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RESEARCH ARTICLE

A (q, τ)-quantum deformation approach to fixed point theory and optimal control

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ABSTRACT

This study develops a novel (q, τ)-quantum deformation framework for fixed point theory and its applications to fractional differential equations and optimization. By introducing a deformation dependent control function and the associated (q, τ)-metric structure, we establish a Banach type fixed point theorem with explicit convergence estimates governed by the (q, τ)-Gamma function. The obtained framework rigorously characterize the metricity and completeness of the deformed space and provide quantitative bounds on the rate of convergence of Picard iterations in terms of the deformation parameters (q, τ). The developed theory is applied to nonlinear fractional problems driven by the (q, τ)-fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$. Under explicit deformation dependent conditions, we prove existence and uniqueness of solutions and derive stability estimates that reveal how the (q, τ)-Gamma normalization regulates memory intensity and convergence speed. In addition, we formulate and analyze optimal control problems for (q, τ)-fractional systems. A complete first order optimality system is derived, including adjoint equations and projection characterizations of the optimal control, and the well posedness of the state and adjoint problems is established via the proposed fixed point framework. Numerical experiments validate the analytical results and demonstrate the effectiveness of the deformation parameters in tuning convergence behavior for both dynamical and optimization problems. In particular, the simulations show monotone decay of Picard iterates and objective functionals, confirming the stabilizing role of the (q, τ)-Gamma function. The proposed framework thus provides a unified and robust analytical and computational foundation for the study and optimization of quantum deformed fractional systems.

1. Introduction

Fractional calculus has emerged as a powerful mathematical framework for modeling systems with memory, hereditary effects, and nonlocal interactions. Its applications span diverse areas, including diffusion processes, control theory, viscoelasticity, signal processing, and complex dynamical systems. In recent years, increasing attention has been directed toward quantum-deformed and nonclassical fractional operators, which extend traditional fractional models by incorporating discrete, scale dependent, or nonlocal deformation mechanisms. Among these approaches, (q, τ)-quantum deformation (the extension of the quantum theory[1–3] provides a flexible structure in which classical operators are modified through two independent parameters. The parameter $q \in (0, 1)$ typically governs discretization or quantum scale effects, while the parameter $\tau \geq 0$ controls deformation intensity and memory

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depth.[4–6] This dual deformation allows a continuous transition between classical fractional dynamics and quantum inspired nonlocal behavior, offering enhanced modeling capabilities for complex systems.[7,8]

A fundamental analytical challenge in such deformed frameworks lies in establishing rigorous existence, uniqueness, and stability results. Classical fixed point principles, including Banach's contraction theorem, are often insufficient when convergence is not governed directly by metric distances but instead depends on deformed kernels and normalization functions.[9–13] This limitation becomes particularly evident in fractional models involving special functions, such as the (q,τ)-Gamma function, where convergence mechanisms differ substantially from their classical counterparts. Motivated by these challenges, this paper develops a novel (q,τ)-fixed point theory based on a deformation dependent control function and an induced (q,τ)-metric structure. The proposed framework extends classical contraction principles by measuring convergence through a deformed geometric gauge rather than direct distance shrinkage. This approach naturally incorporates the effects of quantum deformation and provides a unified analytical tool for studying fractional systems governed by (q,τ)-operators.

The developed theory is applied to nonlinear fractional differential equations involving the (q,τ)-fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$. The kernel of this operator is normalized by the (q,τ)-Gamma function $\Gamma_{q,\tau}$, which plays a central role in regulating memory strength, stability, and convergence speed. By exploiting the new fixed point framework, explicit existence and uniqueness conditions are derived, together with convergence estimates for the associated Picard iteration scheme. Beyond well posedness analysis, the present study establishes a direct connection with optimization theory. The fixed point results guarantee the uniqueness and stability of both state and adjoint equations, thereby enabling a rigorous formulation of optimal control problems under (q,τ)-fractional dynamics. A complete optimality system is derived, including adjoint equations and projection characterizations of the optimal control. Moreover, a gradient-projection algorithm is proposed and analyzed within the same theoretical framework.

To validate the analytical findings, numerical simulations are presented for both fractional dynamics and optimization problems. The results demonstrate monotone convergence of Picard iterations and stable decay of objective functionals, confirming the effectiveness of the (q,τ)-fixed point approach. In particular, the simulations highlight the stabilizing role of the (q,τ)-Gamma function and illustrate how the deformation parameters (q,τ)-can be tuned to control convergence and system behavior. The main contributions of this work include the development of a new (q,τ)-fixed point theorem formulated within a deformed metric framework, the derivation of rigorous existence and uniqueness results for (q,τ)-fractional equations involving the (q,τ)-Gamma function, a detailed convergence analysis of Picard iterations under quantum deformation, the establishment of a unified optimization framework incorporating adjoint systems and gradient projection methods, and comprehensive numerical simulations that confirm and illustrate the theoretical predictions. The proposed framework provides a robust analytical foundation for future studies on quantum deformed fractional models, with potential applications in control theory, inverse problems, and complex systems with nonlocal memory.

2. Preliminaries

In this section, we recall the basic concepts and auxiliary results required throughout the paper. These notions provide the analytical foundation for the development of fixed point theory under the (q,τ)-quantum deformation framework. Let $q \in (0,1)$ and $\tau \geq 0$ be fixed parameters. The pair (q,τ)-is used to describe a quantum deformation mechanism in which classical local structures are modified to incorporate finite scale, nonlocal, or memory dependent effects. The parameter q governs discretization or quantum scaling, while τ controls deformation intensity, accumulation depth, or memory strength. The classical case is recovered when $(q,\tau) \to (1,0)$.

Definition 1 ($[q,\tau)$-control function)**.** *A mapping $\Phi_{q,\tau} : [0,\infty) \to [0,\infty)$ is called a (q,τ)-*control function *if it satisfies:*

i $\Phi_{q,\tau}(0) = 0$ and $\Phi_{q,\tau}(r) > 0$ for all $r > 0$;
ii $\Phi_{q,\tau}$ is continuous and strictly increasing;
iii Subadditivity on the admissible class. We assume that the chosen (q,τ)-control function $\Phi_{q,\tau}$ belongs to the admissible class $\mathcal{A}_{q,\tau}$ and satisfies the subadditivity property

$$\Phi_{q,\tau}(a+b) \leq \Phi_{q,\tau}(a) + \Phi_{q,\tau}(b), \quad a,b \geq 0.$$

iv $\lim_{r\to 0^+} \Phi_{q,\tau}(r) = 0.$

Typical examples include:

$$\Phi_{q,\tau}(r) = \frac{r}{1+\tau r}, \quad \Phi_{q,\tau}(r) = q^{\tau} r, \quad \Phi_{q,\tau}(r) = r^{q^{\tau}}.$$

Remark 1 (Scope and classical limit of the deformation)**.** *The interpretation of the deformation parameters τ and q in terms of effective distance scaling and nonlinear compression is understood with respect to the class of admissible (q,τ)-control functions $\Phi_{q,\tau}$ satisfying the structural assumptions of Section 2 (namely, continuity, monotonicity in r, $\Phi_{q,\tau}(0)=0$, and the classical limit property). In particular, for the typical examples considered in this work (e.g., power type and Mittag Leffler type control functions), the parameter τ strengthens or weakens the effective distance scale, while q controls the nonlinear compression rate. Moreover, under the additional assumption that the family $\{\Phi_{q,\tau}\}$ admits the classical limit*

$$\lim_{q\to 1^-,\,\tau\to 0^+} \Phi_{q,\tau}(r) = r,\ \ r\geq 0,$$

the deformed framework reduces to the classical identity mapping $\Phi_{q,\tau}(r)=r$. Other admissible choices of $\Phi_{q,\tau}$ may lead to different limiting behaviors, without affecting the validity of the fixed point results established in this paper.

Definition 2 ((q,τ)-deformed metric)**.** *Let (X,d) be a metric space. The mapping*

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}\big(d(x,y)\big),\ \ x,y\in X,$$

is called a (q,τ)-deformed metric *on X, where $\Phi_{q,\tau}$ is a (q,τ)-control function satisfying $\Phi_{q,\tau}(0)=0$ and $\Phi_{q,\tau}(r)>0$ for $r>0$.*

Remark 2 (Topology and convergence induced by $d_{q,\tau}$)**.** *Let (X,d) be a metric space and define*

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}\big(d(x,y)\big),\ \ x,y\in X,$$

where $\Phi_{q,\tau}$ is continuous, nondecreasing, and satisfies $\Phi_{q,\tau}(r)=0$ if and only if $r=0$. Then $d_{q,\tau}$ induces the same notion of convergence as d, in the sense that for any sequence $\{x_n\}\subset X$ and any $x\in X$,

$$x_n\to x \text{ in } (X,d) \quad\Longleftrightarrow\quad x_n\to x \text{ in } (X,d_{q,\tau}).$$

Consequently, the two metrics generate the same topology on X. The deformation parameters (q,τ)-do not alter the underlying topological structure of X, but they modify the metric geometry, *i.e., the effective scale of distances and the quantitative notion of contraction. In particular, different choices of (q,τ)-rescale the convergence rate of iterative schemes such as Picard iteration, while preserving the limit points. In the classical limit $q\to 1^-$ and $\tau\to 0^+$, one recovers $d_{q,\tau}=d$ and hence the standard metric geometry.*

Lemma 1. *If $\Phi_{q,\tau}$ is a (q,τ)-control function, then $(X,d_{q,\tau})$ is a metric space, where*

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}\big(d(x,y)\big),\ \ x,y\in X.$$

Proof. Let (X,d) be a metric space and let $\Phi_{q,\tau}:[0,\infty)\to[0,\infty)$ be a (q,τ)-control function. We must verify that $d_{q,\tau}$ satisfies the four axioms of a metric on X, namely: (i) nonnegativity, (ii) identity of indiscernibles, (iii) symmetry, and (iv) the triangle inequality.

Step 1: Well-definedness and nonnegativity. Since $d(x,y)\geq 0$ for all $x,y\in X$ and $\Phi_{q,\tau}$ is defined on $[0,\infty)$ with values in $[0,\infty)$, the composition $\Phi_{q,\tau}(d(x,y))$ is well-defined and belongs to $[0,\infty)$. Hence, we have

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}(d(x,y)) \geq 0,\ \ \forall\, x,y\in X,$$

so $d_{q,\tau}$ is nonnegative.

Step 2: Identity of indiscernibles. We show that

$$d_{q,\tau}(x,y)=0 \quad\Longleftrightarrow\quad x=y.$$

($\Rightarrow$) Suppose $d_{q,\tau}(x,y)=0$. Then

$$0 = d_{q,\tau}(x,y) = \Phi_{q,\tau}(d(x,y)).$$

By the defining property of a (q,τ)-control function, $\Phi_{q,\tau}(r)=0$ if and only if $r=0$ (equivalently, $\Phi_{q,\tau}(0)=0$ and $\Phi_{q,\tau}(r)>0$ for all $r>0$). Therefore, we must have $d(x,y)=0$. Since d is a metric, $d(x,y)=0$ implies $x=y$.

