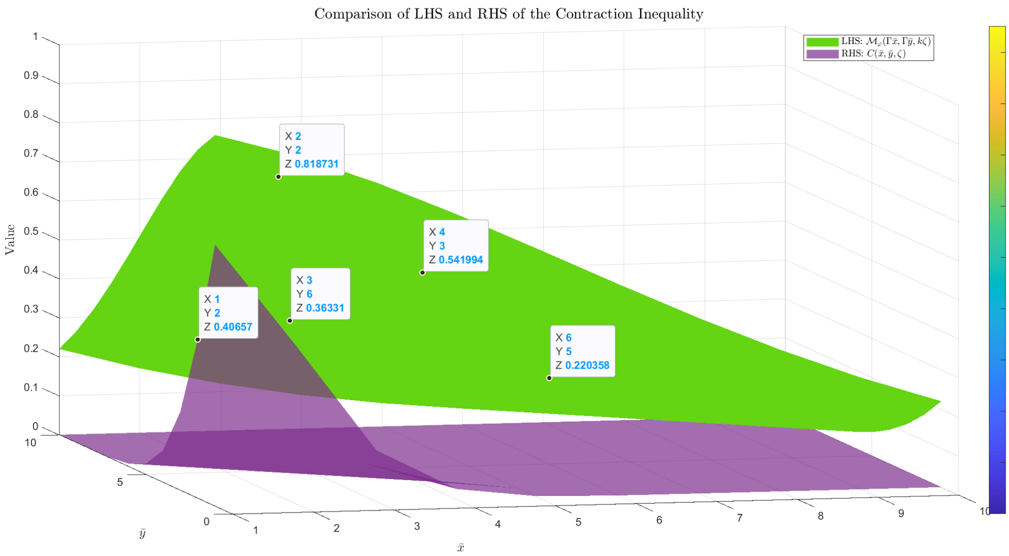

Figure 2. Graphical behavior of the left-hand side and right-hand side of a contraction $\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq C(\sigma, x, \zeta)$ in $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$

Let $\Gamma : \mathcal{X} \to \mathcal{X}$ *be a mapping satisfying:*

$$\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \mathcal{M}_\phi(\sigma, x, \zeta), \ \forall \ \sigma, x \in \mathcal{X}$$

where $0 < k < 1$*, then there is a UFP of*

Proof. By taking $C(\sigma, x, \zeta) = \mathcal{M}(\sigma, x, \zeta)$ in Theorem 1, the proof follows on the same line.

We give an example of an NERFBMLS.

Example 7. *Let* $\mathcal{X} = [0, 1]$*. We define*

$$\mathcal{M}_\phi(\sigma, x, \zeta) = \frac{\zeta}{\zeta + \max\{\sigma, x\}}$$

for all $t > 0$ *with* $\zeta_1 \star \zeta_2 = \zeta_1\zeta_2$ *and* $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, 1/k)$ *by* $\phi(\sigma, x, z) = 3(\sigma^2 + x^2 + z^2 + 1)$*, where* $k \in (0, 1)$*. Then* $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ *is a NERFBMLS. Now, define* $\Gamma : \mathcal{X} \to \mathcal{X}$ *as* $\Gamma\sigma = \frac{\sigma}{6}$*. Now, as*

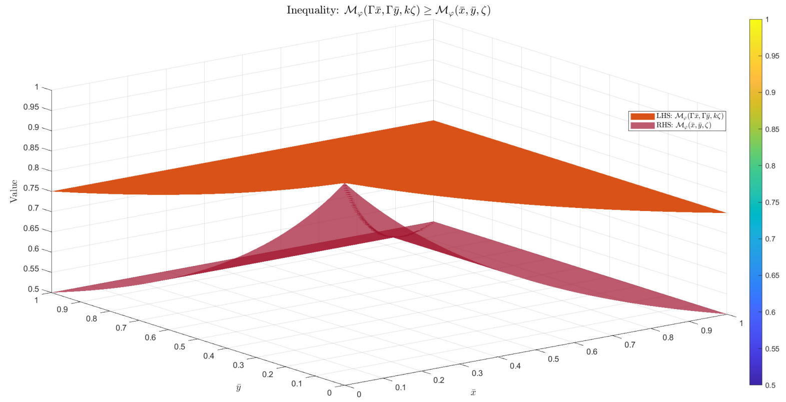

Figure 3. Graphical behavior of the left-hand side (LHS) and right-hand side (RHS) of a contraction $\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \mathcal{M}_\phi(\sigma, x, \zeta)$

$\mathcal{X}$ is complete and we have

$$\begin{aligned}\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) &= \frac{k\zeta}{k\zeta + \max\left\{\frac{\sigma}{6}, \frac{x}{6}\right\}}\\ &= \frac{\zeta}{\zeta + \max\left\{\frac{\sigma}{6k}, \frac{x}{6k}\right\}}\\ &\geq \frac{\zeta}{\zeta + \max\{\sigma, x\}}\\ &= \mathcal{M}_\phi(\sigma, x, \zeta)\end{aligned}$$

Thus, all the conditions of Theorem 2 are satisfied, and 0 is the UFP of Γ.

