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RESEARCH ARTICLE

# On different generalized interpolative proximal-type contractions in metric spaces with applications

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ABSTRACT

In this work, we establish the conditions for ensuring the existence and uniqueness of common best proximity points for non-self-mappings defined on the general metric spaces. A unified theoretical framework is formulated to cover a broad class of contraction mappings. We describe the required conditions on the real-valued functions $(\aleph, \Phi) : [0, \infty) \to \mathbb{R}$ and verify that these secure the existence of common best proximity points for $(\aleph, \Phi)-$interpolative contractions in complete metric spaces. The study further extends this concept by examining multiple forms of interpolative proximal-type contractions, such as proximal, Ćirić—Reich—Rus, Kannan, and Hardy-Rogers variants, through the use of the auxiliary functions $(\aleph, \Phi)$. Several illustrated examples are included to demonstrate the applicability of our findings. Finally, we conclude with an application involving a nonlinear fractional differential equation, showing that it fully satisfies the assumption of our main result.



## 1. Introduction and preliminaries

The classical fixed-point (FP) problem seeks a solution of a non-linear equations of the form $\tilde{\mathtt{f}}(\tilde{\mathtt{l}}) = \tilde{\mathtt{l}}$ where $\tilde{\mathtt{f}}$ is a self-map. Although fixed-point techniques are widely used, a unique solution is not always guaranteed. In situations where FPs do not exist, the best approximation and proximity-point principles provide meaningful alternatives by identifying points that minimize the distance between a mapping and the underlying set. The foundational contribution to this area is due to Fan's, [1] whose 1969 best approximation theorem has become a central tool in optimization and fixed-point theory. Since then, numerous researchers have extended these ideas to more general settings and broader mathematical structures. The Banach contraction principle, introduced by Banach [2] gave a fundamental concept of the Banach contraction principle in FP theory which guarantees the existence and uniqueness of a FP for a self-mapping defined on a complete metric space. Later, Kannan [3] proposed a distinct type of contraction that does not rely on the classical Lipschitz condition.

The best proximity point (BPP) theory addresses problems in which exact solutions cannot be found, typically because the involved mapping is not a self-map. In such cases, a BPP serves as the closest feasible approximation to a FP. These ideas are relevant in functional analysis, optimization, and game theory. When the mapping is a

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self-map, FPs and BPP coincide, showing that BPP theory naturally generalizes classical fixed-point analysis.

Several significant developments have strengthened the field in recent years. Basha [4] examined the role of BPPs in global optimization, with a particular emphasis on multi-objective problems, and demonstrated how these principles help manage competing performance criteria. Deep and Batra [5] further advanced the area by introducing the notion of common best proximity points (CBPP) through the use of proximal $F$-weak dominance, thereby widening the applicability of proximity-based methods in complete MS (CMS). Later Mondel and Dey [6] established BPP results within CMS, and generalize earlier frameworks. Younis and Abdou [7] developed the fuzzy contractions, graph mappings, and Kannan mappings to create a brand-new notion known as "Kannan-graph-fuzzy contraction."

Furthermore, Basha [8] made further progress and explored globally minimal solutions, while Shazad et al. [9] extended CBPP concepts to highlight application global optimization. Recently, Altun and Aysenur [10] proposed new fixed-point structures motivated by applications in nonlinear analysis and mathematical modeling and introduced refined contraction techniques that yield stronger proximity outcomes. Adhikari [11] proposed a novel technique for solving Rhodes' discontinuity problem by exploiting the features of a self-mapping that has an FP but is not continuous at that point within a partial MS (MS). Moreover, they investigated some geometric properties of FP under interpolative-type contractions and establish a few results related to fixed-discs and fixed-circles. Proinov [12] broadened fixed-point theory through auxiliary functions, generating a versatile class of results with implications in optimization and game theory. Malkawi [13] presented new FP theorems in MR–MS using integral-type contractions. Makran et al. [14] proved a general common FP theorem for a pair of multi-valued mappings in Hausdorff modular fuzzy $b$–MS. Ishtiaq et al. [15] adapted Proinov-type contractions to bipolar and fuzzy bipolar MS, enabling treatment of more general fixed-point configurations. An additional contribution by Karapinar et al. [16] emphasized the influence of various generalized contractions on both the FP and the BPP theorems. Younis et al. [17] presented the novel concept of graphical bipolar MS and proved that every bipolar metric space is graphical bipolar metric space but the converse may not be true.

Recent studies further support the findings of the previous studies. Ishtiaq et al. [18] introduced functional conditions for $(\psi, \Phi) : (0, 1] \to \mathbb{R}$ to investigate $(\aleph, \Phi)$ fuzzy interpolative contractions in fuzzy MS (FMS). Saleem et al. [19] established BPP theorems for interpolative Ćirić–Reich–Rus-type contractions using $\omega-$admissibility in CMS. Deng et al. [20] analyzed $(\tilde{\mathtt{p}} - \psi)$ interpolative proximal contractions and identified criteria ensuring BPP existence in MS. Ishtiaq et al. [21] formulated fuzzy versions of iterative mappings and derived corresponding BPP conditions in generalized fuzzy $(\aleph, \Phi)$-iterative settings. Younis and Bahuguna [22] presented the notion of controlled graphical metric type spaces, which integrates the concepts of controlled metric type spaces, extended $b$ metric type spaces, metric type spaces, and graphical type spaces. Further, Ishtiaq et al. [23] examined iterative conditions yielding existence and uniqueness of BPPs results for iterative conditions within a complete fuzzy multiplicative metric space. A complementary perspective appears in Joonaghany et al. [24] who employed $\Psi$-simulation functions to study Suzuki-type contractions, thus expanding the framework of classical contractive mappings. Additional related works can be found in Refs. [25–27]

To prepare for the main results, we recall essential notation and preliminaries. Throughout the paper, $\mathcal{W}$ and $\mathcal{Y}$ denote subsets of a metric space $(\mathcal{V}, \mathfrak{N})$ and $\aleph$ and $\Phi$ are mappings from $[0, \infty)$ to $\mathbb{R}$. We define

$$\mathcal{W}_0 := \Big\{ \mathtt{x} \in \mathcal{W} : \mathfrak{N}(\mathtt{x}, \mathtt{s}) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \text{ for some } \mathtt{s} \in \mathcal{Y} \Big\},$$

$$\mathcal{Y}_0 := \Big\{ \mathtt{z} \in \mathcal{Y} : \mathfrak{N}(\mathtt{x}, \mathtt{z}) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \text{ for some } \mathtt{x} \in \mathcal{W} \Big\}. \tag{1}$$

Recently, Younis and Öztürk [28] introduced the BPP for proximal contractions in the context of extended $b$-MS. Karapinar et al. [29] found some FP results for interpolative Hardy–Rogers and Boyd–Wong type contractions. Babu and Koduru [30] introduced the concept of interpolative contractions for a pair of maps in $b$-metric space and utilized the idea of interpolation in complete $b$-MSs to prove the existence and uniqueness of common FP results. Younis et al. [31] found the concept of some novel convergence results to the Helmholtz problem with mixed boundary conditions and demonstrated the existence and uniqueness of the solution by applying graph-controlled contractions. Ma et al. [32] introduced the concept of topology optimization of cooling elements for worm wheel gear grinding machine tool beds under non-uniform heat sources. Xu et al. [33] developed an integrated thermodynamics economy multi-physics coupling framework that quantifies cross-scale interactions among thermal, electrical, gas, and liquid. Yue [34] studied the existence of nonuniform polynomial dichotomy for discrete evolution families in Banach spaces. Some necessary and sufficient conditions for the nonuniform polynomial dichotomy concept of discrete evolution families with respect to invariant projection sequences and strongly invariant projection sequences are obtained, respectively.

Jahangeer et al. [35] presented several types of interpolative proximal contraction mappings, including Reich–Rus–Ćirić type interpolative-type contractions and Kannan-type interpolative-type contractions in the setting of bipolar MS. Ahmad et al. [36] obtained FP results using an interpolative extended F B-Ćirić-–Reich-–Rus contraction framework. They proved the validity of their FP results by showing examples that demonstrate these contractions were effective. Ali et al. [37] found the notion of F-$\alpha$-proximal contractions for Hardy–Rogers-type mappings as well as for Ä†iriÄ‡-type mappings. Unni et al. [38] introduced and analyzed new concepts such as proximally reciprocal continuous, proximally weak reciprocal continuous, and R-proximally weak commuting of types $M\Lambda$ and $M\Gamma$ for non-self mappings. Vaithilingam and Anisha [39] provided sufficient conditions for the existence of a BPP of F, without requiring $X$ to be strictly convex or the sets $A$ and $B$ to be compact. Pragadeeswarar and Gopi [40] introduced a new concept called proximal E.A. property for single and multi-valued mappings. They proved the existence of a proximally coincidence point for such a class of mappings. Girgin [41] investigated and encompassed a meticulous examination of FP results within the context of non-Archimedean modular MS, which are characterized by their distinctive structural properties that diverge from those of conventional MS.

In particular, Table 1 illustrates how the proposed interpolative proximal-type contraction extends existing approaches by combining interpolation techniques with proximal structures in a generalized setting.

We also recall classical definitions of proximal commutative and proximal dominance, approximate compactness, and CBPP, adapting the formulations from Refs. [4,8]

**Definition 1.** We say that $\mathfrak{P}: \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R}: \mathcal{W} \to \mathcal{Y}$ *commute proximally* (in short, CP) if the following condition is satisfied: [8]

$$[\mathfrak{N}(\mathtt{z}, \mathfrak{P}\mathrm{e}) = \mathfrak{N}(\mathrm{y}, \mathfrak{R}\mathrm{e}) = \mathfrak{N}(\mathcal{W}, \mathcal{Y})] \Rightarrow \mathfrak{P}\mathtt{y} = \mathfrak{R}\mathtt{z}$$

for all $\mathrm{e}, \mathtt{z}, \mathtt{y} \in \mathcal{W}$.

**Definition 2.** We say that $\mathfrak{R}: \mathcal{W} \to \mathcal{Y}$ *dominates a mapping* $\mathfrak{P}: \mathcal{W} \to \mathcal{Y}$ *proximally* if $\exists\ 0 \le \tilde{\mathrm{p}} < 1$ s.t. [8]

$$\mathfrak{N}(\mathrm{x}_1, \mathrm{x}_2) \le \tilde{\mathrm{p}}\mathfrak{N}(\mathtt{z}_1, \mathtt{z}_2),$$

for all $\mathrm{x}_1, \mathrm{x}_2, \mathtt{z}_1, \mathtt{z}_2, \mathrm{e}_1, \mathrm{e}_2 \in \mathcal{W}$, and also, we have

$$\mathfrak{N}(\mathrm{x}_1, \mathfrak{P}(\mathrm{e}_1)) = \mathfrak{N}(\mathrm{x}_2, \mathfrak{P}(\mathrm{e}_2)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}(\mathtt{z}_1, \mathfrak{R}(\mathrm{e}_1)) = \mathfrak{N}(\mathtt{z}_2, \mathfrak{R}(\mathrm{e}_2)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

**Definition 3.** We say that the set $\mathcal{W}$ is *approximately compact (APC) w.r.t.* $\mathcal{Y}$ whenever every sequence $\{\mathfrak{N}_n\} \subset \mathcal{W}$ satisfying $\mathfrak{N}(\mathrm{y}, \mathfrak{N}_n) \to \mathfrak{N}(\mathrm{y}, \mathcal{W})$, for some $\mathtt{y} \in \mathcal{Y}$, admits a convergent subsequence. [4]

**Definition 4.** Assume $(\mathcal{V}, \mathfrak{N})$ is is a metric space and $\mathcal{W}, \mathcal{Y}$ are the nonempty subsets of $\mathcal{V}$. Consider two mappings $\mathfrak{P}: \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R}: \mathcal{W} \to \mathcal{Y}$. A point $\mathtt{s}^* \in \mathcal{W}$ is called a CBPP of $\mathfrak{R}$ and $\mathfrak{P}$ if [4]

$$\mathfrak{N}(\mathrm{e}^*, \mathfrak{P}\mathrm{e}^*) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\mathrm{e}^*, \mathfrak{R}\mathrm{e}^*).$$

This work introduces several new classes of interpolative proximal-type contractions, including the $(\aleph, \Phi)$-proximal contractions, $(\aleph, \Phi)$–Ciri'c–Reich–Rus' contractions, $(\aleph, \Phi)$-Kannan contractions, and $(\aleph, \Phi)$-Hardy–Rogers contractions.

These contractions generalize familiar fixed-point principles through an interpolative mechanism that enhances their flexibility. Furthermore, we establish the existence and uniqueness of CBPP results under these generalized mappings. The theoretical developments are supplemented with nontrivial examples and an application in a fractional differential equation, illustrating the breadth of applicability of the proposed framework.

The remainder of the paper is organized as follows. Section 2 develops the theory of generalized interpolative contractions and presents main results concerning CBPP for several $(\aleph, \Phi)$interpolative contraction types, including the proximal, Kannan, Ciric–Reich–Rus, and Hardy–Rogers variants. Various examples are presented to substantiate these concepts and results. Section 3 provides applications in fractional analysis. Section 4 concludes with a summary of the major contributions.

## 2. Main Results

We begin by formulating several classes of interpolative contractions, including the $(\aleph, \Phi)$-proximal contractions, $(\aleph, \Phi)$-Ćirić-–Reich-–Rus type, $(\aleph, \Phi)$-Kannan type, and $(\aleph, \Phi)$Hardy-Rogers type. Each class is introduced with a precise definition followed by explanation that highlight the role played by the auxiliary functions $(\aleph, \Phi)$ in shaping the contractive behavior of the mappings. To clarify these concepts, we present

**Table 1.** Comparison of contraction-type conditions and corresponding frameworks

| **Contraction type** | **Underlying space** | **Control function / condition** | **Representative reference** |
|---|---|---|---|
| Banach contraction | MS | Constant contraction factor | Banach [2] |
| Kannan contraction | MS | Distance between images and points | Kannan [3] |
| Interpolative contraction | MS | Interpolative control function | Karapinar et al. [29] |
| Proximal contraction | MS with a binary relation | Proximal distance condition | Basha et al. [4] |
| Generalized proximal-type contraction | Generalized MS | Control via auxiliary function | Recent literature [21, 23] |
| Interpolative proximal-type contraction | Generalized MS | Interpolation and proximal structure | Present work |
| Application to fractional differential equations | Function spaces | Fixed-point framework | Present work |

illustrative examples and discuss their relevance to the existence of CBPP. These examples demonstrate how the generalized interpolative structure broadens the scope of proximity-type results in metric settings.