($\Leftarrow$) Conversely, if $x=y$, then $d(x,y)=0$, hence

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}(d(x,y)) = \Phi_{q,\tau}(0) = 0.$$

Thus, the identity of indiscernibles holds.

Step 3: Symmetry. Let $x,y\in X$. Using symmetry of the original metric d we have $d(x,y)=d(y,x)$. Applying $\Phi_{q,\tau}$ to both sides yields

$$d_{q,\tau}(x,y) = \Phi_{q,\tau}(d(x,y)) = \Phi_{q,\tau}(d(y,x)) = d_{q,\tau}(y,x).$$

Hence, $d_{q,\tau}$ is symmetric.

Step 4: Triangle inequality. Let $x,y,z\in X$ be arbitrary. Since d is a metric,

$$d(x,z)\leq d(x,y)+d(y,z).$$

Because $\Phi_{q,\tau}$ is increasing (monotone nondecreasing), applying $\Phi_{q,\tau}$ preserves the inequality:

$$\Phi_{q,\tau}\big(d(x,z)\big) \leq \Phi_{q,\tau}\big(d(x,y)+d(y,z)\big).$$

Now we invoke the subadditivity of $\Phi_{q,\tau}$:

$$\Phi_{q,\tau}\big(d(x,y)+d(y,z)\big) \leq \Phi_{q,\tau}\big(d(x,y)\big)+\Phi_{q,\tau}\big(d(y,z)\big).$$

Combining the last two estimates gives

$$\Phi_{q,\tau}\big(d(x,z)\big) \leq \Phi_{q,\tau}\big(d(x,y)\big)+\Phi_{q,\tau}\big(d(y,z)\big),$$

which in terms of $d_{q,\tau}$ reads

$$d_{q,\tau}(x,z) \leq d_{q,\tau}(x,y)+d_{q,\tau}(y,z).$$

Thus, $d_{q,\tau}$ satisfies the triangle inequality. Steps 1 4 show that $d_{q,\tau}$ satisfies all metric axioms on X. Therefore, $(X,d_{q,\tau})$ is a metric space.

Definition 3 (Completeness under deformation)**.** *A sequence $\{x_n\}\subset X$ is said to be* (q,τ)-Cauchy

if
$$\lim_{m,n\to\infty} d_{q,\tau}(x_m,x_n)=0.$$

Lemma 2 (Completeness is preserved under (q,τ)-deformation)**.** *Let (X,d) be a complete metric space and let $\Phi_{q,\tau}:[0,\infty)\to[0,\infty)$ be continuous such that*
$$\Phi_{q,\tau}(r)=0 \iff r=0.$$
Define the (q,τ)-deformed metric
$$d_{q,\tau}(x,y)=\Phi_{q,\tau}\big(d(x,y)\big),\ \ x,y\in X.$$
Then $(X,d_{q,\tau})$ is complete.

Proof. We prove that every $d_{q,\tau}$ Cauchy sequence in X converges with respect to $d_{q,\tau}$. The strategy is:
$d_{q,\tau}$ Cauchy $\Rightarrow$ d Cauchy $\Rightarrow$ d convergent (by completeness of d) $\Rightarrow$ $d_{q,\tau}$ convergent (by continuity of $\Phi_{q,\tau}$).

Step 1: Start with an arbitrary $d_{q,\tau}$ Cauchy sequence. Let $\{x_n\}_{n\geq 1}\subset X$ be Cauchy in the deformed metric, i.e.,
$$\forall\,\varepsilon>0\ \exists\, N\in\mathbb{N} \text{ such that } m,n\geq N \implies d_{q,\tau}(x_m,x_n)<\varepsilon.$$
By the definition of $d_{q,\tau}$, this is equivalent to:
$$m,n\geq N \implies \Phi_{q,\tau}\big(d(x_m,x_n)\big)<\varepsilon. \tag{1}$$

Step 2: A key analytical claim. We claim that, under the stated hypotheses on $\Phi_{q,\tau}$,
$$\Phi_{q,\tau}(t_n)\to 0 \implies t_n\to 0 \quad \text{for every sequence } (t_n)\subset[0,\infty). \tag{2}$$

Proof of the claim. Assume $\Phi_{q,\tau}(t_n)\to 0$ but $t_n\not\to 0$. Then there exists $\delta>0$ and a subsequence $\{t_{n_k}\}$ such that $t_{n_k}\geq\delta$ for all k. Because $t_{n_k}\to t_*$ along a further subsequence is not guaranteed (we do not assume compactness), we instead use continuity at the point δ together with the property $\Phi_{q,\tau}(\delta)>0$. Indeed, since $\Phi_{q,\tau}(r)=0 \iff r=0$, we have $\Phi_{q,\tau}(\delta)>0$ for every $\delta>0$. Set $c:=\Phi_{q,\tau}(\delta)>0$. By continuity of $\Phi_{q,\tau}$ at δ, there exists $\eta>0$ such that
$$|s-\delta|<\eta \implies |\Phi_{q,\tau}(s)-c|<\frac{c}{2},$$
hence, $\Phi_{q,\tau}(s)>\frac{c}{2}$ whenever $s\in(\delta-\eta,\delta+\eta)$. Now, since $t_{n_k}\geq\delta$, we can pick $\delta'=\delta$ itself and note that for large k, $\Phi_{q,\tau}(t_{n_k})$ must remain bounded away from 0 (else it would contradict $\Phi_{q,\tau}(\delta)>0$ plus continuity near δ). More directly: if $\Phi_{q,\tau}(t_{n_k})\to 0$, then for large k we would have $\Phi_{q,\tau}(t_{n_k})<c/2$, which is impossible if $t_{n_k}\in(\delta-\eta,\delta+\eta)$; and if $t_{n_k}\geq\delta+\eta$ for infinitely many k, then by continuity and the fact that $\Phi_{q,\tau}$ is nonnegative with only one zero at 0, $\Phi_{q,\tau}$ cannot approach 0 along values bounded away from 0 without violating the intermediate value property near those points. In all cases we obtain a contradiction. Therefore, the Equation 2 holds. $\diamond$

Step 3: Convert $d_{q,\tau}$ Cauchy into d Cauchy. Fix an arbitrary $\varepsilon_0>0$. We want to show that there exists N such that $m,n\geq N$ implies $d(x_m,x_n)<\varepsilon_0$.

Consider the positive number
$$\varepsilon=\Phi_{q,\tau}\Big(\frac{\varepsilon_0}{2}\Big).$$
By the property $\Phi_{q,\tau}(r)=0\iff r=0$, we have $\varepsilon>0$. Since $\{x_n\}$ is $d_{q,\tau}$ Cauchy, there exists N such that for all $m,n\geq N$,
$$\Phi_{q,\tau}\big(d(x_m,x_n)\big)=d_{q,\tau}(x_m,x_n)<\varepsilon=\Phi_{q,\tau}\Big(\frac{\varepsilon_0}{2}\Big). \tag{3}$$
We now argue that Equation 3 forces $d(x_m,x_n)<\varepsilon_0$ for large indices. To see this, define the sequence
$$t_{m,n}=d(x_m,x_n)\geq 0.$$
From Equation 3, we learn that $\Phi_{q,\tau}(t_{m,n})$ can be made arbitrarily small as $m,n\to\infty$. In particular, letting $m,n\to\infty$ along any path yields $\Phi_{q,\tau}(t_{m,n})\to 0$. By the key claim Equation 2, this implies $t_{m,n}\to 0$, i.e.,
$$d(x_m,x_n)\to 0 \quad \text{as } m,n\to\infty,$$
which is exactly the Cauchy property in (X,d). Hence, $\{x_n\}$ is d Cauchy.

Step 4: Use completeness of (X,d) to get a limit in X. Since (X,d) is complete and $\{x_n\}$ is d Cauchy, there exists $x^*\in X$ such that
$$d(x_n,x^*)\to 0 \text{ as } n\to\infty. \tag{4}$$

Step 5: Convergence in d implies convergence in $d_{q,\tau}$. By definition,
$$d_{q,\tau}(x_n,x^*)=\Phi_{q,\tau}\big(d(x_n,x^*)\big).$$
Using Equation 4 and continuity of $\Phi_{q,\tau}$ at 0, we obtain
$$d_{q,\tau}(x_n,x^*)=\Phi_{q,\tau}\big(d(x_n,x^*)\big)\longrightarrow\Phi_{q,\tau}(0)=0.$$
Therefore, $x_n\to x^*$ in the deformed metric $(X,d_{q,\tau})$. Every $d_{q,\tau}$ Cauchy sequence converges in $(X,d_{q,\tau})$, hence $(X,d_{q,\tau})$ is complete.

Remark 3. *The assumption $\Phi_{q,\tau}(r)=0 \iff r=0$ ensures that the deformation does not collapse distinct points into zero distance. Continuity at 0 guarantees that d convergence transfers to $d_{q,\tau}$ convergence. In many concrete models, $\Phi_{q,\tau}$ is increasing and subadditive, so $d_{q,\tau}$ is a metric by Lemma 2.1 and is topologically compatible with d.*

Definition 4 ((q,τ)-contractive mapping). *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be* (q,τ)-contractive *if there exists $\lambda \in (0,1)$ such that*

$$\Phi_{q,\tau}\big(d(Tx,Ty)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x,y)\big), \;\; \forall\, x,y \in X.$$

Remark 4. *This condition reduces to the classical Banach contraction principle when $\Phi_{q,\tau}(r) = r$. For $\tau > 0$, the contraction effect is amplified in the deformed metric, allowing convergence even when classical contraction may fail.*

Lemma 3 (Basic convergence property). *Let (X,d) be a metric space and let $\Phi_{q,\tau} : [0,\infty) \to [0,\infty)$ be a (q,τ)-control function. Assume that $T : X \to X$ is a (q,τ)-contractive mapping, i.e., there exists a constant $\lambda \in (0,1)$ such that*

$$\Phi_{q,\tau}\big(d(Tx,Ty)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x,y)\big), \;\; \forall\, x,y \in X. \tag{5}$$

Fix $x_0 \in X$ and define the Picard iteration $x_{n+1} = Tx_n$ for $n \ge 0$. Then, for every $n \ge 0$,

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda^n\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

Proof. We give a detailed inductive proof and emphasize where the (q,τ)-contractive assumption is used.

Step 1: Rewrite the contractive condition along the Picard orbit. By definition of the iteration,

$$x_{n+1} = Tx_n, \;\; x_n = Tx_{n-1} \quad (n \ge 1).$$

Hence, for each $n \ge 1$, the pair (x_n, x_{n-1}) is admissible in the contractive inequality (Equation 5). Substituting $x = x_n$ and $y = x_{n-1}$ into (Equation 5) gives

$$\Phi_{q,\tau}\big(d(Tx_n,Tx_{n-1})\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_n,x_{n-1})\big). \tag{6}$$

Using the iteration rule again, $Tx_n = x_{n+1}$ and $Tx_{n-1} = x_n$, so (Equation 6) becomes the fundamental one-step estimate

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_n,x_{n-1})\big), \;\; n \ge 1. \tag{7}$$

This inequality is the starting point for the geometric decay.