We also used this metric to prove a Kannan-type result and its consequences.

Theorem 3. *Let $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ be a complete NERFBMLS, $\star$ be a t-norm defined by $\star : min\{\sigma_1, \sigma_2\}$ and $\mathcal{M}_\phi$ is strictly increasing in the variable, and let $\Gamma : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying:*

$$\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) \star \mathcal{M}_\phi(x, \Gamma x, \zeta),\ \forall\ \sigma, x \in \mathcal{X}$$

where $t > 0$ and $0 < k < 1$, then there is a UFP of Γ.

Proof. Fix $\sigma_0 \in \mathcal{X}$. Let $\{\sigma_n\}$ be a sequence in $\mathcal{X}$ so that $\sigma_n = \Gamma\sigma_{n-1}$, $(n \in \mathbb{N})$. Now:

$$\begin{aligned}\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, , k\zeta) &= \mathcal{M}_\phi(\Gamma\sigma_{n-1}, \Gamma\sigma_n, k\zeta)\\ &\geq \mathcal{M}_\phi(\sigma_{n-1}, \Gamma\sigma_{n-1}, \zeta) \star \mathcal{M}_\phi(\sigma_n, \Gamma\sigma_n, \zeta)\\ &= \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta) \star \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta).\end{aligned}$$

Since $\mathcal{M}_\phi(\sigma, x, \zeta)$ is a strictly increasing function and $k\zeta < \zeta$, we cannot write:

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) \geq \mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta).$$

Therefore:

$$\begin{aligned}\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, k\zeta) &\geq \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)\\ &= \mathcal{M}_\phi(\Gamma\sigma_{n-1}, \Gamma\sigma_{n-2}, \zeta)\\ &\geq \mathcal{M}_\phi(\sigma_{n-1}, \Gamma\sigma_{n-1}, \zeta/k) * \mathcal{M}_\phi(\sigma_{n-2}, \Gamma\sigma_{n-2}, \zeta/k)\\ &\geq \mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta/k) * \mathcal{M}_\phi(\sigma_{n-2}, \sigma_{n-1}, \zeta/k)\\ &\geq \mathcal{M}_\phi(\sigma_{n-2}, \sigma_{n-1}, \zeta/k)\\ &\geq \mathcal{M}_\phi(\sigma_0, \sigma_1, \zeta/k^{n-1}), \quad \forall\zeta > 0\end{aligned}$$

or equivalently:

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta) \geq \mathcal{M}_\phi(\sigma_0, \sigma_1, \zeta/k^n), \quad \forall t > 0. \tag{3}$$

Case(1): When p is odd, say $p = 2m+1$, for $m \geq 1$, by using $(4bM_\phi)$, we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)\star \\
& \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)\star \\
& \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+2m+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right), \\
\geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+2m+1}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta}{3^3} / \prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{\zeta}{3^3} / \prod_{j=0}^{2} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})\right) \\
& \vdots \\
& \star \mathcal{M}_\phi\left(\sigma_{n+2m}, \sigma_{n+2m+1}, \frac{\zeta}{3^m} / \prod_{j=0}^{m-1} \phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m+1})\right).
\end{aligned}$$

Now, using Equation (3), we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \tfrac{\zeta}{3\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})k^n}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \tfrac{\zeta}{3\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})k^{n+1}}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \tfrac{\zeta}{3^2\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})\phi(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m+1})k^{n+2}}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \tfrac{\zeta}{3^2\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})\phi(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m+1})k^{n+3}}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m+1})k^{n+4}}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m+1})k^{n+5}}\right) \\
& \vdots \\
& \star\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^m\prod\limits_{j=0}^{m-1}\phi(\sigma_{n+2j},\sigma_{n+2j+1},\sigma_{n+2m+1})k^{n+2m}}\right).
\end{aligned}$$

Case(2): When p is even, say $p = 2m$, for $m \geq 1$, by using $(4bM_\phi)$, we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta}{3^2\prod\limits_{j=0}^{1}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta}{3^2\prod\limits_{j=0}^{1}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right) \\
& \star\mathcal{M}_\phi\left(\sigma_{n+6}, \sigma_{n+2m}, \frac{\zeta}{3^3\prod\limits_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right) \\
& \vdots \\
& \star\mathcal{M}_\phi\left(\sigma_{n+2m-2}, \sigma_{n+2m}, \frac{\zeta}{3^{m-1}\prod\limits_{j=0}^{m-2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right).
\end{aligned}$$

Again, by using Equation (3), we have:

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq M_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})k^n}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})k^{n+1}}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})k^{n+2}}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})k^{n+3}}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})k^{n+4}}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})k^{n+5}}\right)$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^3\prod_{j=0}^{2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})k^{n+6}}\right)$$

$$\vdots$$

$$\star\, \mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^{m-1}\prod_{j=0}^{m-2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})k^{n+2m-2}}\right).$$

Now, since $0 < k < 1$ and as $n \to \infty$, we have $k^n \to 0$ as $\zeta \to \infty$. Therefore, we have,

$$\lim_{n\to\infty}\mathcal{M}_\phi(\sigma_n, \sigma_{n+p}, \zeta) \geq 1 \star 1 \star \cdots \star 1 = 1.$$

This implies that $\lim_{n\to\infty}\mathcal{M}_\phi(\sigma_n, \sigma_{n+p}, t) \geq 1$. It shows that $\{\sigma_n\}$ is a CS in $\mathcal{X}$. Since $\mathcal{X}$ is complete, there exists a point $\sigma \in \mathcal{X}$ such that $\lim_{n\to\infty}\sigma_n = \sigma$. Now:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) &\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \Gamma\sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right)\\
&\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right)\\
&\quad \star \mathcal{M}_\phi\left(\Gamma\sigma_n, \Gamma\sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right)\\
&\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_n, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)k}\right).
\end{aligned}$$

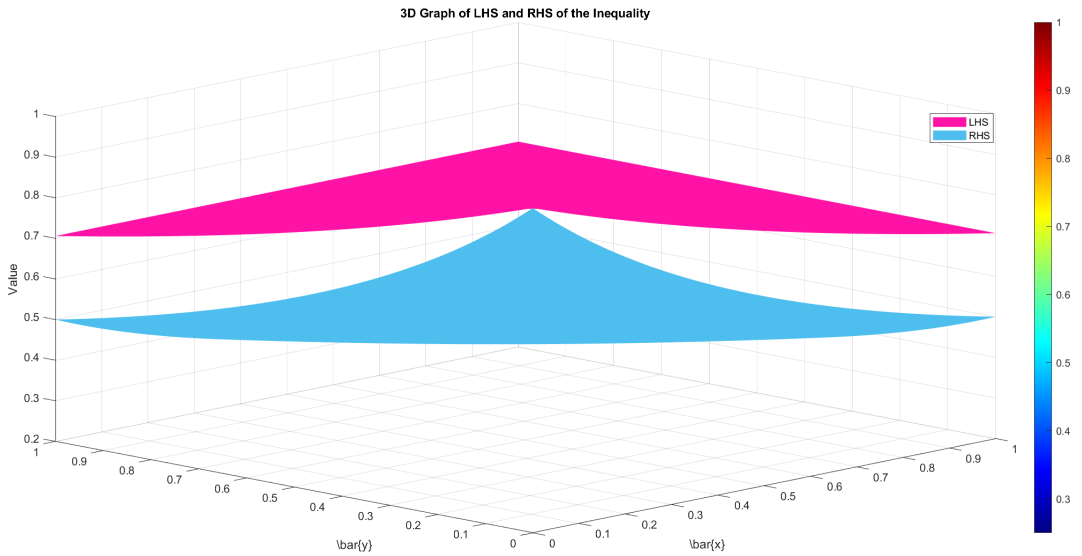

Figure 4. Graphical behavior of the left-hand side (LHS) and right-hand side (RHS) of a contraction $\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) \star \mathcal{M}_\phi(x, \Gamma x, \zeta)$

When $n \to \infty$, we have:

$$\begin{aligned} \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) &\geq \mathcal{M}_\phi(\sigma, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma)}) \star \mathcal{M}_\phi(\sigma, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma)}) \star \mathcal{M}_\phi(\sigma, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma)k}) \\ &\geq 1 \star 1 \star 1 = 1 \\ &\implies \Gamma\sigma = \sigma. \end{aligned}$$

Uniqueness: Let σ and σ^*be two FP of a mapping Γ, so $\Gamma\sigma = \sigma$ and $\Gamma x^* = x^*$ that is $M_\phi(\Gamma\sigma, \sigma, \zeta) = 1$ and $\mathcal{M}_\phi(\Gamma\sigma^*, \sigma^*, \zeta) = 1$. Now:

$$\begin{aligned} \mathcal{M}_\phi(\sigma, \sigma^*, \zeta) &= \mathcal{M}_\phi(\Gamma\sigma, \Gamma\sigma^*, \zeta) \\ &\geq \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta/k) \star \mathcal{M}_\phi(\sigma^*, \Gamma\sigma^*, \zeta/k) \\ &= 1 \star 1 = 1. \end{aligned}$$

Hence, $\sigma = \sigma^*$.