### 2.1. Proximal contraction

This section extends the contribution Refs. [4–6,8,9] by developing the results $(\aleph, \Phi)$-proximal contractions under the non-self-mappings. This framework is well suited for problem involving interactions between two distinct subsets of a metric space, or more generally, for situations where mappings operate across coupled structures. By employing $(\aleph, \Phi)$-proximal contractions, we derive new conditions that guarantee the convergence and stability of iterative procedures.

**Definition 5.** *Assume* $(\mathcal{V}, \mathfrak{N})$ *is an and* $\mathcal{W}, \mathcal{Y}$ *be the nonempty subsets of* $\mathcal{V}$. *The mappings* $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$ *and* $\mathfrak{P} : \mathcal{W} \to \mathcal{Y}$ *are known as* $(\aleph, \Phi)$-proximal *if for all* $\mathrm{x}_1, \mathrm{x}_2, \mathrm{s}_1, \mathrm{s}_2, \mathrm{e}_1, \mathrm{e}_2 \in \mathcal{W}$, *the following conditions hold:*

$$\begin{aligned} &\mathfrak{N}(\mathrm{x}_1, \mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathrm{x}_2, \mathfrak{P}\mathrm{e}_2) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}), \qquad \mathrm{x}_1 \neq \mathrm{x}_2 \\ &\Rightarrow \quad \mathrm{s}_1 \neq \mathrm{s}_2, \\ &\mathfrak{N}(\mathrm{s}_1, \mathfrak{R}\mathrm{e}_1) = \mathfrak{N}(\mathrm{s}_2, \mathfrak{R}\mathrm{e}_2) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \end{aligned} \tag{2}$$

*and*

$$\aleph\left(\mathfrak{N}(\mathrm{x}_1, \mathrm{x}_2)\right) \leq \Phi\left(\mathfrak{N}(\mathrm{s}_1, \mathrm{s}_2)\right). \tag{3}$$

The following example illustrates the applicability of the proposed generalized interpolative proximal-type contraction and verifies the hypothesis of the corresponding theorem hold in a concrete setting.

**Example 1.** Consider $\mathbb{R}^2$ with the Euclidean metric. Let

$$\mathcal{W} = \{(\mathrm{e}, \mathrm{z}) : \mathrm{e} \leq 0\},$$

$$\mathcal{Y} = \{(\mathrm{e}, \mathrm{z}) : \mathrm{e} \geq 1\}.$$

Then, $\mathfrak{N}(\mathcal{W}, \mathcal{Y}) = 1$, $\mathcal{W}_0 = \{(0, \mathrm{z}) : \mathrm{z} \in \mathbb{R}\}$, $\mathcal{Y}_0 = \{(1, \mathrm{z}) : \mathrm{z} \in \mathbb{R}\}$. Define mappings $\mathfrak{P} : \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$ as

$$\mathfrak{P}(\mathrm{e}, \mathrm{z}) = \left(-2\mathrm{e} + 1, \frac{\mathrm{z}}{3}\right),$$

$$\mathfrak{R}(\mathrm{e}, \mathrm{z}) = \left(-3\mathrm{e} + 1, \frac{\mathrm{z}}{2}\right).$$

Clearly, $\mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$ and $\mathfrak{R}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$. Let the functions $\aleph, \Phi : [0, \infty) \to \mathbb{R}$ be given by

$$\aleph(\mathrm{x}) = \frac{\mathrm{x}}{2} \text{ and } \Phi(\mathrm{x}) = \mathrm{x} - 1, \ \mathrm{x} \in [0, \infty).$$

We show that $\mathfrak{P}$ and $\mathfrak{R}$ are $(\Phi, \aleph)$-proximal. Consider $\mathrm{x}_1 = (0, 0)$, $\mathrm{x}_2 = (0, 2)$, $\mathrm{z}_1 = (0, 0)$, $\mathrm{z}_2 = (0, 3)$, and $\mathrm{e}_1 = (0, 0)$, $\mathrm{e}_2 = (0, 6)$. Then

$$\mathfrak{N}((0, 0), \mathfrak{P}(0, 0)) = \mathfrak{N}(\mathcal{W}, B) = \mathfrak{N}((0, 2), \mathfrak{P}(0, 6)),$$

$$\mathfrak{N}((0, 0), \mathfrak{R}(0, 0)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}((0, 3), \mathfrak{R}(0, 6));$$

i.e.,

$$\mathfrak{N}(\mathrm{x}_1, \mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\mathrm{x}_2, \mathfrak{P}\mathrm{e}_2),$$

$$\mathfrak{N}(\mathrm{z}_1, \mathfrak{R}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\tilde{\mathrm{l}}_2, \mathfrak{R}\mathrm{e}_2).$$

This implies

$$\aleph\left(\mathfrak{N}\left(\mathtt{x}_1,\mathtt{x}_2\right)\right)\leq\Phi\left(\mathfrak{N}\left(\tilde{\mathtt{l}}_1,\mathtt{z}_2\right)\right),$$

$$\aleph\left(\mathfrak{N}\left((0,0),(0,2)\right)\right)\leq\Phi\left(\mathfrak{N}\left((0,0),(0,3)\right)\right),$$

$$\aleph\left(2\right)\leq\Phi\left(3\right),$$

$$1\leq 2.$$

Same as all other cases are easily verified. This shows that the mappings $\mathfrak{P}$ and $\mathfrak{R}$ are $(\aleph,\Phi)$-proximal. The parameters involved in this example are chosen to ensure that the proximal contraction condition holds. Now, consider $\mathtt{x}_1=(0,0),\ \mathtt{x}_2=(0,2),\mathtt{z}_1=(0,0),\mathtt{z}_2=(0,3)$, and $\mu=\frac{1}{6}$; this implies

$$\mathfrak{N}\left(\mathtt{x}_1,\mathtt{x}_2\right)\leq\mu\left(\mathfrak{N}\left(\tilde{\mathtt{l}}_1,\mathtt{z}_2\right)\right)$$

$$\mathfrak{N}\left((0,0),(0,2)\right)\leq\mu\left(\mathfrak{N}\left((0,0),(0,3)\right)\right)$$

$$(2)\leq\mu\left(3\right)$$

$$2\leq\frac{1}{2}.$$

This leads to a contradiction. Therefore, $\mathfrak{P}$ and $\mathfrak{R}$ cannot be considered proximal contractions in the absence of $(\aleph,\Phi)$.

The lemmas provided in the following sections will be used to prove the important results presented in this paper. By applying these foundational lemmas, we hope to establish the validity of the main theorems.

**Lemma 1.** *Assume $(\mathcal{V},\mathfrak{N})$ as a metric space, and let $\{\mathtt{x}_p\}\subset\mathcal{V}$ be a sequence s.t.*

$$\lim_{p\to\infty}\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p+1})=0.$$

*If $\{\mathtt{x}_p\}$ is not Cauchy, there exist two subsequences like $\{\mathtt{x}_{p_l}\}$ and $\{\mathtt{x}_{q_l}\}$, along with a constant $\mathtt{s}^*>0$, such that*

$$\lim_{l\to\infty}\mathfrak{N}\big(x_{p_l+1},\,x_{q_l+1}\big)=\mathtt{s}^{*+},\tag{4}$$

*and*

$$\lim_{l\to\infty}\mathfrak{N}(\mathtt{x}_{p_l},\mathtt{x}_{q_l})=\mathfrak{N}(\mathtt{x}_{p_l+1},\mathtt{x}_{m_l})=\mathfrak{N}(\mathtt{x}_{p_l},\mathtt{x}_{q_l+1})=\mathtt{s}^*.\tag{5}$$

**Lemma 2.** *Let $\aleph:[0,\infty)\to\mathbb{R}$ be a given function. The following conditions are equivalent:*

(*i*) *For each $\varepsilon>0$, we have $\inf_{\mathtt{x}>\varepsilon}\aleph(\mathtt{x})>-\infty$.*
(*ii*) *For each $\varepsilon>0$, $\liminf_{\mathtt{x}\to\varepsilon^+}\aleph(\mathtt{x})>-\infty$.*
(*iii*) *If $\lim_{p\to\infty}\aleph(\mathtt{x}_p)=-\infty$, then $\lim_{p\to\infty}\mathtt{x}_p=0$.*

**Lemma 3.** *Assume $\{\mathtt{x}_p\}$ is a sequence in $(\mathcal{V},\mathfrak{N})$ such that $\lim_{n\to\infty}\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p+1})=0$, and assume the mappings $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ satisfy Equation (2). If the functions $\aleph,\Phi:[0,\infty)\to\mathbb{R}$ satisfy*

(*i*) $\limsup_{\mathtt{x}\to\varepsilon^+}\aleph(\mathtt{x})<\Phi(\varepsilon^+),\ \forall\,\varepsilon>0,$

*then $\{\mathtt{x}_p\}$ is a Cauchy sequence.*

***Proof.*** Assume $\{\mathtt{x}_p\}$ is not a Cauchy sequence. According to Lemma 1, there exist two subsequences $\{\mathtt{x}_{p_l}\}$ and $\{\mathtt{x}_{q_l}\}$ of $\{\mathtt{x}_p\}$, along with a constant $\varepsilon>0$, such that the Equations (4) and (5) are satisfied. From Equation (4), it follows that $\mathfrak{N}(\mathtt{x}_{p_l+1},\mathtt{x}_{q_l+1})>\varepsilon$. Since the elements $\mathtt{x}_{p_l}$, $\mathtt{x}_{p_l+1}$, $\mathtt{x}_{q_l}$, $\mathtt{x}_{q_l+1}$, $\mathrm{e}_{p_l}$, $\mathrm{e}_{q_l}$, $\mathrm{e}_{p_l+1}$, and $\mathrm{e}_{q_l+1}$, all belong to $\mathcal{W}$, then

$$\mathfrak{N}(\mathtt{x}_{p_l+1},\mathfrak{P}(\mathrm{e}_{p_l+1}))=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}\left(\mathtt{x}_{q_l+1},\mathfrak{P}\left(\mathrm{e}_{q_l+1}\right)\right),$$

$$\mathfrak{N}(\mathtt{x}_{p_l},\mathfrak{R}(\mathrm{e}_{p_l+1}))=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}\left(\mathtt{x}_{q_l},\mathfrak{R}\left(\mathrm{e}_{q_l+1}\right)\right).$$

By Equation (2), we have

$$\aleph(\mathfrak{N}(\mathtt{x}_{p_l+1},\mathtt{x}_{q_l+1}))\leq\Phi(\mathfrak{N}(\mathtt{x}_{p_l},\mathtt{x}_{q_l})),$$

for all $l\geq 1$. Let $\mathtt{u}_l=\mathfrak{N}(\mathtt{x}_{p_l+1},\mathtt{x}_{q_l+1})$ and $\mathtt{u}_{l-1}=\mathfrak{N}(\mathtt{x}_{p_l},\mathtt{x}_{q_l})$. We have

$$\aleph(\mathtt{u}_l)\leq\Phi(\mathtt{u}_{l-1}),\forall\,l\geq 1.\tag{6}$$

Equations (4) and (5) give $\lim_{l\to\infty}\mathtt{u}_l=\varepsilon^+$ and $\lim_{l\to\infty}\mathtt{u}_{l-1}=\varepsilon$. Thus, Equation (6) yields that

$$\aleph(\varepsilon^+)=\lim_{l\to\infty}\aleph(\mathtt{u}_l)\leq\limsup_{l\to\infty}\Phi(\mathtt{u}_{l-1})\leq\limsup_{c\to\varepsilon}\Phi(c).\tag{7}$$

This contradicts to the hypothesis (*i*), and lastly, $\{\mathtt{x}_p\}$ is Cauchy in $\mathcal{W}$.

Define two functions $\aleph,\Phi:[0,\infty)\to\mathbb{R}$ to ensure the existence of CBPPs. Let $\mathfrak{P}\colon\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ be two mappings. The next properties are needed:

(*i*) $\mathfrak{R}$ dominates $\mathfrak{P}$, $(\aleph,\Phi)$-proximally;
(*ii*) $\mathfrak{P}$ and $\mathfrak{R}$ commute proximally;
(*iii*) $\aleph$ is non-decreasing and $\limsup_{\mathtt{x}\to\epsilon^+}\aleph(\mathtt{x})<\Phi(\epsilon^+),\ \forall\,\epsilon>0$;
(*iv*) $\mathfrak{P}$ and $\mathfrak{R}$ are continuous;
(*v*) $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$ and $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathfrak{R}(\mathcal{W}_0)$;
(*vi*) $\aleph$ is non-decreasing and $\{\Phi(\mathtt{x}_p)\}$ and $\{\aleph(\mathtt{x}_p)\}$ are convergent sequences such that if $\lim_{p\to\infty}\Phi(\mathtt{x}_p)=\lim_{p\to\infty}\aleph(\mathtt{x}_p)$, then $\lim_{p\to\infty}\mathtt{x}_p=0$.