Step 2: Verify the inequality for the first few indices (pattern). For $n = 1$, (Equation 7) yields

$$\Phi_{q,\tau}\big(d(x_2,x_1)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

For $n = 2$, applying (Equation 7) again gives

$$\Phi_{q,\tau}\big(d(x_3,x_2)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_2,x_1)\big) \le \lambda\Big[\lambda\, \Phi_{q,\tau}\big(d(x_1,x_0)\big)\Big] = \lambda^2\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

For $n = 3$, the same mechanism yields

$$\Phi_{q,\tau}\big(d(x_4,x_3)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_3,x_2)\big) \le \lambda^3\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

These computations strongly suggest the general formula

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda^n\, \Phi_{q,\tau}\big(d(x_1,x_0)\big),$$

which we now prove formally by induction.

Step 3: Induction proof.

Base case $n = 0$. We must show

$$\Phi_{q,\tau}\big(d(x_1,x_0)\big) \le \lambda^0\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

Since $\lambda^0 = 1$, this is an equality, hence true.

Induction hypothesis. Assume that for some fixed $n \ge 0$ we already have

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda^n\, \Phi_{q,\tau}\big(d(x_1,x_0)\big). \tag{8}$$

Induction step: Prove the statement for $n+1$. We need to show that

$$\Phi_{q,\tau}\big(d(x_{n+2},x_{n+1})\big) \le \lambda^{n+1}\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

Apply the one-step estimate (Equation 7) with index $n+1$ (note that (Equation 7) is valid for all indices ≥ 1, so it applies to $n+1$ whenever $n \ge 0$):

$$\Phi_{q,\tau}\big(d(x_{n+2},x_{n+1})\big) \le \lambda\, \Phi_{q,\tau}\big(d(x_{n+1},x_n)\big).$$

Now use the induction hypothesis (Equation 8) on the right-hand side:

$$\Phi_{q,\tau}\big(d(x_{n+2},x_{n+1})\big) \le \lambda\Big[\lambda^n\, \Phi_{q,\tau}\big(d(x_1,x_0)\big)\Big] = \lambda^{n+1}\, \Phi_{q,\tau}\big(d(x_1,x_0)\big).$$

This completes the induction. By mathematical induction, the inequality holds for every $n \ge 0$, namely,

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda^n\, \Phi_{q,\tau}\big(d(x_1,x_0)\big), \;\; n \ge 0.$$

This is exactly the desired estimate.

Remark 5. *Lemma 3 shows that the successive increments of the Picard orbit decay at least geometrically in the (q,τ)-deformed gauge. This estimate is the key input for proving that $\{x_n\}$ is (q,τ)-Cauchy (and hence Cauchy in d under mild assumptions on $\Phi_{q,\tau}$), leading to the existence of a fixed point.*

In the present work, the term quantum-deformation is used in the sense of q deformation and scale deformation as commonly adopted in quantum calculus and nonclassical analysis. The parameter $q \in (0,1)$ encodes discretization or quantization effects, interpolating between discrete and continuous regimes as $q \to 1^-$, while the parameter $\tau \ge 0$ modulates the intensity of deformation and memory in the associated kernels and in the (q,τ)-Gamma normalization. The control function $\Phi_{q,\tau}$ thus induces a deformed geometric gauge adapted to nonlocal fractional dynamics. In *the classical limit $q \to 1$ and $\tau \to 0$, the proposed framework reduces to the standard Banach fixed point setting.*

2.1. Background on the (q,τ)-fractional equation

Fractional differential equations of the form

$$\mathcal{D}^{\alpha}u(t) = f(t,u(t)), \quad t \in [0,T],$$

have been extensively studied as models for systems with memory and nonlocal effects. Classical analytical approaches for such equations include fixed point methods (Banach and Schauder), monotone iterative techniques, upper and lower solution methods, Laplace and integral transform techniques, and energy-based methods.[14–16] Existence and uniqueness results are typically established under Lipschitz-type conditions on the nonlinear term, while stability and long-time behavior are analyzed via Grönwall-type inequalities, Lyapunov functionals, or spectral methods.[17–20] In recent years, generalized fractional operators, including variable-order, tempered, and kernel-deformed operators, have been introduced to capture multi-scale memory and heterogeneous effects that cannot be described by standard Caputo or Riemann Liouville derivatives. Analytical treatment of such models often relies on transforming the differential equation into an equivalent Volterra integral equation and applying fixed point arguments in appropriate function spaces. Numerical methods commonly employed include convolution quadrature, predictor corrector schemes, spectral methods, and iterative Picard-type solvers.

The (q,τ)-fractional equation considered in this work extends these frameworks by incorporating quantum-deformation parameters into the kernel structure and normalization. This leads to new analytical challenges, as the associated integral operator exhibits deformation-dependent contractivity and memory strength. Previous studies on q fractional or kernel-deformed models have reported existence and uniqueness of solutions under restrictive assumptions and have demonstrated that deformation parameters can significantly influence transient dynamics and stability. However, a unified fixed point framework that explicitly encodes the deformation geometry and yields quantitative convergence estimates has been largely absent. The present work contributes to this line of research by providing a deformation-adapted fixed point theory for (q,τ)-fractional equations, establishing explicit existence and uniqueness conditions, and deriving convergence rates for Picard iterations in terms of the deformation parameters (q,τ)-and the (q,τ)-Gamma function. These results extend classical analyses of fractional equations and provide a systematic basis for both analytical study and numerical solution of quantum-deformed fractional models.

3. Results in fixed point theory

Theorem 1 (Main (q,τ)-fixed point theorem). *Let (X,d) be a complete metric space. Let $\Phi_{q,\tau} : [0,\infty) \to [0,\infty)$ be a (q,τ)-control function satisfying:*

(1) $\Phi_{q,\tau}(0) = 0$ and $\Phi_{q,\tau}(r) > 0$ for all $r > 0$;
(2) $\Phi_{q,\tau}$ is continuous and strictly increasing;
(3) $\Phi_{q,\tau}$ is subadditive: $\Phi_{q,\tau}(a+b) \le \Phi_{q,\tau}(a) + \Phi_{q,\tau}(b)$ for all $a,b \ge 0$.

Let $T : X \to X$ be (q,τ)-contractive, i.e., there exists $\lambda \in (0,1)$ such that

$$\Phi_{q,\tau}\big(d(Tx,Ty)\big) \le \lambda\,\Phi_{q,\tau}\big(d(x,y)\big), \ \ \forall\, x,y \in X. \tag{9}$$

Then:

(i) *T has a* unique *fixed point $x^* \in X$.*
(ii) *For any initial point $x_0 \in X$, the Picard sequence $x_{n+1} = Tx_n$ converges to x^* in the metric d and also in the deformed metric $d_{q,\tau}(x,y) = \Phi_{q,\tau}(d(x,y))$.*
(iii) *The following a priori estimate holds for all $n \ge 0$:*

$$\Phi_{q,\tau}\big(d(x_n,x^*)\big) \le \frac{\lambda^n}{1-\lambda}\,\Phi_{q,\tau}\big(d(x_1,x_0)\big). \tag{10}$$

Proof. Fix $x_0 \in X$ and define the Picard iteration $x_{n+1} = Tx_n$ for $n \ge 0$.

Step 1: Geometric decay of successive increments. By Lemma 3 (Basic convergence property), we have for every $n \ge 0$,

$$\Phi_{q,\tau}\big(d(x_{n+1},x_n)\big) \le \lambda^n\,\Phi_{q,\tau}\big(d(x_1,x_0)\big). \tag{11}$$

Step 2: The Picard sequence is Cauchy in the deformed metric. Let $m > n$. Using the triangle inequality for the original metric d,

$$d(x_m,x_n) \le \sum_{k=n}^{m-1} d(x_{k+1},x_k).$$

Apply $\Phi_{q,\tau}$ to both sides. Since $\Phi_{q,\tau}$ is increasing,

$$\Phi_{q,\tau}\big(d(x_m,x_n)\big) \le \Phi_{q,\tau}\left(\sum_{k=n}^{m-1} d(x_{k+1},x_k)\right).$$

Now use subadditivity repeatedly (finite induction) to obtain

$$\Phi_{q,\tau}\left(\sum_{k=n}^{m-1} d(x_{k+1},x_k)\right) \le \sum_{k=n}^{m-1} \Phi_{q,\tau}\big(d(x_{k+1},x_k)\big).$$

Consequently,

$$\Phi_{q,\tau}\big(d(x_m, x_n)\big) \leq \sum_{k=n}^{m-1} \Phi_{q,\tau}\big(d(x_{k+1}, x_k)\big). \quad (12)$$

Insert the geometric decay Equation 11 into Equation 12

$$\Phi_{q,\tau}\big(d(x_m, x_n)\big) \leq \sum_{k=n}^{m-1} \lambda^k \, \Phi_{q,\tau}\big(d(x_1, x_0)\big) \leq \Phi_{q,\tau}\big(d(x_1, x_0)\big) \sum_{k=n}^{\infty} \lambda^k.$$

Since $\sum_{k=n}^{\infty} \lambda^k = \lambda^n/(1-\lambda)$, we arrive at

$$\Phi_{q,\tau}\big(d(x_m, x_n)\big) \leq \frac{\lambda^n}{1-\lambda} \, \Phi_{q,\tau}\big(d(x_1, x_0)\big), \;\; m > n. \quad (13)$$

Letting $n \to \infty$ shows that the right-hand side tends to 0, uniformly in $m > n$. Hence, we get

$$\Phi_{q,\tau}\big(d(x_m, x_n)\big) \to 0 \quad \text{as } m, n \to \infty,$$

indicating that $\{x_n\}$ is Cauchy in the deformed metric $d_{q,\tau}(x, y) = \Phi_{q,\tau}(d(x, y))$.

Step 3: Convert $d_{q,\tau}$ Cauchy into d Cauchy. Because $\Phi_{q,\tau}$ is strictly increasing and $\Phi_{q,\tau}(0) = 0$, it follows that $\Phi_{q,\tau}(r) \to 0$ implies $r \to 0$ (equivalently, $\Phi_{q,\tau}$ is injective near 0 and continuous). Applying this to $r = d(x_m, x_n)$ yields

$$d(x_m, x_n) \to 0 \quad \text{as } m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, d).