Example 8. *Let $\mathcal{X} = [0, 1]$. We define:*

$$\mathcal{M}_\phi(\sigma, x, \zeta) = \frac{\zeta}{\zeta + \max\{\sigma, x\}}$$

for all $\zeta > 0$ with $\zeta_1 \star \zeta_2 = \zeta_1\zeta_2$ and $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, 1/k)$ by $\phi(\sigma, x, \rho) = \sigma^2 + x^2 + \rho^2 + 1$, where $k \in (0, \frac{1}{2})$. Then $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ is an NERFBMLS. Now, define $\Gamma : \mathcal{X} \to \mathcal{X}$ as $\Gamma\sigma = \frac{\sigma}{8}$. Thus, all the conditions of Theorem 3 are satisfied, and 0 is the UFP of Γ.

When $\star$ is assumed to be minimum in above theorem, some consequences are presented as follows.

Corollary 1. *Let $(\mathcal{X}, D_\phi)$ be a complete NERBMLS and $\Gamma : \mathcal{X} \to \mathcal{X}$ be a self mapping, such that:*

$$D_\phi (\Gamma\sigma, \Gamma y) \leq k \max[D_\phi (\sigma, \Gamma\sigma), D_\phi(y, \Gamma y)]$$

for all $\sigma, y \in \mathcal{X}$ and some $k \in \left(0, \frac{1}{2\phi}\right)$. Then Γ has a UFP.

Proof. Define $\mathcal{M}_\phi : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0, 1]$ by:

$$\mathcal{M}_\phi (\sigma, x, \zeta) = \begin{cases} \frac{\zeta}{\zeta + D_\phi(\sigma, x)} & \text{if } \zeta > 0, \\ 0 & \text{if } \zeta = 0 \end{cases}$$

Clearly, $\mathcal{M}_\phi (\sigma, x, t)$ is a complete NERFBMLS as $(\mathcal{X}, D_\phi)$ is complete. Consider:

$$D_\phi (\Gamma\sigma, \Gamma x) \leq k \max[D_\phi (\sigma, \Gamma\sigma), D_\phi(x, \Gamma x)].$$

If:

$$\max[D_\phi(\sigma, \Gamma\sigma), D_\phi(x, \Gamma x)] = D_\phi(\sigma, \Gamma\sigma),$$

then:

$$D_\phi(\Gamma\sigma, \Gamma x) \leq k D_\phi(\sigma, \Gamma\sigma).$$

This implies by adding $k\zeta$ on both sides:

$$k\zeta + d_\phi(\Gamma\sigma, \Gamma y) \leq k\zeta + k d_\phi(\sigma, \Gamma\sigma).$$

Further:

$$\begin{aligned} \frac{k\zeta}{k\zeta + D_\phi(\Gamma\sigma, \Gamma y)} &\geq \frac{\zeta}{\zeta + D_\phi(\sigma, \Gamma\sigma)}, \\ \mathcal{M}_\phi(\Gamma\sigma, \Gamma y, k\zeta) &\geq \mathcal{M}_\phi(x, \Gamma x, \zeta). \end{aligned}$$

Similarly, for:

$$\max[D_\phi(\sigma, \Gamma\sigma), D_\phi(x, \Gamma x)] = D_\phi(x, \Gamma x),$$

we have:

$$\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, kt) \geq \mathcal{M}_\phi(x, \Gamma x, t).$$

Combining both cases, we get $\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \min\{M_\phi(\sigma, \Gamma\sigma, \zeta), \mathcal{M}_\phi(x, \Gamma x, \zeta)\}$, or $\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, k\zeta) \geq \mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) \star \mathcal{M}_\phi(x, \Gamma x, \zeta)$. Hence, all the conditions of the above theorem are satisfie to obtain a UFP of Γ.

By using the main theorem with Corollary 1 and Theorem 3, we obtained the following result.

Corollary 2. *Let $(\mathcal{X}, D_\phi)$ be a complete NERBMLS and $\Gamma : \mathcal{X} \to \mathcal{X}$ be a self-mapping, such that:*

$$D_\phi(\Gamma\sigma, \Gamma x) \leq k \max[D_\phi(\sigma, x), D_\phi(\sigma, \Gamma\sigma), D_\phi(x, \Gamma x)],$$

for all $\sigma, x \in \mathcal{X}$ and some $k \in \left(0, \frac{1}{2\phi}\right)$. Then Γ has a UFP.

We now present a theorem in an NERFBMLS using a fuzzy contractive mapping.

Definition 12. *Let $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ be an NERFBMLS. A mapping $\Gamma : \mathcal{X} \to \mathcal{X}$ is said to be a fuzzy contractive mapping if:*

$$\frac{1}{\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, \zeta)} - 1 \leq k\left\{\frac{1}{\mathcal{M}_\phi(\sigma, x, \zeta)} - 1\right\}$$

for all $\sigma, x \in \mathcal{X}$, $\zeta > 0$ and $\frac{1}{2} < k < 1$.