**Theorem 1.** *Let $(\mathcal{V},\mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V},\mathfrak{N})$ with $\mathcal{Y}$ being APC with respect to $\mathcal{W}$. And also, $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation (1), be non-empty. Suppose $\mathfrak{P}\colon\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}\colon\mathcal{W}\to\mathcal{Y}$ satisfy Equation (2) and properties (i)–(v) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e}\in\mathcal{W}$ such that*

$$\mathfrak{N}\left(\mathrm{e},\mathfrak{R}\mathrm{e}\right)=\mathfrak{N}\left(\mathcal{W},\mathcal{Y}\right)\quad\text{and}$$
$$\mathfrak{N}\left(\mathrm{e},\mathfrak{P}\mathrm{e}\right)=\mathfrak{N}\left(\mathcal{W},\mathcal{Y}\right).$$

***Proof.*** Let $\mathrm{e}_0$ be an element of $\mathcal{W}_0$. Since $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathfrak{R}(\mathcal{W}_0)$, there exist elements $\mathrm{e}_1$ and $\mathrm{e}_2$

in $\mathcal{W}_0$ such that $\mathfrak{P}\mathrm{e}_0 = \mathfrak{R}\mathrm{e}_1$ and $\mathfrak{P}\mathrm{e}_1 = \mathfrak{R}\mathrm{e}_2$. This procedure generates a sequence $\{\mathrm{e}_n\} \subseteq \mathcal{W}_0$ satisfying

$$\mathfrak{P}\mathrm{e}_{n-1} = \mathfrak{R}\mathrm{e}_n,$$

for every positive integer $p$. Since $\mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$, for each $p \in \mathbb{N}$, there exists an element $\mathtt{x}_p \in \mathcal{W}_0$ s.t.

$$\mathfrak{N}(\mathtt{x}_p, \mathfrak{P}\mathrm{e}_p) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

In view of the selection of $\mathrm{e}_p$ and $\mathtt{x}_p$, it gives

$$\mathfrak{N}(\mathtt{x}_{p+1}, \mathfrak{P}(\mathrm{e}_{p+1})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\mathtt{x}_p, \mathfrak{P}\mathrm{e}_p\right),$$

$$\mathfrak{N}\left(\mathtt{x}_p, \mathfrak{R}\left(\mathrm{e}_{p+1}\right)\right) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\mathtt{x}_{p-1}, \mathfrak{R}\left(\mathrm{e}_p\right)\right);$$

i.e.,

$$\mathfrak{N}\left(\mathtt{x}_p, \mathfrak{P}\mathrm{e}_p\right) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\mathtt{x}_{p-1}, \mathfrak{R}\left(\mathrm{e}_p\right)\right). \quad (8)$$

If $\mathtt{x}_p = \mathtt{x}_{p-1}$, then from Equation (8), the point $\mathtt{x}_p$ is a CBPP of the mappings $\mathfrak{P}$ and $\mathfrak{R}$. Alternatively, suppose $\mathtt{x}_{p-1} \neq \mathtt{x}_p \ \forall\, p \in \mathbb{N}$. Then, using Equation (8), we get

$$\mathfrak{N}(\mathtt{x}_{p+1}, \mathfrak{P}(\mathrm{e}_{p+1})) = \mathfrak{N}(\mathtt{x}_p, \mathfrak{P}(\mathrm{e}_p)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}(\mathtt{x}_p, \mathfrak{R}(\mathrm{e}_p)) = \mathfrak{N}(\mathtt{x}_{p-1}, \mathfrak{R}(\mathrm{e}_{p-1})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

By applying Equation (3), it follows that

$$\aleph(\mathfrak{N}(\mathtt{x}_{p+1}, \mathtt{x}_p)) \leq \Phi(\mathfrak{N}(\mathtt{x}_p, \mathtt{x}_{p-1})), \quad (9)$$

for all $\mathtt{x}_{p-1}, \mathtt{x}_p, \mathtt{x}_{p+1}, \mathrm{e}_{p-1}, \mathrm{e}_p, \mathrm{e}_{p+1} \in \mathcal{W}$. Let $\mathfrak{N}(\mathtt{x}_{p+1}, \mathtt{x}_p) = \mathtt{u}_p$. Then we have

$$\aleph(\mathtt{u}_p) \leq \Phi(\mathtt{u}_{p-1}) < \Phi(\mathtt{u}_{p-1}).$$

Since $\aleph$ is not decreasing, it follows $\mathtt{u}_p < \mathtt{u}_{p-1}, \forall\, p \in \mathbb{N}$. Therefore, $\{\mathtt{u}_p\}$ is a strictly decreasing and positive sequence converging to a limit point $\mathtt{u} \geq 0$. Suppose, for the sake of contradiction, that $\mathtt{u} > 0$. Then, from Equation (9), we have

$$\aleph\left(\varepsilon^+\right) = \lim_{p\to\infty} \aleph\left(\mathtt{u}_p\right) \leq \lim_{p\to\infty} \Phi\left(\mathtt{u}_{p-1}\right) \leq \limsup_{\mathtt{u}\to\varepsilon^+} \Phi\left(\mathtt{u}\right).$$

This contradicts the property (iii), and so, $p = 0$ and $\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathtt{x}_{p+1}) = 0$. By the property (iii) and Lemma 3, the sequence $\{\mathtt{x}_p\}$ is Cauchy. Since $(\mathcal{V}, \mathfrak{N})$ is a CMS and $\mathcal{W}$ is a closed subset of $(\mathcal{V}, \mathfrak{N})$, so $\exists\, \mathtt{x}^* \in \mathcal{W}$ s.t.

$$\lim_{n\to\infty} \mathfrak{N}(\mathtt{x}_n, \mathtt{x}^*) = 0.$$

Next, we have the following inequalities:

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathrm{e}_p)) \leq \mathfrak{N}(\mathtt{x}^*, \mathtt{x}_p) + \mathfrak{N}(\mathtt{x}_p, \mathfrak{P}(\mathrm{e}_p)),$$

and

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathrm{e}_p)) \leq \mathfrak{N}(\mathtt{x}^*, \mathtt{x}_p) + \mathfrak{N}(\mathtt{x}_p, \mathfrak{R}(\mathrm{e}_p)).$$

Therefore, as $p \to \infty$,

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathrm{e}_p)) \to \mathfrak{N}(\mathtt{x}^*, \mathcal{Y}),$$

and

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathrm{e}_p)) \to \mathfrak{N}(\mathtt{x}^*, \mathcal{Y}).$$

Since $\mathfrak{P}$ and $\mathfrak{R}$ are continuous, we conclude that $\mathfrak{R}(\mathtt{x}^*) = \mathfrak{P}(\mathtt{x}^*)$. Moreover, as $\mathcal{Y}$ is an APC with respect to $\mathcal{W}$, there exist subsequences $\{\mathfrak{R}(\mathrm{e}_{n_l})\}$ of $\{\mathfrak{R}(\mathrm{e}_n)\}$ and $\{\mathfrak{P}(\mathrm{e}_{n_l})\}$ of $\{\mathfrak{P}(\mathrm{e}_n)\}$, such that

$$\mathfrak{R}(\mathrm{e}_{n_l}) \to \mathtt{s}^* \in \mathcal{Y}, \quad \mathfrak{P}(\mathrm{e}_{n_l}) \to \mathtt{s}^* \in \mathcal{Y}, \quad \text{as} \quad l \to \infty.$$

Thus, taking the limit as $l \to \infty$ in the following equations, we obtain

$$\mathfrak{N}(\mathtt{s}^*, \mathfrak{P}(\mathrm{e}_{n_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

and

$$\mathfrak{N}(\mathtt{s}^*, \mathfrak{R}(\mathrm{e}_{n_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

We have

$$\mathfrak{N}(\mathtt{s}^*, \mathtt{x}^*) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

Since $\mathtt{x}^* \in \mathcal{W}_0$, we know that $\mathfrak{P}(\mathtt{x}^*) \in \mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$ and $\mathfrak{R}(\mathtt{x}^*) \in \mathfrak{R}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$. Therefore, there exists an element $\mathtt{s}^* \in \mathcal{W}_0$ such that

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathtt{x}^*)) = \mathfrak{N}(\mathtt{s}^*, \mathfrak{P}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathtt{s}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}). \quad (10)$$

Now, considering Equations (9) and (10), we apply Equation (2) and obtain

$$\aleph(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)) \leq \Phi(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)) < \Phi(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)).$$

As $\aleph$ is a non-decreasing function, it follows that

$$\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*) \leq \tilde{\mathrm{p}}\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*) < \Phi(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)),$$

which implies that $\mathtt{x}^*$ and $\mathtt{s}^*$ are identical. Finally, using Equation (8), we have

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathtt{x}^*)).$$

Thus, $\mathtt{x}^*$ is a CBPP of the mappings $\mathfrak{R}$ and $\mathfrak{P}$. This completes the proof.

**Theorem 2.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V}, \mathfrak{N})$ with $\mathcal{Y}$ being APC with respect to $\mathcal{W}$. Assume that $\mathcal{W}_0$ and $\mathcal{Y}_0$, introduced in Equation (1), are non-empty, and $\mathfrak{P} : \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$ satisfy Equation (2) and properties (i), (ii), and (iv)–(vi) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e} \in \mathcal{W}$ s.t.*

$$\mathfrak{N}\left(\mathrm{e}, \mathfrak{R}\mathrm{e}\right) = \mathfrak{N}\left(\mathcal{W}, \mathcal{Y}\right) \quad \textit{and} \quad \mathfrak{N}\left(\mathrm{e}, \mathfrak{P}\mathrm{e}\right) = \mathfrak{N}\left(\mathcal{W}, \mathcal{Y}\right).$$

***Proof.*** Proceeding with the preliminary steps of Theorem 1, we consider the following inequality

$$\aleph(\mathtt{u}_p) \leq \Phi(\mathtt{u}_{p-1}) < \aleph(\mathtt{u}_{p-1}). \quad (11)$$

From Equation (11), it follows that the sequence $\{\aleph(\mathtt{u}_p)\}$ is strictly decreasing. There are two possible cases: either the sequence is bounded below, or it is not.

Case 1: $\{\aleph(\mathtt{u}_p)\}$ is bounded below. In such a case, the sequence $\{\aleph(\mathtt{u}_p)\}$ is convergent. Since the sequence is strictly decreasing, it must converge to a limit. Given the relationship in Equation (11), the sequences $\aleph(\mathtt{u}_p)$ and $\Phi(\mathtt{u}_{p-1})$ converge to the same limit. By the property (iii), we have that either $\lim_{p\to\infty} q_p = 0$ or $\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathrm{e}_{p+1}) = 0$ for any sequence $\{\mathtt{x}_p\} \subset \mathcal{W}$. Following the reasoning from the proof of Theorem 1, we obtain

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathtt{x}^*)).$$

This shows that the point $\mathtt{x}^*$ is a CBPP of both $\mathfrak{R}$ and $\mathfrak{P}$.

Case 2: $\{\aleph(\mathtt{u}_p)\}$ is not bounded below.

In this case, we have

$$\inf_{\mathtt{u}_p > \varepsilon} \aleph(\mathtt{u}_p) > -\infty \ \forall \varepsilon > 0,\ p \in \mathbb{N}.$$

On the basis of Lemma 2, it follows that $\mathtt{u}_p \to 0$ as $p \to \infty$. This completes the proof.

**Remark 1.** If $\aleph(\mathtt{x}) = \mathtt{x}$ and $\Phi(\mathtt{x}) = \lambda\mathtt{x}$, where $\lambda \in (0,1)$, then Definition 5 reduces to the definition of proximal contraction, and each of Theorems 1 and 2 reduces to the Banach interpolative contraction principle.

**Corollary 1.** *Suppose that $(\mathcal{V}, \mathfrak{N})$ is a CMS and $\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathtt{z}_p) = 0$, $\forall\ \mathtt{x}_p, \mathtt{z}_p \in \mathcal{W}$, and also, $\mathfrak{R}$ and $\mathfrak{P}$ are self-mappings fulfilling Equation (2) and properties (i)–(v) given above. Then $(\mathcal{V}, \mathfrak{N})$ includes a common fixed point for $\mathfrak{R}$ and $\mathfrak{P}$.*

## 2.2. Interpolative Ćirić–Reich–Rus–type proximal contraction

In the present subsection, we establish conditions for determining the CBPP for Ćirić-–Reich-–Rus-type contractions and extend the results Refs. [19,21,23] to the structure of MS. This extension provides a comprehensive framework for solving CBPP problems in CMSs, introducing novel iterative algorithms and convergence studies that build upon and generalize the previous findings.

**Definition 6.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS, and $\mathcal{W}$ and $\mathcal{Y}$ be non-empty subsets of $\mathcal{V}$. The mappings $\mathfrak{R}: \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{P}: \mathcal{W} \to \mathcal{Y}$ are said to be a* $(\aleph, \Phi)$ interpolative Ćirić–Reich–Rus type proximal contraction *if for all* $\mathtt{x}_1, \mathtt{x}_2, \mathtt{z}_1, \mathtt{z}_2, \mathrm{e}_1, \mathrm{e}_2 \in \mathcal{W}$

$$\left.\begin{array}{l} \mathfrak{N}(\mathtt{x}_1, \mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathtt{x}_2, \mathfrak{P}\mathrm{e}_2) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}), \quad \mathtt{x}_1 \neq \mathtt{x}_2 \\ \mathfrak{N}(\mathtt{z}_1, \mathfrak{R}\mathrm{e}_1) = \mathfrak{N}(\mathtt{z}_2, \mathfrak{R}\mathrm{e}_2) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \end{array}\right\} \Rightarrow \quad \mathtt{z}_1 \neq \mathtt{z}_2, \tag{12}$$

*and*

$$\aleph(\mathfrak{N}(\mathtt{x}_1, \mathtt{x}_2)) \leq \Phi\left((\mathfrak{N}(\mathtt{z}_1, \mathtt{z}_2))^{\tilde{\mathrm{p}}} (\mathfrak{N}(\tilde{\mathtt{l}}_1, \mathtt{x}_1))^{\tilde{\mathrm{q}}} (\mathfrak{N}(\tilde{\mathtt{l}}_2, \mathtt{x}_2))^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}}\right), \tag{13}$$

*for some* $\tilde{\mathrm{p}} > 0$, $\tilde{\mathrm{q}} > 0$ *and* $\tilde{\mathrm{p}} + \tilde{\mathrm{q}} < 1$.