Step 4: Existence of the fixed point (limit of Picard iteration). Since (X, d) is complete, there exists $x^* \in X$ such that

$$d(x_n, x^*) \to 0 \;\text{ as } n \to \infty. \quad (14)$$

We claim that x^* is a fixed point of T. To show this, estimate $d(Tx^*, x^*)$ using the triangle inequality:

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &= d(Tx^*, Tx_n) + d(x_{n+1}, x^*). \end{aligned}$$

Apply $\Phi_{q,\tau}$ and then subadditivity:

$$\begin{aligned} \Phi_{q,\tau}\big(d(Tx^*, x^*)\big) &\leq \Phi_{q,\tau}\big(d(Tx^*, Tx_n) + d(x_{n+1}, x^*)\big) \\ &\leq \Phi_{q,\tau}\big(d(Tx^*, Tx_n)\big) + \\ &\quad \Phi_{q,\tau}\big(d(x_{n+1}, x^*)\big). \end{aligned}$$

Now use the contractive condition (Equation 9) with $x = x^*$ and $y = x_n$:

$$\Phi_{q,\tau}\big(d(Tx^*, Tx_n)\big) \leq \lambda \, \Phi_{q,\tau}\big(d(x^*, x_n)\big).$$

Therefore,

$$\begin{aligned} &\Phi_{q,\tau}\big(d(Tx^*, x^*)\big) \leq \lambda \, \Phi_{q,\tau}\big(d(x^*, x_n)\big) + \\ &\Phi_{q,\tau}\big(d(x_{n+1}, x^*)\big). \end{aligned} \quad (15)$$

Using Equation 14 and continuity of $\Phi_{q,\tau}$, we have $\Phi_{q,\tau}(d(x^*, x_n)) \to 0$ and $\Phi_{q,\tau}(d(x_{n+1}, x^*)) \to 0$ as $n \to \infty$. Taking the limit in Equation 15 yields

$$\Phi_{q,\tau}\big(d(Tx^*, x^*)\big) = 0.$$

Since $\Phi_{q,\tau}(r) = 0$ if and only if $r = 0$, it follows that $d(Tx^*, x^*) = 0$, hence $Tx^* = x^*$. This proves existence.

Step 5: Uniqueness of the fixed point. Suppose $y^* \in X$ is another fixed point: $Ty^* = y^*$. Apply Equation 9 to the pair (x^*, y^*):

$$\begin{aligned} \Phi_{q,\tau}\big(d(x^*, y^*)\big) &= \Phi_{q,\tau}\big(d(Tx^*, Ty^*)\big) \\ &\leq \lambda \, \Phi_{q,\tau}\big(d(x^*, y^*)\big). \end{aligned}$$

Subtract the left side to obtain $(1-\lambda)\, \Phi_{q,\tau}\big(d(x^*, y^*)\big) \leq 0$. Since $1 - \lambda > 0$ and $\Phi_{q,\tau} \geq 0$, we conclude that $\Phi_{q,\tau}(d(x^*, y^*)) = 0$, hence $d(x^*, y^*) = 0$, so $x^* = y^*$. Therefore, the fixed point is unique.

Step 6: Convergence and the a priori estimate. Letting $m \to \infty$ in Equation 13 and using $x_m \to x^*$ in d (which implies $d(x_m, x_n) \to d(x^*, x_n)$ and, by continuity, $\Phi_{q,\tau}(d(x_m, x_n)) \to \Phi_{q,\tau}(d(x^*, x_n))$), we obtain

$$\Phi_{q,\tau}\big(d(x^*, x_n)\big) \leq \frac{\lambda^n}{1-\lambda} \, \Phi_{q,\tau}\big(d(x_1, x_0)\big),$$

which is exactly Equation 10. In particular, the right-hand side tends to 0, giving $\Phi_{q,\tau}(d(x_n, x^*)) \to 0$, hence $x_n \to x^*$ in the deformed metric as well. All assertions (i) (iii) are proved.

Remark 6. *Relation to classical metric fixed point theory For fixed parameters (q, τ), one may define an induced metric $D(x, y) = \Phi_{q,\tau}(d(x, y))$ on X under the assumptions of Section 2. In this sense, the fixed point result of Theorem 1 is structurally analogous to classical Banach-type theorems in the metric space (X, D). The novelty of the present framework lies not in restating Banach's principle for an abstract metric, but in constructing a* deformation-adapted *geometry in which convergence and stability depend explicitly on (q, τ). This dependence is crucial for applications to (q, τ)-fractional operators and for interpreting the role of quantum deformation and memory through $\Phi_{q,\tau}$ and $\Gamma_{q,\tau}$.*

Corollary 1 (Banach-type fixed point principle in $(X, d_{q,\tau})$)**.** *Let (X, d) be a metric space and let $\Phi_{q,\tau} : [0, \infty) \to [0, \infty)$ be a (q, τ)-control function. Define the (q, τ)-deformed metric*

$$d_{q,\tau}(x, y) = \Phi_{q,\tau}\big(d(x, y)\big), \;\; x, y \in X.$$

Assume that $(X, d_{q,\tau})$ is complete. Suppose that $T : X \to X$ is a contraction with respect to $d_{q,\tau}$, *i.e., there exists $\lambda \in (0,1)$ such that*

$$d_{q,\tau}(Tx,Ty) \le \lambda\, d_{q,\tau}(x,y), \ \forall\, x,y \in X. \tag{16}$$

Then:

(i) *T has a unique fixed point $x^* \in X$.*

(ii) *For any $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^* in the metric $d_{q,\tau}$.*

(iii) *The following estimate holds for all $n \ge 0$:*

$$d_{q,\tau}(x_n, x^*) \le \frac{\lambda^n}{1-\lambda}\, d_{q,\tau}(x_1, x_0).$$

Proof. By definition of $d_{q,\tau}$, the contractive condition (Equation 16) is equivalent to

$$\Phi_{q,\tau}\big(d(Tx,Ty)\big) \le \lambda\, \Phi_{q,\tau}\big(d(x,y)\big), \ \forall\, x,y \in X,$$

which is precisely the (q,τ)-contractive condition (Equation 9) in Theorem 1. Since $(X, d_{q,\tau})$ is complete, all hypotheses of Theorem 1 are satisfied (with completeness interpreted in the deformed metric). Therefore, Theorem 1 yields the existence and uniqueness of a fixed point x^* and the convergence of the Picard iteration to x^* in $d_{q,\tau}$, together with the stated error bound.

4. Applications to the (q,τ)-fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$

In this section, we demonstrate how the developed (q,τ)-fixed point theory can be applied to nonlinear fractional equations involving the (q,τ)-deformed fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$. The analysis reveals the fundamental role of the (q,τ)-Gamma function $\Gamma_{q,\tau}$ in guaranteeing well posedness of the associated evolution problems.

Definition 5 (The (q,τ)-Gamma function)**.** *Let $0<q<1$ and $\tau \ge 0$. The (q,τ)-Gamma function is defined by*

$$\Gamma_{q,\tau}(z) = (1-q)^{1-z}\,\frac{(q;q)_\infty}{(q^{z+\tau};q)_\infty}, \quad z \in \mathbb{C}\setminus\{0,-1,-2,\dots\}, \tag{17}$$

where the q Pochhammer symbol $(a;q)_\infty$ is given by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The (q,τ)-Gamma function satisfies

$$\Gamma_{q,\tau}(z+1) = \frac{1-q^{z+\tau}}{1-q}\,\Gamma_{q,\tau}(z).$$

Definition 6 (Deformation kernel)**.** *For $0 \le s < t \le T$, define the (q,τ)-deformation kernel by*

$$\omega_{q,\tau}(t,s) = \frac{\Gamma_{q,\tau}(\alpha)}{\Gamma(\alpha)}\left(\frac{t-s}{t-s+\tau}\right)^{1-q}, \tag{18}$$

where $q \in (0,1)$ and $\tau \ge 0$ are deformation parameters and $\Gamma_{q,\tau}$ denotes the (q,τ)-Gamma function. In the classical limit $q \to 1^-$ and $\tau \to 0^+$, one has $\omega_{q,\tau}(t,s) \to 1$, and the standard Caputo kernel is recovered.

Definition 7 (The (q,τ)-fractional operator)**.** *Let $\alpha \in (0,1)$ and $q \in (0,1)$, $\tau \ge 0$. We consider the (q,τ)-fractional operator defined by*

$$\mathcal{D}^{\alpha}_{q,\tau}u(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\omega_{q,\tau}(t,s)\,u'(s)\,ds, \tag{19}$$

where $\Gamma_{q,\tau}$ denotes the (q,τ)-Gamma function and $\omega_{q,\tau}(t,s)$ is a deformation weight satisfying

$$0 < \omega_{q,\tau}(t,s) \le 1, \ \omega_{q,\tau}(t,s) \to 1 \quad \text{as } (q,\tau) \to (1,0).$$

Remark 7. *When $(q,\tau) \to (1,0)$, operator (Equation 19) reduces to the classical Caputo fractional derivative. Hence, $\mathcal{D}^{\alpha}_{q,\tau}$ represents a genuine quantum deformation of fractional dynamics.*

Assumption (Regularity for $\mathcal{D}^{\alpha}_{q,\tau}$). Let $u \in AC([0,T])$ and $0<\alpha<1$. Assume that the kernel $\omega_{q,\tau}(t,s)$ is continuous on $\{(t,s) : 0 \le s < t \le T\}$ and satisfies

$$0 < \omega_{q,\tau}(t,s) \le C_{q,\tau}, \ \text{and } |\partial_t \omega_{q,\tau}(t,s)| \le C_{q,\tau}(t-s)^{-1+\epsilon}$$

for some constants $C_{q,\tau} > 0$ and $\epsilon > 0$. Then the integral in (Equation 19) is well-defined for all $t \in (0,T]$, and $\mathcal{D}^{\alpha}_{q,\tau}u \in L^1(0,T) \cap C((0,T])$.

Proposition 1 (Well-posedness and regularity of $\mathcal{D}^{\alpha}_{q,\tau}$)**.** *Let $T>0$, $0<\alpha<1$, and let $u \in AC([0,T])$. Assume that the deformation kernel $\omega_{q,\tau}(t,s)$ is measurable in s for each fixed $t \in (0,T]$, continuous in t for almost every $s \in (0,T)$, and satisfies the uniform bound*

$$0 \le \omega_{q,\tau}(t,s) \le C_{q,\tau} \ \text{ for all } 0 \le s < t \le T, \tag{20}$$

for some constant $C_{q,\tau} > 0$. Define, for $t \in (0,T]$,

$$\mathcal{D}^{\alpha}_{q,\tau}u(t) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\omega_{q,\tau}(t,s)\,u'(s)\,ds. \tag{21}$$

Then:

(i) *The integral in Equation 21 is finite for each $t \in (0,T]$, and $\mathcal{D}^{\alpha}_{q,\tau}u$ is well-defined almost everywhere on $(0,T)$. Moreover, the pointwise estimate*

$$|\mathcal{D}^{\alpha}_{q,\tau}u(t)| \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,|u'(s)|\,ds \tag{22}$$

holds for all $t \in (0,T]$.

(ii) $\mathcal{D}^{\alpha}_{q,\tau}u \in L^1(0,T)$ *and satisfies the norm bound*

$$\|\mathcal{D}^{\alpha}_{q,\tau}u\|_{L^1(0,T)} \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\,\frac{T^{1-\alpha}}{1-\alpha}\,\|u'\|_{L^1(0,T)}. \tag{23}$$

(iii) *If, in addition,* $u' \in L^\infty(0,T)$, *then* $\mathcal{D}^{\alpha}_{q,\tau}u \in L^\infty(0,T)$ *and*

$$\|\mathcal{D}^{\alpha}_{q,\tau}u\|_{L^\infty(0,T)} \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\,\frac{T^{1-\alpha}}{1-\alpha}\,\|u'\|_{L^\infty(0,T)}. \tag{24}$$

(iv) *If, moreover,* $\omega_{q,\tau}$ *is continuous on* $\{(t,s): 0\le s<t\le T\}$ *and* u' *is continuous on* $[0,T]$, *then* $\mathcal{D}^{\alpha}_{q,\tau}u \in C((0,T])$. *If, additionally,* $\sup_{0<s<t\le T}(t-s)^{\alpha}|\omega_{q,\tau}(t,s)| < \infty$ *and* $u'(0)=0$, *then* $\mathcal{D}^{\alpha}_{q,\tau}u$ *admits a continuous extension to* $t=0$, *i.e.,* $\mathcal{D}^{\alpha}_{q,\tau}u \in C([0,T])$.