Theorem 4. *Let $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ be an NERFBMLS with $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, 1/k)$ and $\Gamma : \mathcal{X} \to \mathcal{X}$ be a fuzzy contractive mapping, such that $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma, x, \zeta) = 1$, then Γ has a UFP.*

Proof. Fix $\sigma_0 \in \mathcal{X}$. Let $\{\sigma_n\}$ be a sequence in $\mathcal{X}$ so that:

$$\sigma_n = \Gamma^n \sigma_0 = \Gamma\sigma_{n-1}, \ (n \in \mathbb{N}).$$

For $\zeta > 0$ and using fuzzy contractive mapping, we have:

$$\begin{aligned} \frac{1}{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta)} - 1 &= \frac{1}{\mathcal{M}_\phi(\Gamma\sigma_{n-1}, \Gamma\sigma_n, \zeta)} - 1 \\ &\leq k\{\frac{1}{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta)} - 1\}. \end{aligned}$$

We have:

$$\begin{aligned} \frac{1}{\mathcal{M}_\phi(\sigma_n, \sigma_{n+1}, \zeta)} &\leq \frac{k}{\mathcal{M}_\phi(\sigma_{n-1}, \sigma_n, \zeta)} + 1 - k \\ &= \frac{k}{\mathcal{M}_\phi(\Gamma\sigma_{n-2}, \Gamma\sigma_{n-1}, \zeta)} + 1 - k \\ &\leq k\left[\frac{k}{\mathcal{M}_\phi(\sigma_{n-2}, \sigma_{n-1}, \zeta)} + 1 - k\right]. \end{aligned}$$

So we have:

$$\frac{k}{\mathcal{M}_\phi(\sigma_{n-1},\sigma_n,\zeta)} \leq \frac{k^2}{\mathcal{M}_\phi(\sigma_{n-2},\sigma_{n-1},\zeta)} + k(1-k) + (1-k).$$

Continuing in this way, we have:

$$\begin{aligned}\frac{1}{\mathcal{M}_\phi(\sigma_n,\sigma_{n+1},\zeta)} - 1 &\leq \frac{k^n}{\mathcal{M}_\phi(\sigma_0,\sigma_1,\zeta)} + k^{n-1}(1-k) + k^{n-2}(1-k) + \cdots + k(1-k) + (1-k) \\ &= \frac{k^n}{\mathcal{M}_\phi(\sigma_0,\sigma_1,\zeta)} + (k^{n-1} + k^{n-2} + \cdots + k + 1)(1-k) \\ &= \frac{k^n}{\mathcal{M}_\phi(\sigma_0,\sigma_1,\zeta)} + (1-k^n).\end{aligned}$$

We have:

$$\mathcal{M}_\phi(\sigma_n,\sigma_{n+1},\zeta) \geq \frac{1}{\frac{k^n}{\mathcal{M}_\phi(\sigma_0,\sigma_1,\zeta)} + (1-k^n)}. \tag{4}$$

In a similar way , we can prove that:

$$\frac{1}{\frac{k^{n-2}}{\mathcal{M}_\phi(\sigma_0,\sigma_2,\zeta)} + (1-k^{n-2})} \leq \mathcal{M}_\phi(\sigma_{n-2},\sigma_n,\zeta).$$

Let σ_n be a sequence in $\mathcal{X}$, then we have the following two cases:
Case (1): When p is odd , say $p = 2m+1$, for $m \geq 1$, by using $(4bM_\phi)$, we have:

$$\begin{aligned}\mathcal{M}_\phi(\sigma_n,\sigma_{n+2m+1},\zeta) \geq \mathcal{M}_\phi&\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+1},\sigma_{n+2},\frac{\zeta/3}{\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+2},\sigma_{n+2m+1},\frac{\zeta/3}{\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})}\right) \\ \geq \mathcal{M}_\phi&\left(\sigma_n,\sigma_{n+1},\frac{\zeta/3}{\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+1},\sigma_{n+2},\frac{\zeta/3}{\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+2},\sigma_{n+3},\frac{\zeta}{3^2\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})\,\phi(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+3},\sigma_{n+4},\frac{\zeta}{3^2\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})\,\phi(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m+1})}\right) \\ &\star\, \mathcal{M}_\phi\left(\sigma_{n+4},\sigma_{n+2m+1},\frac{\zeta}{3^2\phi(\sigma_n,\sigma_{n+1},\sigma_{n+2m+1})\,\phi(\sigma_{n+2},\sigma_{n+3},\sigma_{n+2m+1})}\right).\end{aligned}$$

$$
\begin{aligned}
&\geq \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{t/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m+1})}\right)\\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+6}, \sigma_{n+2m+1}, \frac{\zeta/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m+1})}\right)\\
&\qquad \vdots\\
&\quad \star \mathcal{M}_\phi\Bigg(\sigma_{n+2m}, \sigma_{n+2m+1},\\
&\qquad \frac{\zeta/3^m}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})\cdots\phi(\sigma_{n+2m-2}, \sigma_{n+2m-1}, \sigma_{n+2m+1})}\Bigg).
\end{aligned}
$$