The following example establishes the applicability of the proposed Ä†iriÄ‡-type contraction mapping and confirms that the hypotheses of the corresponding theorem hold in a concrete setting.

**Example 2.** Let $\mathcal{V} = \mathbb{R}$ be endowed with the metric $\mathfrak{N}(\mathrm{e}, \mathtt{z}) = \mid \mathrm{e} - \mathtt{z} \mid$. Let $\mathcal{W} = \{0, 2, 4, 6, 8, 10\}$ and $\mathcal{Y} = \{1, 3, 5, 7, 9, 11\}$. Moreover, $\mathfrak{P}: \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R}: \mathcal{W} \to \mathcal{Y}$ are formulated by

$$\mathfrak{P}(0) = 3,\ \mathfrak{P}(2) = 5,\ \mathfrak{P}(4) = 7,\ \ \mathfrak{P}(6) = 3,\ \mathfrak{P}(8) = 9,\ \mathfrak{P}(10) = 11,$$

and

$$\mathfrak{R}(0) = 3,\ \mathfrak{R}(2) = 1,\ \mathfrak{R}(4) = 9,\ \mathfrak{R}(6) = 7,\ \mathfrak{R}(8) = 5,\ \mathfrak{R}(10) = 11.$$

Evidently, $\mathfrak{N}(\mathcal{W}, \mathcal{Y}) = 1$, $\mathcal{W}_0 = \mathcal{W}$, and $\mathcal{Y}_0 = \mathcal{Y}$. Clearly, $\mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$ and $\mathfrak{R}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$. Let the functions $\aleph, \Phi: [0, \infty) \to \mathbb{R}$ be defined as follows

$$\aleph(\mathtt{x}) = \mathtt{x} \text{ and } \Phi(\mathtt{x}) = \frac{\mathtt{x}}{2}, \quad \mathtt{x} \in [0, \infty).$$

We show that $\mathfrak{P}$ and $\mathfrak{R}$ are $(\aleph, \Phi)$-interpolative Ćirić-Reich-Rus type proximal. Consider $\mathtt{x}_1 = 4$, $\mathtt{x}_2 = 6, \mathtt{z}_1 = 0, \mathtt{z}_2 = 8$ and $\mathrm{e}_1 = 2, \mathrm{e}_2 = 4$, $\tilde{\mathrm{p}} = \frac{1}{2}$, $\tilde{\mathrm{q}} = \frac{1}{3}$. Then

$$\mathfrak{N}(\mathtt{x}_1, \mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\mathtt{x}_2, \mathfrak{P}\mathrm{e}_2),$$

$$\mathfrak{N}(\mathtt{z}_1, \mathfrak{R}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\tilde{\mathtt{l}}_2, \mathfrak{R}\mathrm{e}_2\right).$$

We have

$$\mathfrak{N}(4, \mathfrak{P}(2)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(6, \mathfrak{P}(4)),$$

$$\mathfrak{N}(0, \mathfrak{R}(2)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(8, \mathfrak{R}(4)).$$

This implies

$$\aleph(\mathfrak{N}(\mathtt{x}_1, \mathtt{x}_2)) \leq \Phi\left((\mathfrak{N}(\mathtt{z}_1, \mathtt{z}_2))^{\tilde{\mathrm{p}}} (\mathfrak{N}(\tilde{\mathtt{l}}_1, \mathtt{x}_1))^{\tilde{\mathrm{q}}} (\mathfrak{N}(\mathtt{z}_2, \mathtt{x}_2))^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}}\right),$$

$$\aleph(\mathfrak{N}(4, 6)) \leq \Phi\left((\mathfrak{N}(0, 8))^{\frac{1}{2}} (\mathfrak{N}(0, 4))^{\frac{1}{3}} (\mathfrak{N}(8, 6))^{1-\frac{1}{2}-\frac{1}{3}}\right),$$

$$\aleph(2) \leq \Phi(5.0398),$$

$$2 \leq 2.5199.$$

The remaining cases can be verified in a similar manner. Hence, the mappings $\mathfrak{P}$ and $\mathfrak{R}$ are $(\aleph, \Phi)$-interpolative Ćirić—Reich—Rus-type proximal. The parameters involved in this example are chosen to ensure that the interpolative Ćirić-–Reich-–Rus -type proximal contraction condition holds. On the other hand, consider $\mathtt{x}_1 = 4$, $\mathtt{x}_2 = 6, \mathtt{z}_1 = 0, \mathtt{z}_2 = 8$ and $\mathrm{e}_1 = 2, \mathrm{e}_2 = 4$, $\tilde{\mathrm{p}} = \frac{1}{2}$, $\tilde{\mathrm{q}} = \frac{1}{3}$ with $\mu = \frac{1}{6}$. Thus,

$$\mathfrak{N}(\mathtt{x}_1, \mathtt{x}_2) \leq \mu\left((\mathfrak{N}(\mathtt{z}_1, \mathtt{z}_2))^{\tilde{\mathrm{p}}} (\mathfrak{N}(\tilde{\mathtt{l}}_1, \mathtt{x}_1))^{\tilde{\mathrm{q}}} (\mathfrak{N}(\mathtt{z}_2, \mathtt{x}_2))^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}}\right),$$

$$\mathfrak{N}(4, 6) \leq \mu\left((\mathfrak{N}(0, 8))^{\frac{1}{2}} (\mathfrak{N}(0, 4))^{\frac{1}{3}} (\mathfrak{N}(8, 6))^{1-\frac{1}{2}-\frac{1}{3}}\right),$$

$$2 \leq \mu(5.0398),$$

$$2 \leq 0.8399.$$

This leads to a contradiction. Therefore, $\mathfrak{P}$ and $\mathfrak{R}$ are not Ćirić-–Reich-–Rus-type proximal contractions without $(\aleph, \Phi)$.

**Theorem 3.** *Let $(\mathcal{V},\mathfrak{N})$ be a CMS and $\mathcal{W},\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V},\mathfrak{N})$ so that $\mathcal{Y}$ is APC with respect to $\mathcal{W}$. Also, suppose that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation (1), are non-empty. Let $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ be satisfied the Equation (12) and the properties (i)–(v) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e}\in\mathcal{W}$ s.t.*

$$\mathfrak{N}(\mathrm{e},\mathfrak{R}\mathrm{e})=\mathfrak{N}(\mathcal{W},\mathcal{Y}) \quad \textit{and}$$
$$\mathfrak{N}(\mathrm{e},\mathfrak{P}\mathrm{e})=\mathfrak{N}(\mathcal{W},\mathcal{Y}).$$

***Proof.*** Proceeding with the preliminary steps of Theorem 1, we have

$$\mathfrak{N}(\mathtt{x}_{p+1},\mathfrak{P}(\mathrm{e}_{p+1}))=\mathfrak{N}(\mathtt{x}_p,\mathfrak{P}(\mathrm{e}_p))=\mathfrak{N}(\mathcal{W},\mathcal{Y}),$$

$$\mathfrak{N}(\mathtt{x}_p,\mathfrak{R}(\mathrm{e}_p))=\mathfrak{N}(\mathtt{x}_{p-1},\mathfrak{R}(\mathrm{e}_{p-1}))=\mathfrak{N}(\mathcal{W},\mathcal{Y}).$$

Thus, by Equation (12), we have

$$\aleph(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))\le\Phi\left((\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}}\right);$$

i.e.,

$$\aleph(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))<\ \aleph\left((\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{q}}}\right),\tag{14}$$

for all $\mathtt{x}_{p-1},\mathtt{x}_p,\mathtt{x}_{p+1},\mathrm{e}_{p-1},\mathrm{e}_p,\mathrm{e}_{p+1}\in\mathcal{W}$ and some $\tilde{\mathrm{p}}>0$, $\tilde{\mathrm{q}}>0$, $\tilde{\mathrm{p}}+\tilde{\mathrm{q}}<1$. Since $\aleph(\mathtt{x})<\Phi(\mathtt{x})$ for all $\mathtt{x}>0$, by Equation (14) we have

$$\aleph(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))\le\ \Phi\left((\mathtt{x}(\mathtt{x}_{p+1},\mathtt{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{q}}}\right).$$

Accordingly,

$$\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p)<(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{q}}}.$$

This implies

$$(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))^{1-\tilde{\mathrm{q}}}<(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{q}}};$$

i.e.,

$$(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))<(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1})).$$

The sequence $\{\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p)\}$ is strictly positive and decreasing, so it converges to some limit $\mathtt{u}\ge0$. We will prove that $\mathtt{u}=0$. To reach a contradiction, assume instead that $\mathtt{u}>0$. Then, using Equation (14), we obtain

$$\aleph\left(\varepsilon^+\right)=\lim_{p\to\infty}\aleph(\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))\le\lim_{p\to\infty}$$
$$\Phi\left((\mathfrak{N}(\mathtt{x}_{p+1},\mathtt{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p-1}))^{1-\tilde{\mathrm{q}}}\right)$$
$$\le\limsup_{\mathtt{x}\to\varepsilon+}\Phi(\mathtt{x}).$$

This contradicts the assumption (iii), which implies that $\mathtt{u}=0$, and therefore,

$$\lim_{p\to\infty}\mathfrak{N}(\mathtt{x}_p,\mathtt{x}_{p+1})=0.$$

Combining (iii) with Lemma 3, it follows that $\{\mathtt{x}_p\}$ is a Cauchy sequence. As $(\mathcal{V},\mathfrak{N})$ is a CMS and $\mathcal{W}$ is a closed subset of $\mathcal{V}$, the sequence converges to some $\mathtt{x}^*\in\mathcal{W}$. In other words,

$$\lim_{p\to\infty}\mathfrak{N}(\mathtt{x}_p,\mathtt{x}^*)=0.$$

Moreover,

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{P}(\mathrm{e}_p))\le\mathfrak{N}(\mathtt{x}^*,\mathtt{x}_p)+\mathfrak{N}(\mathtt{x}_p,\mathfrak{P}(\mathrm{e}_p)),$$

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{R}(\mathrm{e}_p))\le\mathfrak{N}(\mathtt{x}^*,\mathtt{x}_p)+\mathfrak{N}(\mathtt{x}_p,\mathfrak{R}(\mathrm{e}_p)).$$

Given that $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$, we have

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{R}(\mathrm{e}_p))\to\mathfrak{N}(\mathtt{x}^*,\mathcal{Y}) \quad \text{and} \quad \mathfrak{N}(\mathtt{x}^*,\mathfrak{P}(\mathrm{e}_p))\to\mathfrak{N}(\mathtt{x}^*,\mathcal{Y}), \quad \text{as} \quad p\to\infty.$$

Since $\mathfrak{P}$ and $\mathfrak{R}$ are CP, it follows that $\mathfrak{R}\mathtt{x}^*=\mathfrak{P}\mathtt{x}^*$. Because $\mathcal{Y}$ is APC with respect to $\mathcal{W}$, there exist subsequences $\{\mathfrak{R}(\mathrm{e}_{p_l})\}$ from $\{\mathfrak{R}(\mathrm{e}_p)\}$ and $\{\mathfrak{P}(\mathrm{e}_{p_l})\}$ from $\{\mathfrak{P}(\mathrm{e}_p)\}$ such that

$$\mathfrak{R}(\mathrm{e}_{p_l})\to\mathtt{z}^*\in\mathcal{Y} \quad \text{and} \quad \mathfrak{P}(\mathrm{e}_{p_l})\to\mathtt{z}^*\in\mathcal{Y}, \quad \text{as} \quad l\to\infty.$$

Taking the limit as $l\to\infty$ in these equations,

$$\mathfrak{N}(\mathtt{z}^*,\mathfrak{P}(\mathrm{e}_{p_l}))=\mathfrak{N}(\mathcal{W},\mathcal{Y}),$$

$$\mathfrak{N}(\mathtt{z}^*,\mathfrak{R}(\mathrm{e}_{n_l}))=\mathfrak{N}(\mathcal{W},\mathcal{Y}),\tag{15}$$

and we have

$$\mathfrak{N}(\mathtt{z}^*,\mathtt{x}^*)=\mathfrak{N}(\mathcal{W},\mathcal{Y}).$$

As $\mathtt{x}^*\in\mathcal{W}_0$, we have $\mathfrak{P}(\mathtt{x}^*)\in\mathcal{Y}_0$ and $\mathfrak{R}(\mathtt{x}^*)\in\mathcal{Y}_0$. There exists $\mathtt{s}^*\in\mathcal{W}_0$ such that

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{P}(\mathtt{x}^*)=\mathfrak{N}(\mathtt{s}^*,\mathfrak{P}(\mathtt{x}^*))=\mathfrak{N}(\mathcal{W},\mathcal{Y}),\tag{16}$$

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{R}(\mathtt{x}^*))=\mathfrak{N}(\mathtt{s}^*,\mathfrak{R}(\mathtt{x}^*))=\mathfrak{N}(\mathcal{W},\mathcal{Y}).$$

Using Equations (15) and (16), along with (12), we obtain

$$\aleph(\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*))\le\Phi\Big((\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*))^{\tilde{\mathrm{q}}}$$
$$(\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*))^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}}\le\Phi(\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*))<\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*).$$

As $\aleph$ is non-decreasing, we get

$$\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*)\le\tilde{\mathrm{p}}\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*)<\mathfrak{N}(\mathtt{x}^*,\mathtt{s}^*),$$

which implies $\mathtt{x}^*=\mathtt{s}^*$. From Equation (16), it follows that

$$\mathfrak{N}(\mathtt{x}^*,\mathfrak{R}(\mathtt{x}^*))=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}(\mathtt{x}^*,\mathfrak{P}(\mathtt{x}^*)).$$

Therefore, $\mathtt{x}^*$ is a CBPP for $\mathfrak{R}$ and $\mathfrak{P}$. This completes the proof.