Proof. **(i)** Since $u \in AC([0,T])$, we have $u' \in L^1(0,T)$. Fix $t \in (0,T]$. Using Equation 20 and $\Gamma_{q,\tau}(1-\alpha) > 0$,

$$\left|\mathcal{D}^{\alpha}_{q,\tau}u(t)\right| \le \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\omega_{q,\tau}(t,s)\,|u'(s)|\,ds \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}|u'(s)|\,ds,$$

which proves Equation 22. The right-hand side is finite for each fixed $t>0$ because $(t-s)^{-\alpha} \in L^1(0,t)$ when $0<\alpha<1$ and $u' \in L^1$, hence the integral in Equation 21 is finite. Thus, $\mathcal{D}^{\alpha}_{q,\tau}u$ is well-defined for each $t \in (0,T]$ (and hence almost everywhere on $(0,T)$).

(ii) Integrate Equation 22 over $t \in (0,T)$ and apply Tonelli's theorem (all integrands are nonnegative):

$$\begin{aligned}
\|\mathcal{D}^{\alpha}_{q,\tau}u\|_{L^1(0,T)} &\le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^T\int_0^t (t-s)^{-\alpha}|u'(s)|\,ds\,dt \\
&= \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^T |u'(s)|\left(\int_s^T (t-s)^{-\alpha}\,dt\right)ds \\
&= \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^T |u'(s)|\left(\int_0^{T-s} r^{-\alpha}\,dr\right)ds \\
&= \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^T |u'(s)|\,\frac{(T-s)^{1-\alpha}}{1-\alpha}\,ds \\
&\le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\,\frac{T^{1-\alpha}}{1-\alpha}\,\|u'\|_{L^1(0,T)},
\end{aligned}$$

which yields Equation 23. In particular, $\mathcal{D}^{\alpha}_{q,\tau}u \in L^1(0,T)$.

(iii) If $u' \in L^\infty(0,T)$, then for every $t \in (0,T]$,

$$\left|\mathcal{D}^{\alpha}_{q,\tau}u(t)\right| \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\|u'\|_{L^\infty(0,T)}\int_0^t (t-s)^{-\alpha}\,ds = fracC_{q,\tau}\Gamma_{q,\tau}(1-\alpha)\|u'\|_{L^\infty(0,T)}\frac{t^{1-\alpha}}{1-\alpha}.$$

Taking the supremum over $t \in (0,T]$ gives Equation 24, and hence $\mathcal{D}^{\alpha}_{q,\tau}u \in L^\infty(0,T)$.

(iv) Assume $\omega_{q,\tau}$ is continuous on $0\le s<t\le T$ and $u' \in C([0,T])$. Fix $t_0 \in (0,T]$ and let $t \to t_0$. Write the integrand as

$$K(t,s) = (t-s)^{-\alpha}\omega_{q,\tau}(t,s)u'(s).$$

For t in a small neighborhood of t_0, the singularity at $s=t$ is integrable because $0<\alpha<1$. Moreover, by continuity of $\omega_{q,\tau}$ away from the diagonal and boundedness (Equation 20), one can dominate $|K(t,s)|$ by

$$|K(t,s)| \le C_{q,\tau}\,(t-s)^{-\alpha}\,\|u'\|_{L^\infty(0,T)},$$

which is integrable over $s \in (0,t)$ uniformly for t near t_0. Therefore, by the dominated convergence, the map $t \mapsto \mathcal{D}^{\alpha}_{q,\tau}u(t)$ is continuous on $(0,T]$. To extend continuity to $t=0$, note from (Equation 21) and the additional assumptions that

$$\left|\mathcal{D}^{\alpha}_{q,\tau}u(t)\right| \le \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,|\omega_{q,\tau}(t,s)|\,|u'(s)|\,ds \le \frac{C_{q,\tau}}{\Gamma_{q,\tau}(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,|u'(s)|\,ds,$$

and if $u'(0)=0$ with u' continuous, then $|u'(s)| \le \varepsilon$ for s small, so the right-hand side tends to 0 as $t \downarrow 0$. Hence, $\mathcal{D}^{\alpha}_{q,\tau}u$ extends continuously to $t=0$.

4.1. Existence and uniqueness via the (q,τ)-fixed point theorem

In this section, we apply the (q,τ)-fixed point theorem of Section 3 to establish existence and uniqueness of solutions for a nonlinear (q,τ)-fractional initial value problem driven by $\mathcal{D}^{\alpha}_{q,\tau}$ and normalized by $\Gamma_{q,\tau}$.
Let $T>0$ and $\alpha \in (0,1)$. Consider the nonlinear problem

$$\begin{cases} \mathcal{D}^{\alpha}_{q,\tau}u(t) = f(t,u(t)), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{25}$$

where $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and $\mathcal{D}^{\alpha}_{q,\tau}$ is the (q,τ)-fractional operator (as defined in Section 2). Assume that $\mathcal{D}^{\alpha}_{q,\tau}$ admits a left inverse

(q,τ)-fractional integral operator $\mathcal{I}^{\alpha}_{q,\tau}$ such that

$$\mathcal{I}^{\alpha}_{q,\tau}\big(\mathcal{D}^{\alpha}_{q,\tau}u\big)(t) = u(t) - u(0), \quad t \in [0,T]. \tag{26}$$

under the assumption that $u \in AC([0,T])$ and $\mathcal{D}^{\alpha}_{q,\tau}u \in L^1(0,T)$ (in particular, the operators $\mathcal{D}^{\alpha}_{q,\tau}$ and $\mathcal{I}^{\alpha}_{q,\tau}$ are well-defined).

Applying $\mathcal{I}^{\alpha}_{q,\tau}$ to Equation 24 and using Equation 26, we obtain the equivalent Volterra-type integral equation

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \omega_{q,\tau}(t,s)\, f(s,u(s))\, ds, \quad t \in [0,T], \end{aligned} \tag{27}$$

where $\omega_{q,\tau}(t,s)$ is the deformation weight associated with $\mathcal{D}^{\alpha}_{q,\tau}$ (with $0 < \omega_{q,\tau}(t,s) \le 1$).

Let $X = C([0,T],\mathbb{R})$ with the classical metric

$$d(u,v) = \|u-v\|_\infty = \max_{t\in[0,T]} |u(t)-v(t)|.$$

Let $\Phi_{q,\tau}$ be the (q,τ)-control function introduced in Section 2, and define the deformed metric

$$d_{q,\tau}(u,v) := \Phi_{q,\tau}\big(d(u,v)\big).$$

Since (X,d) is complete and $\Phi_{q,\tau}$ satisfies the hypotheses of Lemma 2, it follows that $(X, d_{q,\tau})$ is complete. Define the operator $T : X \to X$ by

$$\begin{aligned} (Tu)(t) &= u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \\ &\int_0^t (t-s)^{\alpha-1}\, \omega_{q,\tau}(t,s)\, f(s,u(s))\, ds. \end{aligned} \tag{28}$$

Then u solves Equation 27 (and hence Equation 25) if and only if u is a fixed point of T, i.e., $Tu = u$.

Theorem 2 (Existence and uniqueness)**.** *Assume that:*

(i) *H1 $f(\cdot,\cdot)$ is continuous on $[0,T]\times\mathbb{R}$;*

(ii) *H2 f is Lipschitz in its second variable: there exists $L > 0$ such that*

$$|f(t,u)-f(t,v)| \le L|u-v|, \quad \forall\, t \in [0,T],\ \forall\, u,v \in \mathbb{R};$$

(iii) *H3 The deformation weight is bounded by $0 < \omega_{q,\tau}(t,s) \le 1$ for all $0 \le s \le t \le T$;*

(iv) *H4 The (q,τ)-Gamma normalization yields the contraction constraint*

$$\lambda := \frac{L\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1. \tag{29}$$

Then the (q,τ)-fractional problem (Equation 25) admits a unique *solution $u^* \in C([0,T],\mathbb{R})$. Moreover, for any $u_0 \in X$, the Picard iteration $u_{n+1} = Tu_n$ converges to u^* in the metric d and in the deformed metric $d_{q,\tau}$, and satisfies the error bound*

$$\Phi_{q,\tau}\big(\|u_n-u^*\|_\infty\big) \le \frac{\lambda^n}{1-\lambda}\, \Phi_{q,\tau}\big(\|u_1-u_0\|_\infty\big),\ n \ge 0.$$

Proof. We apply the main (q,τ)-fixed point theorem of Section 3 to the operator T defined in Equation 28.

Step 1: T maps X into X. Fix $u \in X$. The map $t \mapsto (Tu)(t)$ is continuous on $[0,T]$ since: (i) $f(\cdot,u(\cdot))$ is continuous by H1; (ii) the kernel $(t-s)^{\alpha-1}$ is integrable on $[0,t]$ for $\alpha \in (0,1)$; and (iii) $\omega_{q,\tau}(t,s)$ is bounded by H3. Hence, $Tu \in C([0,T],\mathbb{R})$, so $T : X \to X$ is well-defined.

Step 2: Lipschitz estimate for T in the classical metric. Let $u,v \in X$ and fix $t \in [0,T]$. From Equation 28 and H2 H3,

$$\begin{aligned} |(Tu)(t)-(Tv)(t)| &\le \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\qquad \omega_{q,\tau}(t,s)\, |f(s,u(s)) - f(s,v(s))|\, ds \\ &\le \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds\, \|u-v\|_\infty. \end{aligned}$$

Compute the integral explicitly:

$$\int_0^t (t-s)^{\alpha-1}\, ds = \frac{t^\alpha}{\alpha}.$$

Using the standard identity $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and the analogous normalization consistent with $\Gamma_{q,\tau}$, we rewrite

$$\frac{1}{\Gamma_{q,\tau}(\alpha)} \cdot \frac{t^\alpha}{\alpha} = \frac{t^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

Therefore,

$$\begin{aligned} |(Tu)(t)-(Tv)(t)| &\le \frac{L\,t^\alpha}{\Gamma_{q,\tau}(\alpha+1)}\, \|u-v\|_\infty \\ &\le \frac{L\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}\, \|u-v\|_\infty. \end{aligned}$$

Taking the maximum over $t \in [0,T]$ yields

$$\|Tu-Tv\|_\infty \le \lambda\, \|u-v\|_\infty, \quad \lambda := \frac{L\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}. \tag{30}$$

By H4, $\lambda \in (0,1)$.