Using Equation (4), we have:

$$
\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq\ & \frac{1}{\frac{k^n}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)} + (1-k^n)}\\
& \star \frac{1}{\frac{k^{n+1}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})}\right)} + (1-k^{n+1})}\\
& \star \frac{1}{\frac{k^{n+2}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})}\right)} + (1-k^{n+2})}\\
& \vdots\\
& \star \frac{1}{\frac{k^{n+p-1}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^m\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m+1})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m+1})\phi(\sigma_{n+2m-2}, \sigma_{n+2m-1}, \sigma_{n+2m+1})}\right)} + (1-k^{n+p-1})}.
\end{aligned}
$$

Taking limit $n \to \infty$, we have $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma_n, \sigma_{n+2m+1}, \zeta) \geq 1 \star 1 \star 1 \star ... \star 1 = 1$.
Case (2): When p is even, say $p = 2m$, for $m \geq 1$, by using $(4bM_\phi)$ we have:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right)\star \\
& \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right)\star \\
& \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+2m}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
\geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+2m+1}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
\geq\ & \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\zeta/3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+3}, \sigma_{n+4}, \frac{\zeta/3^2}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+4}, \sigma_{n+5}, \frac{\zeta/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+5}, \sigma_{n+6}, \frac{\zeta/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m})}\right) \\
& \star \mathcal{M}_\phi\left(\sigma_{n+6}, \sigma_{n+2m}, \frac{\zeta/3^3}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})\phi(\sigma_{n+4}, \sigma_{n+5}, \sigma_{n+2m})}\right) \\
& \cdot \star \mathcal{M}_\phi\left(\sigma_{n+2m-2}, \sigma_{n+2m}, \frac{\zeta/3^{m-1}}{\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})\cdots\phi(\sigma_{n+2m-4}, \sigma_{n+2m-3}, \sigma_{n+2m})}\right).
\end{aligned}$$

Using Equation (4), we have:

$$\mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq \frac{1}{\frac{k^n}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right)} + (1-k^n)}$$
$$\star \frac{1}{\frac{k^{n+1}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})}\right)} + (1-k^{n+1})}$$
$$\star \frac{1}{\frac{k^{n+2}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^2\phi(\sigma_n, \sigma_{n+1}, \sigma_{n+2m})\phi(\sigma_{n+2}, \sigma_{n+3}, \sigma_{n+2m})}\right)} + (1-k^{n+2})}$$
$$\vdots$$
$$\star \frac{1}{\frac{k^{n+p-1}}{\mathcal{M}_\phi\left(\sigma_0, \sigma_1, \frac{\zeta}{3^{m-1}\prod_{j=0}^{m-2}\phi(\sigma_{n+2j}, \sigma_{n+2j+1}, \sigma_{n+2m})}\right)} + (1-k^{n+p-1})}.$$

Taking the limit $n \to \infty$, we have $\lim_{n\to\infty} \mathcal{M}_\phi(\sigma_n, \sigma_{n+2m}, \zeta) \geq 1 \star 1 \star 1 \star \dots \star 1 = 1$. This shows that $\{\sigma_n\}$ is a CS in $\mathcal{X}$. Since $\mathcal{X}$ is complete, there exists a point $\sigma \in \mathcal{X}$ such that $\lim_{n\to\infty} \sigma_n = \sigma$. Consider:

$$\begin{aligned}
\frac{1}{\mathcal{M}_\phi(\Gamma\sigma_n, \Gamma\sigma, \zeta)} - 1 &\leq k\left[\frac{1}{\mathcal{M}_\phi(\sigma_n, \sigma, \zeta)} - 1\right] \\
&= \frac{k}{\mathcal{M}_\phi(\sigma_n, \sigma, \zeta)} - k \\
\frac{1}{\mathcal{M}_\phi(\Gamma\sigma_n, \Gamma\sigma, \zeta)} &\leq \frac{k}{\mathcal{M}_\phi(\sigma_n, \sigma, \zeta)} + 1 - k.
\end{aligned}$$

So:

$$\mathcal{M}_\phi(\Gamma\sigma_n, \Gamma\sigma, \zeta) \geq \frac{1}{k/\mathcal{M}_\phi(\sigma_n, \sigma, \zeta) + (1-k)}.$$

Now:

$$\begin{aligned}
\mathcal{M}_\phi(\sigma, \Gamma\sigma, \zeta) &\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\sigma_n, \sigma_{n+1}, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_{n+1}, \Gamma\sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \\
&\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\Gamma\sigma_{n-1}, \Gamma\sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \\
&\quad \star \mathcal{M}_\phi\left(\Gamma\sigma_n, \Gamma\sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \\
&\geq \mathcal{M}_\phi\left(\sigma, \sigma_n, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)}\right) \star \mathcal{M}_\phi\left(\sigma_n, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)(1-k)}\right) \\
&\quad \star \mathcal{M}_\phi\left(\sigma_n, \sigma, \frac{\zeta/3}{\phi(\sigma, \Gamma\sigma, \sigma_n)k}\right).
\end{aligned}$$

In the limiting case, where $n \to \infty$, we have $\Gamma\sigma = \sigma$. Then σ is an FP of Γ.