**Theorem 4.** *Let $(\mathcal{V},\mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V},\mathfrak{N})$, with $\mathcal{Y}$ being APC relative to $\mathcal{W}$. Assume that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation (1), are non-empty. Let $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ satisfy Equation (12) and properties (i), (ii), and (iv)–(vi) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e}\in\mathcal{W}$ s.t.*

$$\mathfrak{N}(\mathrm{e},\mathfrak{R}\mathrm{e})=\mathfrak{N}(\mathcal{W},\mathcal{Y}) \quad \textit{and}$$
$$\mathfrak{N}(\mathrm{e},\mathfrak{P}\mathrm{e})=\mathfrak{N}(\mathcal{W},\mathcal{Y}).$$

***Proof.*** Proceeding with the preliminary steps of Theorem 3, we get

$$\aleph\left((\mathfrak{N}(x_{p+1},x_p))\right) \le \Phi\left((\mathfrak{N}(x_p,x_{p-1}))^{1-\tilde{q}}(\mathfrak{N}(x_{p+1},x_p))^{\tilde{q}}\right)$$
$$< \aleph\left((\mathfrak{N}(x_p,x_{p-1}))^{1-\tilde{q}}(\mathfrak{N}(x_{p+1},x_p))^{\tilde{q}}\right).$$

By Equation (17), we deduce that $\{\aleph(\mathfrak{N}(x_{p+1},x_p))\}$ is a strictly decreasing sequence. There are two possibilities: either the sequence is bounded below or not. If it is not bounded below, then

$$\inf_{\mathfrak{N}(x_{p+1},x_p)>\varepsilon} \aleph(\mathfrak{N}(x_{p+1},x_p)) > -\infty \text{ for all } \varepsilon>0, p\in\mathbb{N}.$$

By Lemma 2, this implies $\mathfrak{N}(x_{p+1},x_p)\to 0$ as $p\to\infty$. If the sequence is bounded below, it must converge. By assumption (iii), we have $\lim_{p\to\infty}\mathfrak{N}(x_{p+1},x_p)=0$. Similar to the proof of Theorem 3, we get

$$\mathfrak{N}(x^*,\mathfrak{R}(x^*)) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}(x^*,\mathfrak{P}(x^*)).$$

This demonstrates that $x^*$ is a CBPP for the pair of mappings $\mathfrak{R}$ and $\mathfrak{P}$.

**Remark 2.** If $\aleph(x)=x$ and $\Phi(x)=\lambda x$, where $\lambda\in(0,1)$, then Definition 6 reduces to the definition of Ciri'c-Reich-Rus interpolative contraction principle.

**Corollary 2.** *Suppose $(\mathcal{V},\mathfrak{N})$ is a CMS and $\lim_{p\to\infty}\mathfrak{N}(x_p,\tilde{l}_p)=0$, $\forall\, x_p,\tilde{l}_p\in\mathcal{W}$, and also, $\mathfrak{R}$ and $\mathfrak{P}$ are self-mappings fulfilling Equation (12) and properties (i) − (v) above. Then $(\mathcal{V},\mathfrak{N})$ includes a common fixed point for $\mathfrak{R}$ and $\mathfrak{P}$.*

### 2.3. Kannan-type proximal contraction

In this section, we establish conditions for determining the CBPP for Kannan-type contractions and extend the results Refs. [21,23] to the structure of the MSs. This extension provides a comprehensive framework for solving CBPP-problems in CMSs, introducing novel iterative schemes and convergence studies that build upon and generalize the previous findings.

**Definition 7.** *Let $(\mathcal{V},\mathfrak{N})$ be a MS, $\mathcal{W}$ and $\mathcal{Y}$ be two non-empty subsets of $\mathcal{V}$. The mappings $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ are said to be* $(x,\aleph)$-Kannan-type proximal contraction *if for all* $x_1,x_2,z_1,z_2,e_1,e_2\in\mathcal{W}$

$$\left.\begin{array}{l}\mathfrak{N}(x_1,\mathfrak{P}e_1)=\mathfrak{N}(x_2,\mathfrak{P}e_2)\\ =\mathfrak{N}(\mathcal{W},\mathcal{Y}),\\ \\ \mathfrak{N}(z_1,\mathfrak{R}e_1)=(z_2,\mathfrak{R}e_2)\\ =\mathfrak{N}(\mathcal{W},\mathcal{Y}) \text{ for } x_1\ne x_2\end{array}\right\}\Rightarrow\quad z_1\ne z_2, \tag{17}$$

*and*

$$\aleph(\mathfrak{N}(x_1,x_2)) \le \Phi\left((\mathfrak{N}(z_1,x_1))^{\tilde{p}}\left(\mathfrak{N}(\tilde{l}_2,x_2)\right)^{1-\tilde{p}}\right), \tag{18}$$

*for some $\tilde{p}\in(0,1)$.*

**Remark 3.** *If $\aleph(x)=x$ and $\Phi(x)=\lambda x$, where $\lambda\in(0,1)$, then the above definition reduces to the Kannan-type interpolative contraction principle.*

The following example establishes the applicability of the proposed Kannan-type contraction mapping and confirms that the hypotheses of the corresponding theorem hold in a concrete setting.

**Example 3.** Let $\mathcal{V}=\mathbb{R}$ be endowed with the metric $\mathfrak{N}(e,y)=|\ e-y\ |$. Let $\mathcal{W}=\{1,2,3,4,5,6,7,8,9,10\}$ and $\mathcal{Y}=\{0,1,2,3,4,5,6,7,8,9,10,11\}$. Moreover, $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ are formulated by

$$\mathfrak{P}(e)=e-1,$$

and

$$\mathfrak{R}(e)=e+1.$$

Then, $\mathfrak{N}(\mathcal{W},\mathcal{Y})=0$, $\mathcal{W}_0=\mathcal{W}$, and $\mathcal{Y}_0\subseteq\mathcal{Y}$. Then clearly $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$ and $\mathfrak{R}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$. Define the functions $\aleph,\Phi:[0,\infty)\to\mathbb{R}$ by

$$\Phi(x)=\begin{cases} x, & \text{for } x=1,\\ x+4, & \text{otherwise,}\end{cases} \quad \text{and}$$

$$\aleph(x)=\begin{cases} \dfrac{x}{2}, & \text{for } x=1,\\ x+3, & \text{otherwise.}\end{cases}$$

We show that $\mathfrak{P}$ and $\mathfrak{R}$ are $(\Phi,\aleph)$-interpolative Kannan-type proximal. Consider, $x_1=2$, $x_2=1$, $z_1=4$, $z_2=3$ and $e_1=3$, $e_2=2$, for all $\tilde{p}=\frac{1}{2}$ so that

$$\mathfrak{N}(x_1,\mathfrak{P}e_1)=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}(x_2,\mathfrak{P}e_2),$$

$$\mathfrak{N}(z_1,\mathfrak{R}e_1)=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}(\tilde{l}_2,\mathfrak{R}e_2).$$

Then,

$$\mathfrak{N}(2,\mathfrak{P}3)=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}(1,\mathfrak{P}2),$$

$$\mathfrak{N}(4,\mathfrak{R}3)=\mathfrak{N}(\mathcal{W},\mathcal{Y})=\mathfrak{N}(3,\mathfrak{R}2).$$

This implies that

$$\aleph(\mathfrak{N}(x_1,x_2)) \le \Phi\left((\mathfrak{N}(z_1,x_1))^{\tilde{p}}\left(\mathfrak{N}(\tilde{l}_2,x_2)\right)^{1-\tilde{p}}\right),$$

$$\aleph(\mathfrak{N}(2,1)) \le \Phi\left((\mathfrak{N}(4,2))^{\frac{1}{2}}(\mathfrak{N}(3,1))^{\frac{1}{2}}\right),$$

$$\aleph(1)\le\Phi(4.9996),$$

$$1\le 1.9996.$$

The remaining cases can be verified in a similar manner. This shows that mappings $\mathfrak{P}$ and $\mathfrak{R}$ are $(\Phi,\aleph)$-interpolative Kannan-type proximal. The parameters involved in this example are chosen to ensure that the interpolative Kannan-type proximal contraction condition holds. On the other hand, consider $x_1 = 2$, $x_2 = 1, z_1 = 4, z_2 = 3$ and $e_1 = 3, e_2 = 2$, for all $\tilde{p} = \frac{1}{2}$, and $\mu = \frac{1}{6}$; we have

$$\mathfrak{N}(x_1, \mathfrak{P}e_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(x_2, \mathfrak{P}e_2),$$

$$\mathfrak{N}(z_1, \mathfrak{R}e_1) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}(\tilde{1}_2, \mathfrak{R}e_2).$$

This implies that

$$\mathfrak{N}(x_1, x_2) \leq \mu\left((\mathfrak{N}(z_1, x_1))^{\tilde{p}}\left(\mathfrak{N}(\tilde{1}_2, x_2)\right)^{1-\tilde{p}}\right),$$

$$\mathfrak{N}(2,1) \leq \mu\left((\mathfrak{N}(4,2))^{\frac{1}{2}}(\mathfrak{N}(3,1))^{\frac{1}{2}}\right),$$

$$1 \leq \mu(4.9996),$$

$$1 \leq 0.8332.$$

This leads to a contradiction. Thus, $\mathfrak{P}$ and $\mathfrak{R}$ cannot be classified as the interpolative Kannan-type proximal contractions without $(\aleph, \Phi)$.

**Theorem 5.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V}, \mathfrak{N})$, with $\mathcal{Y}$ being APC with respect to $\mathcal{W}$. Assume that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation* (1)*, are non-empty. Let $\mathfrak{P}\colon \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R}\colon \mathcal{W} \to \mathcal{Y}$ satisfy Equation* (17) *and properties (i)–(v) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP* $e \in \mathcal{W}$ *so that*

$$\mathfrak{N}(e, \mathfrak{R}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \quad \textit{and} \quad \mathfrak{N}(e, \mathfrak{P}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

***Proof.*** Proceeding with the preliminary steps of Theorem 3, we obtain

$$\aleph(\mathfrak{N}(x_{p+1}, x_p)) \leq \Phi\left((\mathfrak{N}(x_{p+1}, x_p))^{\tilde{p}}(\mathfrak{N}(x_p, x_{p-1}))^{1-\tilde{p}}\right) \tag{19}$$

for all $x_{p-1}, x_p, x_{p+1}, e_{p-1}, e_p, e_{p+1} \in \mathcal{W}$ and some $\tilde{p} \in (0,1)$. Since, $\aleph(x) < \Phi(x)$ for all $x > 0$, by Equation (19), we have

$$\aleph(\mathfrak{N}(x_{p+1}, x_p)) < \aleph\left((\mathfrak{N}(x_{p+1}, x_p))^{\tilde{p}}(\mathfrak{N}(x_p, x_{p-1}))^{1-\tilde{p}}\right).$$

As $\aleph$ is a non-decreasing function, it follows that

$$\mathfrak{N}(x_{p+1}, x_p) < (\mathfrak{N}(x_{p+1}, x_p))^{\tilde{p}}(\mathfrak{N}(x_p, x_{p-1}))^{1-\tilde{p}}.$$

This implies

$$(\mathfrak{N}(x_{p+1}, x_p))^{1-\tilde{p}} < (\mathfrak{N}(x_p, x_{p-1}))^{1-\tilde{p}}.$$

Let $\mathfrak{N}(x_{p+1}, x_p) = u_p$. We have

$$\Phi(u_p) \leq \aleph\left((u_p)^{\tilde{p}}(u_{p-1})^{1-\tilde{p}}\right) < \Phi((u_p)^{\tilde{p}}(u_{p-1})^{1-\tilde{p}}).$$

Given that $u_n < u_{p-1}$, $\forall p \in \mathbb{N}$, $\{u_p\}$ is a strictly decreasing positive sequence converging to a limit $u \geq 0$. Assume, for contradiction, that $u > 0$. From Equation (19), it follows that

$$\aleph(\varepsilon^+) = \lim_{p\to\infty} \aleph(u_p) \leq \lim_{p\to\infty} \Phi\left((u_p)^{\tilde{p}}(u_{p-1})^{1-\tilde{p}}\right)$$
$$\leq \lim_{x\to u+} \sup \Phi(x).$$

This contradicts the hypothesis (iii), so $u = 0$ and $\lim_{p\to\infty} \mathfrak{N}(x_p, x_{p+1}) = 0$. By assumption (iii) and Lemma 3, $\{x_p\}$ is a Cauchy sequence. We know that $(\mathcal{V}, \mathfrak{N})$ is complete and $\mathcal{W}$ is closed, so $\exists\, x^* \in \mathcal{W}$ s.t. $\lim_{p\to\infty} \mathfrak{N}(x_p, x^*) = 0$, and

$$\mathfrak{N}(x^*, \mathfrak{P}(e_p)) \leq \mathfrak{N}(x^*, x_p) + \mathfrak{N}(x_p, \mathfrak{P}(e_p)).$$

Also,

$$\mathfrak{N}(x^*, \mathfrak{R}(e_p)) \leq \mathfrak{N}(x^*, x_p) + \mathfrak{N}(x_p, \mathfrak{R}(e_p)).$$