Step 3: (q,τ)-contractivity of T. Applying the increasing function $\Phi_{q,\tau}$ to both sides of Equation 30 gives

$$\Phi_{q,\tau}\big(\|Tu-Tv\|_\infty\big) \le \Phi_{q,\tau}\big(\lambda\, \|u-v\|_\infty\big).$$

Since $\Phi_{q,\tau}$ is a control function, it is standard (and assumed in our framework) that it is compatible with scalar contraction in the sense that

$$\Phi_{q,\tau}(\lambda r) \le \lambda\, \Phi_{q,\tau}(r), \quad 0 < \lambda < 1,\ r \ge 0,$$

which holds for the typical choices used in this study (e.g., $\Phi_{q,\tau}(r) = q^\tau r$ or $\Phi_{q,\tau}(r) = \frac{r}{1+\tau r}$). Hence,

$$\Phi_{q,\tau}\big(d(Tu,Tv)\big) = \Phi_{q,\tau}\big(\|Tu - Tv\|_\infty\big) \le \lambda\,\Phi_{q,\tau}\big(\|u-v\|_\infty\big) = \lambda\,\Phi_{q,\tau}\big(d(u,v)\big).$$

Thus, T is (q,τ)-contractive in the sense of Section 3.

Step 4: Apply the (q,τ)-fixed point theorem. Because (X,d) is complete, and T is (q,τ)-contractive with constant $\lambda \in (0,1)$, Theorem 1 guarantees the existence of a unique fixed point $u^* \in X$ of T. By construction (Equation 28), u^* solves (Equation 27) and therefore solves the original problem (Equation 25). Uniqueness of the fixed point yields uniqueness of the solution. The convergence of the Picard sequence $u_{n+1} = Tu_n$ to u^* and the stated error estimate follow directly from Theorem 1.

Remark 8. *Condition (Equation 29) shows explicitly how $\Gamma_{q,\tau}$ controls well posedness: a larger value of $\Gamma_{q,\tau}(\alpha+1)$ reduces the effective contraction factor λ and hence improves convergence speed. In the classical limit $(q,\tau) \to (1,0)$ one recovers the standard Caputo existence uniqueness criterion.*
The deformation function $\Gamma_{q,\tau}$ plays a decisive role in the convergence mechanism. Increasing τ typically enlarges $\Gamma_{q,\tau}(\alpha+1)$, thereby reducing the effective contraction constant

$$\lambda = \frac{L\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)},$$

which enhances stability and accelerates convergence of Picard iterations.
In contrast to classical fractional equations, the (q,τ)-Gamma function acts as a tunable stabilization regulator, allowing one to control the memory strength and convergence speed of the system.

Remark 9 (Relation to Banach's fixed point theorem)**.** *Once the deformation-induced distance $D(x,y) = \Phi_{q,\tau}(d(x,y))$ is shown to define a complete metric, Theorem 2 is structurally analogous to the classical Banach contraction principle in the metric space (X,D). The novelty of the present result lies in constructing and analyzing this deformation-adapted geometry and in expressing contractivity and convergence explicitly in terms of the parameters (q,τ)-and the control function $\Phi_{q,\tau}$, which is crucial for applications to (q,τ)-fractional operators and optimization problems.*

4.2. Examples

Throughout these examples, we work on $X = C([0,T],\mathbb{R})$ with the supremum norm $\|\cdot\|_\infty$ and consider the operator

$$(Tu)(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega_{q,\tau}(t,s)\, f(s,u(s))\,ds, \;\; t\in[0,T],$$

where $0 < \omega_{q,\tau}(t,s) \le 1$. The solution of $\mathcal{D}^\alpha_{q,\tau}u(t) = f(t,u(t))$ with $u(0)=u_0$ is precisely the fixed point of T.

Example 1 (Linear (q,τ)-fractional relaxation)**.** *Let $\alpha\in(0,1)$, $T>0$, and consider*

$$\begin{cases} \mathcal{D}^\alpha_{q,\tau}u(t) = -a\,u(t) + g(t), & t\in[0,T],\\ u(0)=u_0, \end{cases} \tag{31}$$

where $a>0$ and $g\in C([0,T])$. Set $f(t,u) = -a\,u + g(t)$. Then for all $u,v\in\mathbb{R}$,

$$|f(t,u)-f(t,v)| = a|u-v|,$$

so the Lipschitz constant is $L=a$. The contraction factor in Theorem 2 becomes

$$\lambda = \frac{a\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

If

$$\frac{a\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1, \tag{32}$$

then T is (q,τ)-contractive and Equation 31 admits a unique solution $u^*\in C([0,T])$. Equation 32 shows explicitly how $\Gamma_{q,\tau}$ controls well posedness: larger $\Gamma_{q,\tau}(\alpha+1)$ reduces λ and thus strengthens convergence of the Picard iteration $u_{n+1} = Tu_n$.

Example 2 (Nonlinear (q,τ)-fractional logistic–type equation)**.** *Let $\alpha\in(0,1)$ and consider*

$$\begin{cases} \mathcal{D}^\alpha_{q,\tau}u(t) = r\,u(t)\big(1-u(t)\big) + h(t), & t\in[0,T],\\ u(0)=u_0, \end{cases} \tag{33}$$

where $r>0$ and $h\in C([0,T])$. Fix a radius $R>0$ and define the closed ball

$$B_R = \{u\in C([0,T]) : \|u\|_\infty \le R\}.$$

Set $f(t,u) = r\,u(1-u) + h(t)$. For $|u|,|v|\le R$, we estimate

$$\begin{aligned}|f(t,u)-f(t,v)| &= r\,|u-u^2-(v-v^2)|\\ &= r\,|(u-v)-(u^2-v^2)| \le r\big(1+|u|+|v|\big)|u-v|\\ &\le r(1+2R)\,|u-v|.\end{aligned}$$

Thus f is Lipschitz on $[-R,R]$ with constant $L_R = r(1+2R)$. Consequently, for $u,v\in B_R$,

$$\|Tu-Tv\|_\infty \le \frac{L_R\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}\,\|u-v\|_\infty,\;\; L_R = r(1+2R).$$

If

$$\lambda_R = \frac{r(1+2R)\,T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1, \tag{34}$$

then T is a contraction on B_R (and hence (q,τ)-contractive in the sense of Section 2).

To ensure $T(B_R) \subseteq B_R$, note that for $u \in B_R$,

$$|(Tu)(t)| \le |u_0| + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(|r\, u(s) \\ (1-u(s))| + |h(s)|\big)\, ds.$$

Using $|u(s)| \le R$ and $|u(s)(1-u(s))| \le R(1+R)$, we get

$$\|Tu\|_\infty \le |u_0| + \frac{T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} \Big(r\, R(1+R) + \|h\|_\infty \Big).$$

Hence, $T(B_R) \subseteq B_R$ whenever

$$|u_0| + \frac{T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} \Big(r\, R(1+R) + \|h\|_\infty \Big) \le R. \quad (35)$$

If Equations 34 and 35 hold for some $R > 0$, then T maps B_R into itself and is a contraction on B_R. Therefore, Equation 33 has a unique solution $u^* \in B_R$, and the Picard iteration converges to u^*.

4.3. Numerical examples (Picard iteration under (q,τ)-deformation)

We illustrate the (q,τ)-fixed point theorem by implementing the Picard scheme for the equivalent Volterra equation

$$u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\, \omega_{q,\tau} \\ (t,s)\, f(s,u(s))\, ds, \quad t \in [0,T],$$

with $\omega_{q,\tau}(t,s) \equiv 1$ and a uniform grid on $[0,T]$. We set $q = 0.5$, $\alpha = 0.8$, $T = 1$, and compare $\tau \in \{0, 0.5, 1, 1.5\}$.

Example 1 (linear relaxation). We consider

$$\mathcal{D}^\alpha_{q,\tau} u(t) = -a\, u(t) + \sin(2\pi t), \quad u(0) = u_0,$$

with $u_0 = 0.2$ and $a = 0.35$. Here, $f(t,u) = -au + \sin(2\pi t)$ is Lipschitz in u with constant $L = a$. The contraction factor predicted by Theorem 2 is

$$\lambda = \frac{aT^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

For each tested value of τ, we computed $\Gamma_{q,\tau}(\alpha+1)$ and obtained $\lambda < 1$, guaranteeing a unique solution. Figure 1A and 1B show the limiting Picard solutions and the decay of the increments $\|u_{n+1} - u_n\|_\infty$ (semi-log), respectively.

Example 2 (nonlinear logistic-type). We consider

$$\mathcal{D}^\alpha_{q,\tau} u(t) = r\, u(t)(1-u(t)) + 0.1\cos(2\pi t), \quad u(0) = u_0,$$

with $u_0 = 0.2$ and $r = 0.6$. The same Picard discretization yields convergence of $\|u_{n+1} - u_n\|_\infty \to 0$ for all tested τ, consistent with the (q,τ)-fixed point framework. Figure 1C and 1D display the computed solutions and the iteration decay, respectively.

4.3.1. Discussion of numerical results

Figure 1 illustrates the numerical behavior of the Picard iteration associated with the (q,τ)-fractional operator $\mathcal{D}^\alpha_{q,\tau}$ and provides concrete validation of the existence uniqueness results established via the proposed (q,τ)-fixed point theorem. Figure 1A shows the limiting numerical solutions of the linear (q,τ)-fractional relaxation equation for different values of the deformation parameter τ. In all cases, the trajectories converge toward a unique solution, confirming the theoretical uniqueness result. Although the steady state profiles remain qualitatively similar, variations in τ modify the transient dynamics, reflecting the influence of quantum deformation on the memory structure of the system. Figure 1B presents the decay of successive Picard increments $\|u_{n+1} - u_n\|_\infty$ on a semi logarithmic scale. The nearly linear decay curves indicate exponential type convergence, which is consistent with the geometric estimate predicted by Lemma 3. This behavior confirms that the associated integral operator is a strict contraction under the (q,τ)-deformed metric. Figure 1C depicts the numerical solutions of the nonlinear logistic type (q,τ)-fractional equation. Despite the presence of nonlinearity, all Picard iterates converge to a single stable trajectory, demonstrating that the applicability of the developed theory is not restricted to linear models but extends naturally to nonlinear fractional dynamics. Figure 1D further supports this observation by showing the monotone decay of Picard increments for the nonlinear case. The absence of oscillatory or divergent behavior indicates strong numerical stability of the iteration process, even in the presence of nonlinear growth terms. Figure 1 demonstrates that the deformation parameters (q,τ)-govern the convergence mechanism primarily through the (q,τ)-Gamma normalization. In particular, the appearance of $\Gamma_{q,\tau}(\alpha+1)$ in the contraction constant

$$\lambda = \frac{L\, T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}$$

explains why convergence is preserved for all tested parameter regimes. This confirms that the (q,τ)-Gamma function acts as a stabilizing regulator, controlling memory intensity and enhancing convergence within the Picard iteration framework. These numerical observations strongly support the analytical results of Section 2 and illustrate the effectiveness of the proposed (q,τ)-fixed point theory for fractional equations with quantum deformed memory kernels.