Uniqueness: Let σ and σ^* be two FPs of a mapping Γ, so $\Gamma\sigma = \sigma$ and $\Gamma\sigma^* = \sigma^*$, that is, $\mathcal{M}_\phi(\Gamma\sigma, \sigma, \zeta) = 1$ and $\mathcal{M}_\phi(\Gamma\sigma^*, \sigma^*, \zeta) = 1$.

$$\begin{aligned}
\frac{1}{\mathcal{M}_\phi(\sigma, \sigma^*, \zeta)} - 1 &= \frac{1}{\mathcal{M}_\phi(\Gamma\sigma, \Gamma\sigma^*, \zeta)} - 1 \\
&\leq k\left[\frac{1}{\mathcal{M}_\phi(\sigma, \sigma^*, \zeta)} - 1\right] \\
&< \frac{1}{\mathcal{M}_\phi(\sigma, \sigma^*, \zeta)} - 1
\end{aligned}$$

It is a contradiction. Thus, σ is the only FP of Γ.

Example 9. *Let $\mathcal{X} = \mathbb{N}$. Consider $\mathcal{M}_\phi : \mathcal{X} \times \mathcal{X} \times [0, \infty) \to [0, 1]$, such that:*

$$\mathcal{M}_\phi(\sigma, x, \zeta) = \frac{\zeta}{\zeta + \max\{\sigma, x\}}.$$

Also, consider $\phi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [1, \infty)$ as:

$$\phi(\sigma, x, z) = \begin{cases} 1, & \text{if } \sigma = z \text{ and } x \text{ is even or odd} \\ \frac{\sigma z}{\sigma + z}, & \text{if } \sigma \neq z,\ \sigma \text{ and } z \text{ are even and } x \text{ is odd,} \\ \frac{x}{2}, & \text{if } \sigma \neq z,\ \sigma \text{ and } z \text{ are odd and } x \text{ is even,} \\ \frac{3}{2}, & \text{if } \sigma \neq z,\ \sigma, x \text{ and } z \text{ are all even or all odd ,} \\ \frac{\sigma + x(1+\sigma)}{\sigma(1+x)}, & \text{if } \sigma \neq z,\ \sigma \text{and } x \text{ are even and } z \text{ is odd,} \\ \frac{z + x(z+1)}{z(x+1)}, & \text{if } \sigma \neq z,\ \sigma \text{ is odd and } x \text{ and } z \text{ are even,} \\ \frac{2+z}{1+z}, & \text{if } \sigma \neq z,\ \sigma \text{ and } x \text{ are odd and } z \text{ is even,} \\ \frac{\sigma+1}{\sigma}, & \text{if } \sigma \neq z,\ \sigma \text{ is even and } x \text{ and } z \text{ are odd.} \end{cases}$$

Then $(\mathcal{X}, \mathcal{M}_\phi, \star, \phi)$ is an NERFBMLS with $\star$ product t-norm. Now, define $\Gamma : \mathcal{X} \to \mathcal{X}$ as $\Gamma x = \frac{\sigma}{6}$. As $\mathcal{X}$ is complete, we have:

$$\begin{aligned}
\frac{1}{\mathcal{M}_\phi(\Gamma\sigma, \Gamma x, \zeta)} - 1 &= \frac{1}{\frac{\zeta}{\zeta + \max\left\{\frac{\sigma}{6}, \frac{x}{6}\right\}}} - 1 \\
&\leq \frac{1}{\frac{\zeta}{\zeta + \max\{\sigma, x\}}} - 1 \\
&\leq k\left\{\frac{1}{\frac{\zeta}{\zeta + \max\{\sigma, x\}}} - 1\right\} \\
&= k\left\{\frac{1}{\mathcal{M}_\phi(\sigma, x, \zeta)} - 1\right\}
\end{aligned}$$

Thus, all the conditions of Theorem 4 are satisfied and 0 is the UFP of Γ.

3. Applications

To strengthen the discussion of our Volterra integral equation example, we now provide explicit conditions under which it satisfies the assumptions of our main theorem: Consider the Volterra integral equation,

$$\sigma(u) = f(u) + \int_0^u F(u,r,\sigma(r))dr. \forall c \in [0,1] \quad (5)$$

where $u \in I = [0,1]$. We define an NERFBMLS as $\phi(\sigma, x, z) = 3(\sigma + x + z + 1)$ and:

$$\mathcal{M}(\sigma, x, t) = e^{-\frac{\sup_{u\in[0,1]} |(\sigma(u)-x(u))|^2}{t}}$$

for all $\zeta > 0$ and $\sigma, x \in C(I, \mathbb{R})$ (the space of all continuous real-valued functions defined on I), with the product t-norm.