Thus, $\mathfrak{N}(x^*, \mathfrak{R}(e_p)) \to \mathfrak{N}(x^*, \mathcal{Y})$ and $\mathfrak{N}(x^*, \mathfrak{P}(e_p)) \to \mathfrak{N}(x^*, \mathcal{Y})$ as $p \to \infty$. Since $\mathfrak{P}$ and $\mathfrak{R}$ commute proximally, $\mathfrak{R}x^*$ and $\mathfrak{P}x^*$ are equal. Because $\mathcal{Y}$ is APC with respect to $\mathcal{W}$, there exist subsequences $\{\mathfrak{R}(e_{p_l})\}$ of $\{\mathfrak{R}(e_p)\}$ and $\{\mathfrak{P}(e_{p_l})\}$ of $\{\mathfrak{P}(e_p)\}$ such that $\mathfrak{R}(e_{p_l}) \to z^* \in \mathcal{Y}$ and $\mathfrak{P}(e_{p_l}) \to z^* \in \mathcal{Y}$ as $l \to \infty$. Taking $l \to \infty$ in the equations, it gives

$$\mathfrak{N}(z^*, \mathfrak{P}(e_{p_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}(z^*, \mathfrak{R}(e_{p_l})) = \mathfrak{N}, (\mathcal{W}, \mathcal{Y}) \tag{20}$$

and we have

$$\mathfrak{N}(z^*, x^*) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

Since, $x^* \in \mathcal{W}_0$, $\mathfrak{P}(x^*) \in \mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$. There is $s^* \in \mathcal{W}_0$ such that

$$\begin{aligned} \mathfrak{N}(x^*, \mathfrak{P}(x^*)) = \mathfrak{N}(s^*, \mathfrak{P}(z^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}), \\ \mathfrak{N}(x^*, \mathfrak{R}(x^*)) = \mathfrak{N}(s^*, \mathfrak{R}(x^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}). \end{aligned} \tag{21}$$

Using Equations (20) and (21), along with Equation (17), we obtain

$$\Phi(\mathfrak{N}(x^*, s^*)) \leq \aleph\left((\mathfrak{N}(x^*, s^*))^{\tilde{p}}(\mathfrak{N}(x^*, s^*))^{1-\tilde{p}}\right)$$

$$\leq \aleph(\mathfrak{N}(x^*, s^*)) < \mathfrak{N}(x^*, s^*).$$

As $\aleph$ is a non-decreasing function, it follows that

$$\mathfrak{N}(x^*, s^*) \leq \tilde{\mu}\mathfrak{N}(x^*, s^*) < \mathfrak{N}(x^*, s^*).$$

This implies that $x^*$ and $s^*$ are equal. Consequently, from Equation (**??**), we obtain

$$\mathfrak{N}(e, \mathfrak{R}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \quad \text{and}$$
$$\mathfrak{N}(e, \mathfrak{P}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

Thus, $x^*$ is a CBPP for the pair of mappings $\mathfrak{R}$ and $\mathfrak{P}$. This concludes the proof.

**Theorem 6.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS and $\mathcal{W}, \mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V}, \mathfrak{N})$ so that $\mathcal{Y}$ is APC with respect to $\mathcal{W}$. Suppose also that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation* (1)*, are non-empty. Let $\mathfrak{P}\colon \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$ be satisfied the Equation* (17) *and properties (i), (ii), (iv)–(vi) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP* $e \in \mathcal{W}$ *so that*

$$\mathfrak{N}(e, \mathfrak{R}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \quad \textit{and}$$
$$\mathfrak{N}(e, \mathfrak{P}e) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

***Proof.*** Proceeding with the preliminary steps of Theorem 2, we have

$$\begin{aligned}\aleph(\mathtt{u}_p) &\leq \Phi\left((\mathtt{u}_{p-1})^{1-\tilde{\mathtt{q}}}(\mathtt{u}_p)^{\tilde{\mathtt{q}}}\right)\\ &< \aleph\left((\mathtt{u}_{p-1})^{1-\tilde{\mathtt{q}}}(\mathtt{u}_p)^{\tilde{\mathtt{q}}}\right).\end{aligned} \tag{22}$$

From Equation (22), it follows that $\{\aleph(\mathtt{u}_p)\}$ is strictly decreasing. There are two possibilities: either $\{\aleph(\mathtt{u}_p)\}$ is bounded below, or it is not. If it is not bounded below, then

$$\inf_{\mathtt{u}_p>\varepsilon}\aleph(\mathtt{u}_p) > -\infty, \quad \forall\varepsilon>0, n\in\mathbb{N}.$$

By Lemma 1, $\mathtt{u}_p \to 0$ as $p\to\infty$. If the sequence $\{\mathtt{x}(\mathtt{u}_p)\}$ is bounded below, it converges. By Equation (19), the sequence $\{\aleph(\mathtt{u}_p)\}$ also converges to the same limit. On the basis of (iii), we get $\lim_{p\to\infty}\mathtt{u}_p = 0$ or $\lim_{p\to\infty}\mathfrak{N}(\mathtt{x}_p, \mathrm{e}_{p+1}) = 0$ for each sequence $\{\mathtt{x}_p\}$ in $\mathcal{W}$. Using the proof of Theorem 5, we obtain

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathtt{x}^*)).$$

This proves that $\mathtt{x}^*$ is a CBPP for $\mathfrak{R}$ and $\mathfrak{P}$.

**Corollary 3.** *Suppose* $(\mathcal{V},\mathfrak{N})$ *is a CMS and* $\lim_{p\to\infty}\mathfrak{N}(\mathtt{x}_1,\mathtt{x}_2) = 0$, $\forall\ \mathtt{x}_1,\mathtt{x}_2\in\mathcal{W}$. *Suppose* $\mathfrak{R}$ *and* $\mathfrak{P}$ *are self-mappings satisfying Equation* (17) *and (i)–(v) above. Then* $\mathfrak{P}$ *and* $\mathfrak{R}$ *have a CFP.*

### 2.4. Hardy–Rogers–type proximal contraction

In this section, we present conditions for determining the CBPP for Hardy–Rogers–type contractions and extend the results Refs. [21,23] within the MS framework. Hardy–Rogers–type contractions are a generalization of traditional contraction mappings, incorporating additional structural properties that facilitate the analysis of FPs in more complex settings. By extending these results, we provide a robust framework for tackling CBPP problems in CMS. This involves introducing novel iterative schemes and convergence criteria that further enrich and extend previous work.

**Definition 8.** *Let* $(\mathcal{V},\mathfrak{N})$ *be an MS,* $\mathcal{W}$ *and* $\mathcal{Y}$ *be two non-empty subsets of* $\mathcal{V}$. *The mappings* $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ *and* $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ *are called* $(\aleph,\Phi)$-interpolative Hardy–Rogers–type proximal contraction *if for all* $\mathtt{x}_1,\mathtt{x}_2,\mathtt{z}_1,\mathtt{z}_2,\mathrm{e}_1,\mathrm{e}_2\in\mathcal{W}$,

$$\left.\begin{array}{l}\mathfrak{N}(\mathtt{x}_1,\mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathtt{x}_2,\mathfrak{P}\mathrm{e}_2)\\ = \mathfrak{N}(\mathcal{W},\mathcal{Y}),\\[2ex] \mathfrak{N}(\mathtt{z}_1,\mathfrak{R}\mathrm{e}_1) = (\mathtt{z}_2,\mathfrak{R}\mathrm{e}_2)\\ = \mathfrak{N}(\mathcal{W},\mathcal{Y})\ \textit{for}\ \mathtt{x}_1\neq\mathtt{x}_2\end{array}\right\}\quad\Rightarrow\quad \mathtt{z}_1\neq\mathtt{z}_2, \tag{23}$$

*and*

$$\aleph(\mathfrak{N}(\mathtt{x}_1,\mathtt{x}_2)) \leq \Phi\Big(\mathfrak{N}(\mathtt{z}_1,\mathtt{z}_2)^{\tilde{\mathtt{p}}}(\mathfrak{N}(\mathtt{z}_1,\mathtt{x}_1))^{\tilde{\mathtt{q}}} (\mathfrak{N}(\mathtt{z}_2,\mathtt{x}_2))^{\tilde{\gamma}}\left(\frac{1}{2}(\mathfrak{N}(\mathtt{z}_1,\mathtt{x}_2)+\mathfrak{N}(\mathtt{z}_2,\mathtt{x}_1))\right)^{1-\tilde{\mathtt{p}}-\tilde{\mathtt{q}}-\tilde{\gamma}}\Big), \tag{24}$$

*for some* $\tilde{\mathtt{p}},\tilde{\mathtt{q}},\tilde{\mathtt{r}}\in(0,1)$ *with* $\tilde{\mathtt{p}}+\tilde{\mathtt{q}}+\tilde{\mathtt{r}}<1$.

**Remark 4.** If $\aleph(\mathtt{x}) = \mathtt{x}$ and $\Phi(\mathtt{x}) = \lambda\mathtt{x}$, where $\lambda\in(0,1)$, then the previous definition reduces to the definition of the interpolative Hardy–Roger's proximal contractions.

The following example establishes the applicability of the proposed Hardy–Rogers-type contraction mapping and confirms that the hypotheses of the corresponding theorem hold in a concrete setting.

**Example 4.** Consider $\mathbb{R}^2$ with the Euclidean metric. Let

$$\mathcal{W} = \{(0,\mathrm{e}): 0\leq\mathrm{e}<\infty\},$$

$$\mathcal{Y} = \{(1,\mathrm{e}): 0\leq\mathrm{e}<\infty\}\cup\{(-1,0)\}.$$

Define mappings $\mathfrak{P}:\mathcal{W}\to\mathcal{Y}$ and $\mathfrak{R}:\mathcal{W}\to\mathcal{Y}$ as

$$\mathfrak{P}(0,\mathrm{e}) = (1,\mathrm{e}-1),$$

$$\mathfrak{R}(0,\mathrm{e}) = (1,\mathrm{e}+1).$$

Then, $\mathfrak{N}(\mathcal{W},\mathcal{Y}) = 1$, $\mathcal{W}_0 = \mathcal{W}$, and $\mathcal{Y}_0 = \mathcal{Y}$. Then clearly $\mathfrak{P}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$ and $\mathfrak{R}(\mathcal{W}_0)\subseteq\mathcal{Y}_0$. The functions $\aleph,\Phi:[0,\infty)\to\mathbb{R}$ are formulated as

$$\aleph(\mathtt{x}) = \begin{cases}\mathtt{x}+1, & \text{for } \mathtt{x}=2,\\ \mathtt{x}+4, & \text{otherwise,}\end{cases}\quad\text{and}$$

$$\Phi(\mathtt{x}) = \begin{cases}\frac{\mathtt{x}}{2}, & \text{for } \mathtt{x}=2,\\ \mathtt{x}+3, & \text{otherwise.}\end{cases}$$

We show that $\mathfrak{P}$ and $\mathfrak{R}$ are $(\Phi,\aleph)$-interpolative Hardy–Rogers-type proximal. Consider $\mathtt{x}_1 = (0,2)$, $\mathtt{x}_2 = (0,4)$, $\mathtt{z}_1 = (0,4)$, $\mathtt{z}_2 = (0,6)$, $\mathrm{e}_1 = (0,3)$, $\mathrm{e}_2 = (0,5)$, and $\tilde{\mathtt{p}} = 0.1$, $\tilde{\mathtt{q}} = 0.2$, $\tilde{\mathtt{r}} = 0.3$. We have

$$\mathfrak{N}(\mathtt{x}_1,\mathfrak{P}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}(\mathtt{x}_2,\mathfrak{P}\mathrm{e}_2),$$

$$\mathfrak{N}(\mathtt{z}_1,\mathfrak{R}\mathrm{e}_1) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}(\tilde{\mathtt{l}}_2,\mathfrak{R}\mathrm{e}_2).$$

Then

$$\mathfrak{N}((0,2),\mathfrak{P}(0,3)) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}((0,4),\mathfrak{P}(0,5)),$$

$$\mathfrak{N}((0,4),\mathfrak{R}(0,3)) = \mathfrak{N}(\mathcal{W},\mathcal{Y}) = \mathfrak{N}((0,6),T(0,5)).$$

This implies

$$\aleph\left(\mathfrak{N}\left(\mathrm{x}_1, \mathrm{x}_2\right)\right) \leq \Phi\left(\left(\mathfrak{N}\left(\mathrm{z}_1, \mathrm{z}_2\right)\right)^{0.1}\left(\mathfrak{N}\right.\right.$$

$$\left.\left(\mathrm{z}_1, \mathrm{x}_1\right)^{0.2}\left(\mathfrak{N}\left(\mathrm{z}_2, \mathrm{x}_2\right)\right)^{0.3}$$

$$\left.\times\left(\frac{1}{2}\left(\mathfrak{N}\left(\mathrm{z}_1, \mathrm{x}_2\right)+\mathfrak{N}\left(\mathrm{z}_2, \mathrm{x}_1\right)\right)\right)^{0.4}\right),$$

$$\aleph(2) \leq \Phi(2.4698),$$

$$3 \leq 5.4698.$$

The remaining cases can be verified in a similar manner. This shows that mappings $\mathfrak{P}$ and $\mathfrak{R}$ are $(\Phi, \aleph)$-interpolative Hardy–Rogers-type proximal contractions. The parameters involved in this example are chosen to ensure that the interpolative Hardy–Rogers-type proximal contraction condition holds. On the other hand, $\mathrm{x}_1=(0,2)$, $\mathrm{x}_2=(0,4), \mathrm{z}_1=(0,4), \mathrm{z}_2=(0,6)$, $\mathrm{e}_1=(0,3)$, $\mathrm{e}_2=(0,5)$, and $\tilde{\mathrm{p}}=0.1$, $\tilde{\mathrm{q}}=0.2$, $\tilde{\mathrm{r}}=0.3$ with $\mu=\frac{1}{2}$ give

$$\mathfrak{N}\left(\mathrm{x}_1, \mathrm{x}_2\right) \leq \mu\left(\left(\mathfrak{N}\left(\mathrm{z}_1, \mathrm{z}_2\right)\right)^{0.1}\left(\mathfrak{N}\left(\mathrm{z}_1, \mathrm{x}_1\right)\right)^{0.2}\right.$$

$$\left.\times\left(\frac{1}{2}\left(\mathfrak{N}\left(\mathrm{z}_1, \mathrm{x}_2\right)+\mathfrak{N}\left(\mathrm{z}_2, \mathrm{x}_1\right)\right)\right)^{0.4}\right),$$

$$(2) \leq \mu(2.4698),$$

$$2 \leq 1.2349.$$

This leads to a contradiction. Hence, $\mathfrak{P}$ and $\mathfrak{R}$ is not an interpolative Hardy–Rogers-type proximal contraction without $\aleph$ and $\Phi$.