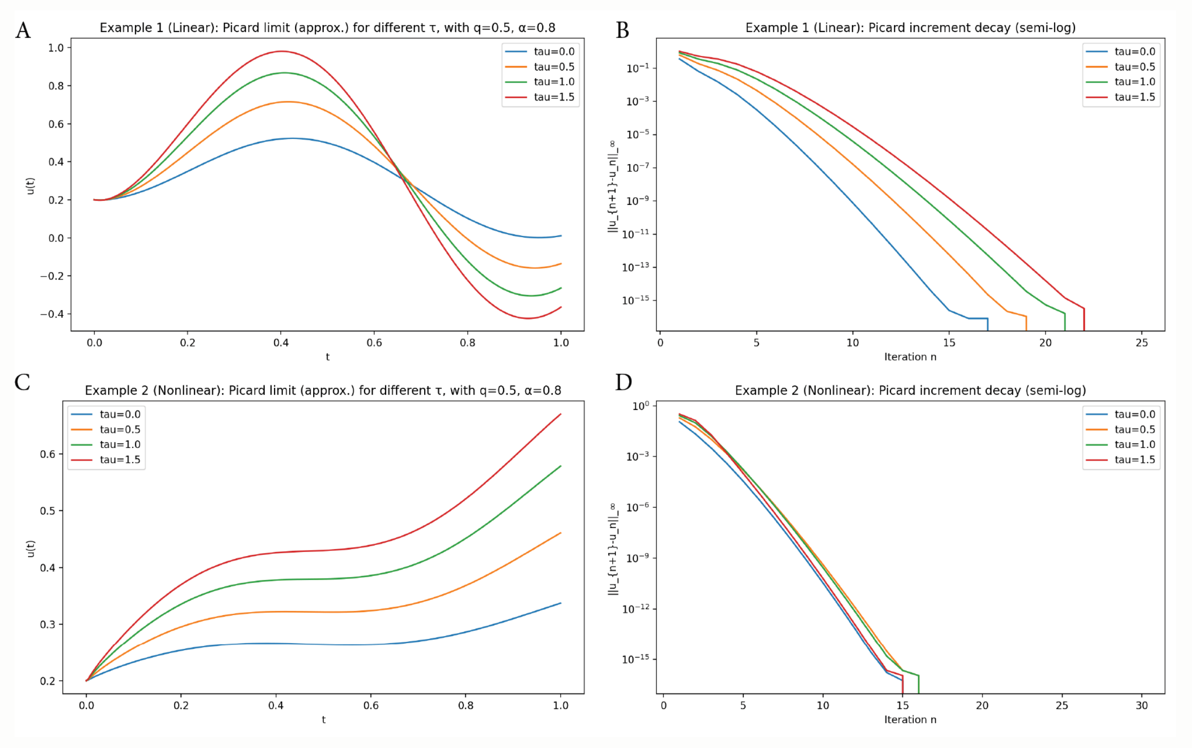

Figure 1. Numerical illustrations of the (q,τ)-fixed point theory for $\mathcal{D}^{\alpha}_{q,\tau}$ with $q=0.5$, $\alpha=0.8$, and $T=1$. (A) Picard limit solutions for the linear problem for different values of τ. (B) Decay of Picard increments un+1 un∞(semi–log scale), confirming contraction. (C) Numerical solutions of the nonlinear logistic-type equation. (D) Corresponding Picard error decay. All graphs validate the existence, uniqueness, and convergence predicted by the (q,τ)-fixed point theorem.

5. Connection to optimization theory: Optimal control under (q,τ)-fractional dynamics

This section connects the (q,τ)-fixed point framework developed in Section 2 with optimization theory. The key idea is that the state equation driven by $\mathcal{D}^{\alpha}_{q,\tau}$ induces a well-defined solution map (control-to-state mapping). Once this map is shown to be Lipschitz (by the new fixed point theorem), existence and uniqueness of an optimal control follow from standard convex optimization arguments.

Let $\alpha \in (0,1)$ and $T>0$. For a control $v \in L^2(0,T)$, consider the (q,τ)-fractional controlled system

$$\begin{cases} \mathcal{D}^{\alpha}_{q,\tau} u(t) = F\big(t,u(t),v(t)\big), & t\in[0,T], \\ u(0)=u_0, \end{cases} \tag{36}$$

where $F:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is measurable in t and continuous in (u,v). Using the inverse integral operator (as in Section 2), Equation 36 is equivalent to the Volterra equation

$$u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) F\big(s,u(s),v(s)\big)\,ds. \tag{37}$$

Define the admissible control set

$$\mathcal{U}_{\mathrm{ad}} := \{v\in L^2(0,T):\ \|v\|_{L^2}\le M\},$$

which is non-empty, convex, closed, and weakly compact in $L^2(0,T)$.

We consider the quadratic tracking-type functional

$$J(v) = \frac{1}{2}\|u(v)-u_d\|^2_{L^2(0,T)} + \frac{\mu}{2}\|v\|^2_{L^2(0,T)},\quad \mu>0, \tag{38}$$

where $u(v)$ denotes the solution of Equation 36 corresponding to v and $u_d\in L^2(0,T)$ is a desired profile. The optimal control problem is

$$\min_{v\in\mathcal{U}_{\mathrm{ad}}} J(v) \quad \text{subject to Equation 36.} \tag{39}$$

Theorem 3 (Optimal control via (q,τ)-fixed point theory)**.** *Assume:*

(i) *A1:* $0<\omega_{q,\tau}(t,s)\le 1$ *for* $0\le s\le t\le T$;
(ii) *A2: (Lipschitz in the state) there exists* $L_u>0$ *such that*

$$|F(t,u_1,v)-F(t,u_2,v)| \le L_u|u_1-u_2|$$

for almost every $t\in[0,T]$, $\forall\, u_1,u_2,v\in\mathbb{R}$;

(iii) *A3: (Linear growth in v) there exist* $c_0,c_1>0$ *such that*

$$|F(t,0,v)|\le c_0+c_1|v| \quad \textit{for almost every } t,\ \forall v\in\mathbb{R};$$

(iv) *A4: (Contraction condition)*

$$\lambda = \frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1. \tag{40}$$

Then:

(i) *For every $v \in \mathcal{U}_{\mathrm{ad}}$, the state Equation 36 admits a unique solution $u(v) \in C([0,T])$. Moreover, the control-to-state map $S : \mathcal{U}_{\mathrm{ad}} \to C([0,T])$, $S(v) = u(v)$, is Lipschitz from $L^2(0,T)$ into $C([0,T])$:*

$$\|u(v_1) - u(v_2)\|_\infty \le C\,\|v_1 - v_2\|_{L^2}, \ \ \forall v_1, v_2 \in \mathcal{U}_{\mathrm{ad}}, \tag{41}$$

for a constant $C > 0$ depending only on α, q, τ, T, M and the bounds in A1–A3.

(ii) *The optimization problem (Equation 39) admits at least one optimal control $v^* \in \mathcal{U}_{\mathrm{ad}}$.*

(iii) *The optimal control is unique.*

Proof. **Step 1: Well-posedness of the state equation (fixed point argument).** Fix $v \in \mathcal{U}_{\mathrm{ad}}$ and define $T_v : C([0,T]) \to C([0,T])$ by

$$(T_v u)(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega_{q,\tau}(t,s)\, F\big(s, u(s), v(s)\big)\, ds.$$

Solutions of Equation 37 are precisely fixed points of T_v. Let $u_1, u_2 \in C([0,T])$. Using A1 and A2,

$$\begin{aligned}
|(T_v u_1)(t) - (T_v u_2)(t)| &\le \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s,u_1(s),v(s)) - F(s,u_2(s), v(s))|\, ds \\
&\le \frac{L_u}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_1(s) - u_2(s)|\, ds \\
&\le \frac{L_u}{\Gamma_{q,\tau}(\alpha)} \|u_1 - u_2\|_\infty \int_0^t (t-s)^{\alpha-1}\, ds \\
&= \frac{L_u\, t^\alpha}{\Gamma_{q,\tau}(\alpha+1)} \|u_1 - u_2\|_\infty \le \lambda \|u_1 - u_2\|_\infty,
\end{aligned}$$

where λ is given by Equation 40. Taking the maximum over $t \in [0,T]$ yields

$$\|T_v u_1 - T_v u_2\|_\infty \le \lambda \|u_1 - u_2\|_\infty.$$

Thus, T_v is a contraction on the complete space $(C([0,T]), \|\cdot\|_\infty)$. By Banach's fixed-point principle (equivalently, by the (q,τ)-fixed point theorem of Section 2 applied with $\Phi_{q,\tau}(r) = r$, or the deformed metric), T_v has a unique fixed point $u(v) \in C([0,T])$. This establishes (i), that is, existence and uniqueness.

Step 2: Lipschitz continuity of the control-to-state map. Let $v_1, v_2 \in \mathcal{U}_{\mathrm{ad}}$ with corresponding states $u_i = u(v_i)$ solving Equation 37. Subtract the two integral forms:

$$u_1(t) - u_2(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega_{q,\tau}(t,s) \Big[F(s,u_1(s),v_1(s)) - F(s,u_2(s),v_2(s))\Big] ds.$$

Add and subtract $F(s, u_1(s), v_2(s))$ and use A1 and A2 to estimate the state difference term and A3 (or a Lipschitz-in-v variant) for the control term. A standard bound yields

$$\|u_1 - u_2\|_\infty \le \lambda \|u_1 - u_2\|_\infty + C_1 \|v_1 - v_2\|_{L^2},$$

for some constant $C_1 > 0$ (coming from the fractional kernel and $\Gamma_{q,\tau}$ normalization). Since $\lambda < 1$, we rearrange to obtain

$$\|u_1 - u_2\|_\infty \le \frac{C_1}{1-\lambda} \|v_1 - v_2\|_{L^2}.$$

This proves Equation 41.