Theorem 5. *Consider an integral operator defined on $C(I,\mathbb{R})$ as:*

$$\Gamma\sigma(u) = f(u) + \int_0^u F(u,r,\sigma(r))dr, f \in C(I,\mathbb{R})$$

where F satisfies the following conditions:
(i): There exists $f : I \times I \to [0,\infty)$ such that $f \in L^1(I,\mathbb{R})$.
(ii): For all $\sigma, x \in C(I,\mathbb{R})$ and $r, u \in I$, we have:

$$|F(u,r,\sigma(r)) - F(u,r,x(r))|^2 \leq f^2(u,r)|\sigma(r) - x(r)|$$

where:

$$\sup_{u\in[0,1]} \int_0^u f^2(u,r)dr \leq k < 1$$

then the integral Equation (5) has a unique solution.

Proof. Consider:

$$\begin{aligned}
\mathcal{M}_\phi(\Gamma\sigma(u), \Gamma x(u), k\zeta) &= e^{-\frac{\sup_{u\in[0,1]} |(\Gamma\sigma(u)-\Gamma x(u))|^2}{k\zeta}} \\
&= e^{-\frac{\sup_{u\in[0,1]} |(\int_0^u F(u,r,\sigma(r)) - \int_0^u F(u,r,x(r))dr)|^2}{kt}} \\
&\geq e^{-\frac{k|\sigma(r)-x(r)|^2}{kt}} \\
&= e^{-\frac{|\sigma(r)-x(r)|^2}{t}} \\
&\geq e^{-\frac{\sup_{r\in[0,1]} |(\sigma(r)-x(r))|^2}{t}} \\
&= \mathcal{M}_\phi(\sigma, x, \zeta),
\end{aligned}$$

for all $\sigma, x \in C([0,a],\mathbb{R})$ and $0 < k < 1$. Since all conditions of the Banach contraction theorem are satisfied, the given integral Equation (5) has a unique solution.

Example 10. *Consider the differential equation:*

$$x''(u) - x(u) = cosu, x(0) = 0, x'(0) = 0$$

which gives a solution as:

$$x(u) = 1 - cosu - \int_0^u (u-r)x(r)dr$$

Here, $F(u,r,\sigma(r)) = (u-r)x(r)$ and take $u - r = f(u,r)$. Now:

$$|F(u,r,\sigma(r)) - F(u,r,x(r))|^2 = |(u-r) \\ \sigma(r) - (u-r)x(r)| = (u-r)|\sigma(r) - x(r)|.$$

Also, we have:

$$\sup_{u\in[0,1]} \int_0^u f^2(u,r)dr \leq 1.$$

Now:

$$\begin{aligned}
\mathcal{M}_\phi(\Gamma x(u), \Gamma x(u), k\zeta) &= e^{-\frac{\sup_{u\in[0,1]} |(\Gamma\sigma(u)-\Gamma x(u))|^2}{k\zeta}} \\
&= e^{-\frac{\sup_{u\in[0,1]} |(\int_0^u F(u,r,\sigma(r)) - \int_0^u F(u,r,x(r))dr)|^2}{kt}} \\
&\geq e^{-\frac{|\sigma(r)-x(r)|^2 \sup_{u\in[0,1]} |\int_0^u f^2(u,r)}{k\zeta}} \\
&= e^{-\frac{|\sigma(r)-x(r)|^2}{3k\zeta}}, \forall\ k \in [1/3, 1). \\
&\geq e^{-\frac{|\sigma(r)-x(r)|^2}{\zeta}} \\
&\geq e^{-\frac{\sup_{r\in[0,1]} |(\sigma(r)-x(r))|^2}{\zeta}} \\
&= \mathcal{M}_\phi(\sigma, x, \zeta).
\end{aligned}$$

Since all the conditions of the theorem are satisfied, the integral equation has a unique solution.

4. Conclusion

In this work, we presented an NERFBMLS. We provided some basic definitions and examples to illustrate that the NERFBMLS has higher generalizability than the previous results. Several FPTs were proved for mappings that satisfy fuzzy Ćirić-type and Banach-type contractions. This study expands classical results by employing a more generalized framework, thereby broadening the applicability of FPT in an NERFBMLS. Additionally, a series of remarks, corollaries, and illustrative non-trivial examples that support and validate our results were provided. Moreover, we presented an application demonstrating the solutions to specific integral equations. This study improves our understanding of FMS and lays the groundwork for future research on generalized fuzzy distance structures. Researchers can expand this subject to include new extended rectangular BMS, new extended fuzzy BMLS, and other sophisticated mathematical structures. In future work, these results may be extended to multi-valued mappings and hybrid contractions in NERFBMLS. Moreover, the developed framework can be applied to analyze fuzzy differential and integral equations, offering potential for modeling uncertainty in dynamic systems.

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Conflict of interest

The authors declare they have no competing interests.

Author contributions

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