**Theorem 7.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $\mathcal{V}$ such that $\mathcal{Y}$ is APC with respect to $\mathcal{W}$. Additionally, suppose that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation* (1), *are non-empty subsets. Consider two functions, $\mathfrak{P}: \mathcal{W} \rightarrow \mathcal{Y}$ and $\mathfrak{R}: \mathcal{W} \rightarrow \mathcal{Y}$, satisfying the conditions described in Equations* (23) *and properties (i)–(v) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e} \in \mathcal{W}$ provided that*

$$\mathfrak{N}(\mathrm{e}, \mathfrak{R}\mathrm{e})=\mathfrak{N}(\mathcal{W}, \mathcal{Y}) \quad \text{and} \quad \mathfrak{N}(\mathrm{e}, \mathfrak{P}\mathrm{e})=\mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

***Proof.*** Applying the initial steps as in the proof of Theorem 5, we have

$$\mathfrak{N}(\mathrm{x}_{p+1}, \mathfrak{P}(\mathrm{e}_{p+1}))=\mathfrak{N}\left(\mathrm{x}_p, \mathfrak{P}\left(\mathrm{e}_p\right)\right)=\mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}(\mathrm{x}_p, \mathfrak{R}(\mathrm{e}_p))=\mathfrak{N}\left(\mathrm{x}_{p-1}, \mathfrak{R}\left(\mathrm{e}_{p-1}\right)\right)=\mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

Thus, we have

$$\aleph(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p)) \leq \Phi\left((\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}\right.$$

$$(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{r}}}$$

$$\left.\times\left(\frac{1}{2}(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_p)+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1}))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}\right) \tag{25}$$

for all $\mathrm{x}_{p-1}, \mathrm{x}_p, \mathrm{x}_{p+1}, \mathrm{e}_{p-1}, \mathrm{e}_p, \mathrm{e}_{p+1} \in \mathcal{W}$. Since, $\aleph(\mathrm{x})<\Phi(\mathrm{x})$ for all $\mathrm{x} \in[0, \infty)$, by Equation (25), we have

$$\aleph(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))<\aleph\left((\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}\right.$$

$$(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{r}}}$$

$$\left.\times\left(\frac{1}{2}(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_p)+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1}))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}\right).$$

We know that $\aleph$ is non-decreasing, so we obtain

$$\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p)<(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}}$$

$$(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{r}}}$$

$$\times\left(\frac{1}{2}((\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_p)+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1})))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\gamma}}$$

$$<(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{r}}}$$

$$\times\left(\frac{1}{2}\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1})\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}},$$

$$<(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{r}}}$$

$$\times\left(\frac{1}{2}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1})+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1}))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}$$

$$<(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}+\tilde{\mathrm{r}}}$$

$$\times\left(\frac{1}{2}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1})+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1}))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}.$$

This implies

$$\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p)<(\mathfrak{N}(\mathrm{x}_p, \mathrm{x}_{p-1}))^{\tilde{\mathrm{p}}}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p))^{\tilde{\mathrm{q}}+\tilde{\mathrm{r}}}$$

$$\times\left(\frac{1}{2}(\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1})+\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_{p-1}))\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}.$$

Set $\mathfrak{N}(\mathrm{x}_{p+1}, \mathrm{x}_p)=\mathrm{u}_p$. Then we have

$$\aleph(\mathrm{u}_p) \leq \Phi\left((\mathrm{u}_{p-1})^{\tilde{\mathrm{p}}}(\mathrm{u}_p)^{\tilde{\mathrm{q}}+\tilde{\mathrm{r}}}\left(\frac{1}{2}(\mathrm{u}_p+\mathrm{u}_p)\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}\right)$$

$$<\aleph\left((\mathrm{u}_{p-1})^{\tilde{\mathrm{p}}}(\mathrm{u}_p)^{\tilde{\mathrm{q}}+\tilde{\mathrm{r}}}\left(\frac{1}{2}(\mathrm{u}_p+\mathrm{u}_p)\right)^{1-\tilde{\mathrm{p}}-\tilde{\mathrm{q}}-\tilde{\mathrm{r}}}\right). \tag{26}$$

For some $n \geq 1$, we take $\mathtt{u}_p \geq \mathtt{u}_{p-1}$. We know that $\aleph$ is non-decreasing, so Equation (26) gives

$$\mathtt{u}_p \geq \left( (\mathtt{u}_{p-1})^{\tilde{\mathtt{p}}} (\mathtt{u}_p)^{\tilde{\mathtt{q}}+\tilde{\gamma}} \left( \frac{1}{2} (\mathtt{u}_p + \mathtt{u}_p) \right)^{1-\tilde{\mathtt{p}}-\tilde{\mathtt{q}}-\tilde{\gamma}} \right).$$

This leads to a contradiction. Hence, we conclude that $\mathtt{u}_p < \mathtt{u}_{p-1}$ for all $p \geq 1$. Therefore, $\{\mathtt{u}_p\}$ converges to some element $\mathtt{u} \geq 0$. We now show that $\mathtt{u} = 0$. Assume, for the sake of contradiction, that $\mathtt{u} > 0$. By Equation (26), we obtain the following

$$\aleph\left(\varepsilon^{+}\right) = \lim_{p\to\infty} \aleph\left(\mathtt{u}_p\right)$$

$$\leq \lim_{p\to\infty} \Phi$$

$$\left( \left( (\mathtt{u}_{p-1})^{\tilde{\mathtt{p}}} (\mathtt{u}_p)^{\tilde{\mathtt{q}}+\tilde{\mathtt{r}}} \left( \frac{1}{2} (\mathtt{u}_p + \mathtt{u}_p) \right)^{1-\tilde{\mathtt{p}}-\tilde{\mathtt{q}}-\tilde{\gamma}} \right) \right)$$

$$\leq \lim_{p\to\infty} \sup \Phi\left(\mathtt{u}_p\right).$$

This contradicts assumption (iii), so we conclude that $\mathtt{u} = 0$ and $\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathtt{x}_{p+1}) = 0$. Taking assumption (iii) and Lemma 3 into account, we deduce that $\{\mathtt{x}_p\}$ is a Cauchy sequence. On the other side, $(\mathcal{V}, \mathfrak{N})$ is a CMS and $\mathcal{W}$ is a closed subset of $\mathcal{V}$. In this case, this Cauchy sequence must converge to some $\mathtt{x}^* \in \mathcal{W}$, meaning that

$$\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathtt{x}^*) = 0.$$

Thus,

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathrm{e}_p)) \leq \mathfrak{N}\left(\mathtt{x}^*, \mathtt{x}_p\right) + \mathfrak{N}\left(\mathtt{x}_p, \mathfrak{P}\left(\mathrm{e}_p\right)\right),$$

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathrm{e}_p)) \leq \mathfrak{N}\left(\mathtt{x}^*, \mathtt{x}_p\right) + \mathfrak{N}\left(\mathtt{x}_p, \mathfrak{R}\left(\mathrm{e}_p\right)\right).$$

Thus, the mappings $\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathrm{e}_p)) \to \mathfrak{N}(\mathtt{x}^*, \mathcal{Y})$ and $\mathfrak{N}(\mathtt{x}^*, \mathfrak{P}(\mathrm{e}_p)) \to \mathfrak{N}(\mathtt{x}^*, \mathcal{Y})$ hold as $p \to \infty$. Given that both $\mathfrak{P}$ and $\mathfrak{R}$ are CP, we have $\mathfrak{R}\mathtt{x}^* = \mathfrak{P}\mathtt{x}^*$. Since $\mathcal{Y}$ is APC with respect to $\mathcal{W}$, there exist subsequences $\{\mathfrak{R}(\mathrm{e}_{p_l})\}$ of $\{\mathfrak{R}(\mathrm{e}_p)\}$ and $\{\mathfrak{P}(\mathrm{e}_{p_l})\}$ of $\{\mathfrak{P}(\mathrm{e}_p)\}$ such that $\mathfrak{R}(\mathrm{e}_{p_l}) \to \mathtt{z}^* \in \mathcal{Y}$ and $\mathfrak{P}(\mathrm{e}_{p_l}) \to \mathtt{z}^* \in \mathcal{Y}$ as $l \to \infty$.

Taking the limit as $l \to \infty$ in the following equations, $\mathfrak{N}(\tilde{\mathtt{l}}^*, \mathfrak{P}(\mathrm{e}_{p_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y})$ and $\mathfrak{N}(\tilde{\mathtt{l}}^*, \mathfrak{R}(\mathrm{e}_{p_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y})$,

we conclude the desired result

$$\begin{aligned} &\mathfrak{N}(\mathtt{z}^*, \mathfrak{P}(\mathrm{e}_{p_l})) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) \textit{ and} \\ &\mathfrak{N}\left(\tilde{\mathtt{l}}^*, \mathfrak{R}\left(\mathrm{e}_{p_l}\right)\right) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}), \end{aligned} \tag{27}$$

and we have

$$\mathfrak{N}\left(\mathtt{z}^*, \mathtt{x}^*\right) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}).$$

Since, $\mathtt{x}^* \in \mathcal{W}_0$, we have $\mathfrak{P}(\mathtt{x}^*) \in \mathfrak{P}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$ and $\mathfrak{R}(a^*) \in \mathfrak{R}(\mathcal{W}_0) \subseteq \mathcal{Y}_0$. There exists $\mathtt{s}^* \in \mathcal{W}_0$ such that

$$\mathfrak{N}(a^*, \mathfrak{P}(\mathtt{x}^*) = \mathfrak{N}(\mathtt{s}^*, \mathfrak{P}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}),$$

$$\mathfrak{N}\left(\mathtt{x}^*, \mathfrak{R}\left(\mathtt{x}^*\right)\right) = \mathfrak{N}\left(\mathtt{s}^*, \mathfrak{R}\left(\mathtt{x}^*\right)\right) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}). \tag{28}$$

Now, considering Equations (27) and (28), we conclude

$$\Phi(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)) \leq \aleph\left( (\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*))^{\tilde{\mathtt{p}}} (\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*))^{\tilde{\mathtt{q}}} \right)$$

$$\leq \aleph(\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*)) < \mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*).$$

As $\aleph$ is a non-decreasing function, it follows that

$$\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*) \leq \tilde{\mathtt{p}}\mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*) < \mathfrak{N}(\mathtt{x}^*, \mathtt{s}^*),$$

and so,

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\mathtt{x}^*, \mathfrak{P}\left(\mathtt{x}^*\right)\right).$$

This shows that the point $\mathtt{x}^*$ is a CBPP of the pair of mappings $\mathfrak{P}$ and $\mathfrak{R}$. This completes the proof of the theorem.

**Theorem 8.** *Let $(\mathcal{V}, \mathfrak{N})$ be a CMS, and let $\mathcal{W}$ and $\mathcal{Y}$ be non-empty closed subsets of $(\mathcal{V}, \mathfrak{N})$, with $\mathcal{Y}$ being APC relative to $\mathcal{W}$. Assume that $\mathcal{W}_0$ and $\mathcal{Y}_0$, defined in Equation (1), are non-empty. Let $\mathfrak{P} : \mathcal{W} \to \mathcal{Y}$ and $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$ satisfy Equation (23) and conditions (i), (ii), and (iv)–(vi) above. Then the mappings $\mathfrak{R}$ and $\mathfrak{P}$ possess a CBPP $\mathrm{e} \in \mathcal{W}$ so that*

$$\mathfrak{N}\left(\mathrm{e}, \mathfrak{R}\mathrm{e}\right) = \mathfrak{N}\left(\mathcal{W}, \mathcal{Y}\right) \quad \textit{and}$$
$$\mathfrak{N}\left(\mathrm{e}, \mathfrak{P}\mathrm{e}\right) = \mathfrak{N}\left(\mathcal{W}, \mathcal{Y}\right).$$

***Proof.*** Proceeding as in the proof of Theorem 7, we have

$$\aleph(\mathtt{u}_p) \leq \Phi\left( (\mathtt{u}_p)^{\tilde{\mathtt{p}}+\tilde{\mathtt{q}}} (\mathtt{u}_{p-1})^{\tilde{\mathtt{r}}} \left( \frac{1}{2} ((\mathtt{u}_p) + (\mathtt{u}_{p-1})) \right)^{1-\tilde{\mathtt{p}}-\tilde{\mathtt{q}}-\tilde{\mathtt{r}}} \right)$$

$$< \aleph\left( (\mathtt{u}_p)^{\tilde{\mathtt{p}}+\tilde{\mathtt{q}}} (\mathtt{u}_{p-1})^{\tilde{\mathtt{r}}} \left( \frac{1}{2} ((\mathtt{u}_p) + (\mathtt{u}_{p-1})) \right)^{1-\tilde{\mathtt{p}}-\tilde{\mathtt{q}}-\tilde{\mathtt{r}}} \right).$$

By Equation (29), we conclude that the sequence $\{\aleph(\mathtt{x}_p)\}$ is strictly decreasing. Two scenarios emerge: either $\{\aleph(\mathtt{x}_p)\}$ is bounded below or it is not. If $\{(\mathtt{x}_p)\}$ is unbounded below, then

$$\inf_{\mathtt{u}_p > \varepsilon} \aleph(\mathtt{u}_p) > -\infty \quad \text{for all } \varepsilon > 0 \text{ and } p \in \mathbb{N}.$$

Using Lemma 1, it follows that $\mathtt{u}_p \to 0$ as $p \to \infty$. If, on the other hand, $\{\aleph(\mathtt{u}_p)\}$ is bounded below, the sequence must converge. By Equation (29), the sequence $\{\aleph(\mathtt{u}_p)\}$ also converges, and both sequences share the same limit. By assumption (iii), we deduce $\lim_{p\to\infty} \mathtt{u}_p = 0$, or equivalently,

$$\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_p, \mathtt{x}_{p+1}) = 0.$$

Proceeding as in the proof of Theorem 7, we obtain

$$\mathfrak{N}(\mathtt{x}^*, \mathfrak{R}(\mathtt{x}^*)) = \mathfrak{N}(\mathcal{W}, \mathcal{Y}) = \mathfrak{N}\left(\mathtt{x}^*, \mathfrak{P}\left(\mathtt{x}^*\right)\right). \tag{29}$$

This demonstrates that the point $\mathtt{x}^\star$ is a CBPP for the pair of mappings $\mathfrak{R}$ and $\mathfrak{P}$.

**Corollary 4.** *Suppose* $(\mathcal{V}, \mathfrak{N})$ *is a CMS and* $\lim_{p\to\infty} \mathfrak{N}(\mathtt{x}_1, \mathtt{z}_1) = 0$, $\forall\, \mathtt{x}_1, \mathtt{z}_1 \in \mathcal{W}$. *Suppose* $\mathfrak{R}$ *and* $\mathfrak{P}$ *are self-mapping verifying Equation* (23) *and the properties (i)–(v) above. Then* $\mathfrak{R}$ *and* $\mathfrak{P}$ *have a common FP.*

## 3. Application

Fixed-point and BPP theorems play a fundamental role in the quantitative analysis of nonlinear problems arising in applied mathematics and related disciplines. In recent years, these methods have been successfully employed in the study of nonlinear integral and differential equations, fractional-order models, dynamic systems, optimization problems, and equilibrium theory. In particular, FP results in generalized metric structures such as FMSs, and extended $b$-MS have been instrumental in establishing the existence, uniqueness, and stability of the solutions to the boundary value problems and functional equations. Motivated by these developments, we present in this section an application of our CBPP results to a class of nonlinear problems, illustrating the practical relevance and applicability of the theoretical framework developed in the preceding sections.

### 3.1. Application to nonlinear fractional differential equations

To find the CBPP, we apply Theorem 1 to a nonlinear differential equation given by

$$\mathbb{D}_0^\delta u(\mathtt{x}) = \mathtt{s}^*(\mathtt{x}, u(\mathtt{x})), \tag{30}$$

where, $(0 \le \mathtt{x} \le 1, 1 < \delta \le 2)$ it is equipped with the integral boundary conditions

$$u(0) = 0, \quad u(1) = \int_0 u(\tilde{\mathtt{v}})\, d\tilde{\mathtt{v}}, \qquad 0 << 1, \tag{31}$$

so that $\mathbb{D}_0^\delta$ represents the Caputo fractional derivative of order $\delta$. Indeed, generally, the Caputo fractional derivative of order $\delta$ is defined by

$$\mathbb{D}_0^\delta u(\mathtt{x}) = \frac{1}{\tilde{\gamma}(m-\delta)} \int_0^{\mathtt{x}} \frac{u^{(m)}(\mathtt{x})}{(\mathtt{x}-\tilde{\mathtt{v}})^{\delta-m+1}}\, d\tilde{\mathtt{v}},$$

where $m-1 < \delta < m$. The space $(\mathcal{V}, \mathfrak{N}, \|\cdot\|_\infty)$ includes the continuous functions defined on $I = [0,1]$, and $\mathcal{V}$ is a Banach space defined by

$$\| u \|_\infty = \sup_{\mathtt{x}\in I} | u(\mathtt{x}) | .$$

Suppose that a CMS $(\mathcal{V}, \mathfrak{N})$ has been defined with the metric

$$\mathfrak{N}(u_1(\mathtt{x}), u_2(\mathtt{x})) = \sup_{\mathtt{x}\in[0,1]} |u_1(\mathtt{x}) - u_2(\mathtt{x})|.$$

$\mathcal{W}$ and $\mathcal{Y}$ are as subsets of a metric space $(\mathcal{V}, \mathfrak{N})$ with this metric $\mathfrak{N}$.

**Theorem 9.** *Let*
**(i)** $\mathtt{s}^* : I \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $\mathtt{x} \in [0,1] := I$ *provided that*

$$| \mathtt{s}^*(\mathtt{x}, u_1(\mathtt{x})) - \mathtt{s}^*(\mathtt{x}, u_2(\mathtt{x})) | \le \frac{\tilde{\mathtt{r}}(\delta+1)}{5} | u_1(\mathtt{x}) - u_2(\mathtt{x}) |, \qquad 1 < \delta \le 2, \tag{32}$$

*for all* $u_1, u_2 \in \mathbb{R}$ *with* $\mathtt{s}^*(\mathtt{x}, u_i) > 0, (i = 1, 2)$*;*

**(ii)** $\exists\, u_0 \in \mathcal{W}$ *such that* $\mathtt{s}^*(u_0(\mathtt{x}), \mathfrak{T}u_0(\mathtt{x}) > 0$ *for all* $\mathtt{x} \in I$*, where* $\mathfrak{T} : \mathcal{W} \to \mathcal{V}$ *is defined by*

$$\mathfrak{T}u(\mathtt{x}) = \frac{1}{\tilde{\mathtt{r}}(\delta)} \int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{v}}, u(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}} - \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)} \int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{v}}, u(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}} + \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)} \int_0 \left( \int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}} - \tilde{\mathtt{q}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{q}}, u(\tilde{\mathtt{q}}))\, d\tilde{\mathtt{q}} \right) d\tilde{\mathtt{v}};$$

**(iii)** *Suppose there exist mappings* $\mathfrak{P} : \mathcal{W} \to \mathcal{Y}$ *and* $\mathfrak{R} : \mathcal{W} \to \mathcal{Y}$*, for some* $\mathcal{W} \subseteq \mathcal{V}$*, such that*

$$\mathfrak{N}(u_1, \mathfrak{R}y_1) = \mathfrak{N}(u_2, \mathfrak{R}y_2), \quad \mathfrak{N}(x_1, \mathfrak{P}y_1) = \mathfrak{N}(x_2, \mathfrak{P}y_2),$$

*with* $u_1 \neq u_2$ *and* $x_1 \neq x_2$*. Here* $x_1, x_2, u_1, u_2, y_1, y_2 \in \mathcal{W}$*.*

**(iv)** $\forall\, u_1, u_2 \in \mathcal{W}$,

$$\aleph(\mathfrak{N}(u_1, u_2)) \le \Phi(\mathfrak{N}(u_1, u_2)),$$

*where* $\aleph(\mathtt{x}) = \mathtt{x}$ *and* $\Phi(\mathtt{x}) = \dfrac{\mathtt{x}}{2}$ *for* $\mathtt{x} \ge 0$*. Then, the boundary value problem of Equations 30 and 31 admits a CBPP if*

$$\frac{1}{5} + \frac{2}{5(2-^2)} + \frac{2^{1+\delta}}{5(2-^2)(1+\delta)} < \frac{1}{2}.$$

***Proof.*** Let $u \in \mathcal{W}$ be a solution of Equations 30 and 31. By the properties of the Caputo derivative, $u$ satisfies the equivalent integral equation

$$\mathfrak{T}u(\mathtt{x}) = \frac{1}{\tilde{\mathtt{r}}(\delta)} \int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{v}}, u(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}} - \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)} \int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{v}}, u(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}} + \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)} \int_0 \left( \int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}} - \tilde{\mathtt{q}})^{\delta-1} \mathtt{s}^*(\tilde{\mathtt{q}}, u(\tilde{\mathtt{q}}))\, d\tilde{\mathtt{q}} \right) d\tilde{\mathtt{v}}.$$

Then Equation (30) is equivalent to find $u^* \in \mathcal{W}$, which is a CBPP of $\mathfrak{R}$ and $\mathfrak{P}$. Now, let

$u_1, u_2 \in \mathcal{W}$. By hypothesis (i), we find

$$\Big|\mathfrak{T}(u_1(\tilde{\mathtt{t}})) - \mathfrak{T}(u_2(\tilde{\mathtt{t}}))\Big| = \Big|\frac{1}{\tilde{\gamma}(\delta)}\int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{v}}, u_1(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}}$$

$$- \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{v}}, u_1(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}}$$

$$+ \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0 \left(\int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}} - \tilde{\mathtt{q}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{q}}, u_1(\tilde{\mathtt{q}}))\, d\tilde{\mathtt{q}}\right) d\tilde{\mathtt{v}}$$

$$- \frac{1}{\tilde{\gamma}(\delta)}\int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{v}}, u_2(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}}$$

$$+ \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{v}}, u_2(\tilde{\mathtt{v}}))\, d\tilde{\mathtt{v}}$$

$$- \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0 \left(\int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}} - \tilde{\mathtt{q}})^{\delta-1}\mathtt{s}^*(\tilde{\mathtt{q}}, u_2(\tilde{\mathtt{q}}))\, d\tilde{\mathtt{q}}\right) d\tilde{\mathtt{v}}\Big|$$

$$\leq \frac{1}{\tilde{\gamma}(\delta)}\int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1}\Big|\mathtt{s}^*(\tilde{\mathtt{v}}, u_1(\tilde{\mathtt{v}})) - \mathtt{s}^*(\tilde{\mathtt{v}}, u_2(\tilde{\mathtt{v}}))\Big| d\tilde{\mathtt{v}}$$

$$+ \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1}\Big|\mathtt{s}^*(\tilde{\mathtt{v}}, u_1(\tilde{\mathtt{v}}))$$
$$- \mathtt{s}^*(\tilde{\mathtt{v}}, u_2(\tilde{\mathtt{v}}))\Big| d\tilde{\mathtt{v}}$$

$$+ \frac{2\mathtt{x}}{(2-^2)\,\tilde{\mathtt{r}}(\delta)}\int_0 \left(\int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}} - \tilde{\mathtt{q}})^{\delta-1}\Big|\mathtt{s}^*(\tilde{\mathtt{q}}, u_1(\tilde{\mathtt{q}}))\right.$$
$$- \mathtt{s}^*(\tilde{\mathtt{q}}, u_2(\tilde{\mathtt{q}}))\Big| d\tilde{\mathtt{q}} d\tilde{\mathtt{v}}$$

$$\leq \frac{1}{\tilde{\gamma}(\delta)}\int_0^{\mathtt{x}} (\tilde{\mathtt{t}} - \tilde{\mathtt{v}})^{\delta-1}\frac{\tilde{\mathtt{r}}(\delta+1)}{5} \mid u_1(\tilde{\mathtt{v}}) - u_2(\tilde{\mathtt{v}}) \mid d\tilde{\mathtt{v}}$$

$$+ \frac{2\tilde{\mathtt{t}}}{(2-^2)\,\tilde{\gamma}(\delta)}\int_0^1 (1-\tilde{\mathtt{v}})^{\delta-1}\frac{\tilde{\mathtt{r}}(\delta+1)}{5} \mid u_1(\tilde{\mathtt{v}}) - u_2(\tilde{\mathtt{v}}) \mid d\tilde{\mathtt{v}}$$

$$+ \frac{2\tilde{\mathtt{t}}}{(2-^2)\,\tilde{\gamma}(\delta)}\int_0 \left(\int_0^{\tilde{\mathtt{v}}} (\tilde{\mathtt{v}}\right.$$
$$- \tilde{\mathtt{q}})^{\delta-1}\frac{\tilde{\mathtt{r}}(\delta+1)}{5}$$
$$\mid u_1(\tilde{\mathtt{q}}) - u_2(\tilde{\mathtt{q}}) \mid d\tilde{\mathtt{q}} d\tilde{\mathtt{v}}$$

$$\leq \frac{\tilde{\gamma}(\delta+1)}{5}\,\frac{1}{\tilde{\mathtt{r}}(1+\delta)}\|u_1 - u_2\|_\infty$$

$$+ \frac{\tilde{\gamma}(\delta+1)}{5}\,\frac{2}{(2-^2)\,\tilde{\gamma}(1+\delta)}\|u_1 - u_2\|_\infty$$

$$+ \frac{\tilde{\gamma}(\delta+1)}{5}\,\frac{2^{1+\delta}}{(2-^2)\,(1+\delta)\tilde{\gamma}(1+\delta)}\|u_1 - u_2\|_\infty$$

$$\leq \left[\frac{1}{5} + \frac{2}{5(2-^2)} + \frac{2^{1+\delta}}{5(2-^2)(1+\delta)}\right]\|u_1 - u_2\|_\infty$$
$$\leq \frac{\|u_1 - u_2\|_\infty}{2}.$$

Since, for some $u_0 \in \mathcal{W}$, $\mathtt{s}^*(u_0(\tilde{\mathtt{t}}) - \mathfrak{T}u_0(\tilde{\mathtt{t}})) > 0$, so for all $\mathtt{x} \in [0, 1]$, we have

$$\aleph(\mathfrak{N}(\mathfrak{T}u_1, \mathfrak{T}u_2)) \leq \frac{\|u_1 - u_2\|_\infty}{2} = \Phi(\mathfrak{N}(u_1, u_2)),$$

for all $u_1, u_2 \in \mathcal{W}$ with $\mathfrak{N}(u_1, u_2) > 0$. Hence, all the conditions of Theorem 1 also show the existence of CBPP for $\mathfrak{R}$ and $\mathfrak{P}$. So, $\exists\, u^* \in \mathcal{W}$ such that $u^* = \mathfrak{T}u^*$ which satisfy Equations (30) and (31).

## 4. Conclusion

In this paper, we developed a comprehensive framework for CBPP results by introducing and analyzing several types of $(\aleph, \Phi)$- interpolative proximal contractions. The study encompassed generalized forms such as the $(\aleph, \Phi)$-proximal, Kannan, Ćirić-–Reich-–Rus, and Hardy-Rogers-type contractions, offering a unified perspective that extends and strengthens numerous existing results in the literature. We established existence and uniqueness theorems for CBPP under non-self mappings and demonstrated how the auxiliary functions $(\aleph, \Phi)$ influence the contractive behavior of the mappings. For each type of contraction, numerous nontrivial examples were given that satisfy our main result. Finally, an application of fractional differential equations was given that satisfies all the conditions of our main results and also show the practical significance of our proposed framework.

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The authors declare they have no competing interests.

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