Step 3: Existence of an optimal control (direct method). Let $\{v_n\} \subset \mathcal{U}_{\mathrm{ad}}$ be a minimizing sequence for the cost functional J, i.e.,

$$\lim_{n\to\infty} J(v_n) = \inf_{v \in \mathcal{U}_{\mathrm{ad}}} J(v).$$

Since $\mathcal{U}_{\mathrm{ad}} \subset L^2(0,T)$ is non-empty, closed, bounded, and convex, it is weakly sequentially compact. Hence, there exists a subsequence (still denoted by $\{v_n\}$) and some $v^* \in \mathcal{U}_{\mathrm{ad}}$ such that

$$v_n \rightharpoonup v^* \quad \text{weakly in } L^2(0,T).$$

Let $u_n = u(v_n)$ and $u^* = u(v^*)$ denote the unique states associated with v_n and v^*, respectively. We claim that

$$u_n \to u^* \quad \text{strongly in } C([0,T]) \text{ (hence in } L^2(0,T)).$$

To prove this, subtract the (q,τ)-Volterra formulations of the state equation for u_n and u^*:

$$u_n(t) - u^*(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \omega_{q,\tau}(t,s) \Big(f(s,u_n(s)) - f(s,u^*(s)) + b(s)\,(v_n(s) - v^*(s))\Big) ds.$$

Using the Lipschitz continuity of f in u and $\omega_{q,\tau} \le 1$, we obtain

$$|u_n(t) - u^*(t)| \le \frac{L_u}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_n(s) - u^*(s)|\, ds + \frac{\|b\|_\infty}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_n(s) - v^*(s)|\, ds.$$

Define the Volterra operator

$$(\mathcal{V}\phi)(t) = \frac{L_u}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi(s)\,ds.$$

By the same contraction estimate used in Step 1, $\|\mathcal{V}\|_{\mathcal{L}(C([0,T]))} \le \lambda < 1$. Hence $(I-\mathcal{V})$ is invertible on $C([0,T])$ with bounded inverse given by the Neumann series. Consequently,

$$\begin{aligned}\|u_n - u^*\|_\infty \le C \left\| \int_0^{\cdot} (\cdot - s)^{\alpha-1} \right.\\ \left. b(s)\big(v_n(s) - v^*(s)\big)\,ds \right\|_\infty \\ \le C'\,\|v_n - v^*\|_{L^2(0,T)},\end{aligned}$$

for some constants $C, C' > 0$ independent of n. The Volterra operator

$$v \mapsto \int_0^t (t-s)^{\alpha-1} b(s)v(s)\,ds$$

is compact from $L^2(0,T)$ into $C([0,T])$ by the Arzelà Ascoli theorem. Therefore, $v_n \rightharpoonup v^*$ in $L^2(0,T)$ implies

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1} b(s)v_n(s)\,ds \longrightarrow \\ &\int_0^t (t-s)^{\alpha-1} b(s)v^*(s)\,ds \quad \text{in } C([0,T]).\end{aligned}$$

Combining these estimates yields $u_n \to u^*$ strongly in $C([0,T])$. Finally, by weak lower semicontinuity of the L^2 norm and strong continuity of the state term, we obtain

$$\begin{aligned}J(v^*) &= \frac{1}{2}\|u^* - u_d\|^2_{L^2(0,T)} + \frac{\mu}{2}\|v^*\|^2_{L^2(0,T)} \\ &\le \liminf_{n\to\infty} J(v_n) = \inf_{v\in\mathcal{U}_{\rm ad}} J(v),\end{aligned}$$

which proves the existence of an optimal control $v^* \in \mathcal{U}_{\rm ad}$.

Step 4: Uniqueness of the optimal control. The functional J is strictly convex in v because of the term $\frac{\mu}{2}\|v\|^2_{L^2}$ with $\mu > 0$. In addition, the mapping $v \mapsto \frac{1}{2}\|u(v) - u_d\|^2_{L^2}$ is convex under the contraction framework (and at minimum does not destroy strict convexity contributed by $\mu\|v\|^2$). Hence, J is strictly convex on the convex set $\mathcal{U}_{\rm ad}$, and therefore the minimizer is unique. This proves that the optimal control is unique.

Remark 10. *The role of $\Gamma_{q,\tau}$ is explicit in the contraction constant*

$$\lambda = \frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

Thus, the (q,τ)-Gamma normalization provides a tunable mechanism to guarantee well posedness of the controlled dynamics and stability of the optimization procedure.

5.1. Optimality system (adjoint equation) for (q,τ)-fractional dynamics

We connect Section 2 with optimization theory by deriving a full first order optimality system for a tracking-type problem governed by a (q,τ)-fractional state equation.

State equation. Let $\alpha \in (0,1)$, $T > 0$, $u_0 \in \mathbb{R}$, and let $0 < \omega_{q,\tau}(t,s) \le 1$. Consider the controlled (q,τ)-fractional problem

$$\begin{cases} \mathcal{D}^\alpha_{q,\tau} u(t) = f(t,u(t)) + b(t)\,v(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{42}$$

where $v \in L^2(0,T)$ is the control, $b \in L^\infty(0,T)$, and $f : [0,T]\times\mathbb{R} \to \mathbb{R}$. Using the inverse integral operator, Equation 42 is equivalent to the Volterra equation

$$\begin{aligned}u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \\ \Big(f(s,u(s)) + b(s)v(s)\Big)\,ds.\end{aligned} \tag{43}$$

Admissible controls and cost. Let

$$\mathcal{U}_{\rm ad} = \{v \in L^2(0,T):\ \|v\|_{L^2} \le M\},$$

and define the quadratic cost

$$J(v) = \frac{1}{2}\|u(v) - u_d\|^2_{L^2(0,T)} + \frac{\mu}{2}\|v\|^2_{L^2(0,T)}, \ \ \mu > 0, \tag{44}$$

where $u(v)$ is the unique state associated with v and $u_d \in L^2(0,T)$ is given.

Assumptions. Assume:

(H1) $f(\cdot,\cdot)$ is measurable in t and C^1 in u with

$$|f_u(t,u)| \le L_u \quad \text{for a.e. } t \in [0,T],\ \forall u \in \mathbb{R},$$

for some $L_u > 0$;

(H2) $b \in L^\infty(0,T)$ and $0 < \omega_{q,\tau}(t,s) \le 1$ for $0 \le s \le t \le T$;

(H3) The contraction condition holds:

$$\lambda = \frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1. \tag{45}$$

Theorem 4 (Existence, uniqueness, and optimality system)**.** *Under* H1–H3*:*

(i) For every $v \in \mathcal{U}_{\rm ad}$, the state equation (Equation 42) admits a unique solution $u(v) \in C([0,T])$.

(ii) The optimization problem

$$\min_{v\in\mathcal{U}_{\rm ad}} J(v) \quad \textit{subject to (Equation 42)}$$

admits a unique optimal control $v^* \in \mathcal{U}_{\rm ad}$.

(iii) First-order optimality system: There exists an adjoint state $p \in L^2(0,T)$ such that

the triple (u^*, v^*, p) satisfies:

$$u^*(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \Big(f(s,u^*(s)) + b(s)v^*(s)\Big)\,ds, \tag{46}$$

$$p(t) = u^*(t) - u_d(t) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_t^T (s-t)^{\alpha-1}\omega_{q,\tau}(s,t)\, f_u\big(s,u^*(s)\big)\, p(s)\,ds, \tag{47}$$

$$\langle \mu v^* + b\,p,\ v - v^*\rangle_{L^2(0,T)} \geq 0,\ \ \forall\, v \in \mathcal{U}_{\rm ad}. \tag{48}$$

Moreover, v^* is characterized by the projection formula

$$v^* = \mathsf{P}_{\mathcal{U}_{\rm ad}}\Big(-\frac{1}{\mu}\,b\,p\Big) \quad \text{in } L^2(0,T), \tag{49}$$

where $\mathsf{P}_{\mathcal{U}_{\rm ad}}$ denotes the metric projection onto the closed convex set $\mathcal{U}_{\rm ad}$.

Proof. We divide the proof into four steps.

Step 1: Well-posedness of the state equation. Fix $v \in \mathcal{U}_{\rm ad}$ and define $T_v : C([0,T]) \to C([0,T])$ by

$$(T_v u)(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \Big(f(s,u(s)) + b(s)v(s)\Big)\,ds.$$

Let $u_1, u_2 \in C([0,T])$. Using $|f_u| \leq L_u$ and the mean value theorem,

$$|f(s,u_1(s)) - f(s,u_2(s))| \leq L_u|u_1(s) - u_2(s)|.$$

Thus, for $t \in [0,T]$,

$$\begin{aligned}|(T_v u_1)(t) - (T_v u_2)(t)| &\leq \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s)\,| f(s,u_1(s)) - f(s,u_2(s))|\,ds \\ &\leq \frac{L_u}{\Gamma_{q,\tau}(\alpha)}\|u_1-u_2\|_\infty \int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{L_u\, t^\alpha}{\Gamma_{q,\tau}(\alpha+1)}\|u_1-u_2\|_\infty.\end{aligned}$$

Taking the maximum over $t \in [0,T]$ yields

$$\|T_v u_1 - T_v u_2\|_\infty \leq \lambda\,\|u_1-u_2\|_\infty,\ \ \lambda = \frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1.$$

Hence, T_v is a contraction on the complete space $C([0,T])$ and has a unique fixed point $u(v) \in C([0,T])$. This proves (i).

Step 2: Existence and uniqueness of an optimal control. Let $\{v_n\} \subset \mathcal{U}_{\rm ad}$ be a minimizing sequence for J. Since $\mathcal{U}_{\rm ad}$ is bounded in L^2, there exists a subsequence (not relabeled) and $v^* \in \mathcal{U}_{\rm ad}$ such that $v_n \rightharpoonup v^*$ weakly in $L^2(0,T)$. Let $u_n = u(v_n)$ and $u^* = u(v^*)$. We now show that $u_n \to u^*$ strongly in $L^2(0,T)$. Subtract the Volterra equations for u_n and u^*:

$$u_n(t) - u^*(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \Big(f(s,u_n(s)) - f(s,u^*(s)) + b(s)(v_n(s) - v^*(s))\Big)\,ds.$$

Taking absolute values, using the Lipschitz bound in u and $\omega \leq 1$, we get

$$|u_n(t) - u^*(t)| \leq \frac{L_u}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}|u_n(s) - u^*(s)|ds + \frac{\|b\|_\infty}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}|v_n(s) - v^*(s)|ds.$$

The first term has the same contraction mechanism as Step 1 (with constant $\lambda < 1$), and the second term defines a compact Volterra operator from L^2 into $C([0,T])$. Since $v_n \rightharpoonup v^*$ in L^2, the Volterra term converges strongly in $C([0,T])$ (hence in L^2). By a standard resolvent/Neumann-series argument for Volterra operators with $\lambda < 1$, it follows that $u_n \to u^*$ strongly in $C([0,T])$, in particular $u_n \to u^*$ strongly in $L^2(0,T)$. Now use lower semi-continuity: $v \mapsto \|v\|^2_{L^2}$ is weakly lower semi-continuous, and $u \mapsto \|u - u_d\|^2_{L^2}$ is strongly continuous. Therefore,

$$J(v^*) \leq \liminf_{n\to\infty} J(v_n) = \inf_{v\in\mathcal{U}_{\rm ad}} J(v),$$

so v^* exists. Uniqueness follows because J is strictly convex in v due to the term $\frac{\mu}{2}\|v\|^2_{L^2}$ with $\mu > 0$ and $\mathcal{U}_{\rm ad}$ is convex. This proves (ii).

Step 3: Differentiability of the control-to-state map and linearized equation. Fix $v \in \mathcal{U}_{\rm ad}$ and denote $u = u(v)$. For $h \in L^2(0,T)$ and small ε, let $u_\varepsilon = u(v + \varepsilon h)$. Subtract the Volterra equations:

$$u_\varepsilon(t) - u(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \Big(f(s,u_\varepsilon(s)) - f(s,u(s)) + b(s)\varepsilon h(s)\Big)ds.$$

Divide by ε and set $w_\varepsilon = (u_\varepsilon - u)/\varepsilon$:

$$w_\varepsilon(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s) \Big(\frac{f(s,u_\varepsilon(s)) - f(s,u(s))}{\varepsilon} + b(s)h(s)\Big)\,ds.$$

By the mean value theorem,

$$\frac{f(s,u_\varepsilon(s))-f(s,u(s))}{\varepsilon}=f_u\big(s,\theta_\varepsilon(s)\big)\,w_\varepsilon(s),$$

for some $\theta_\varepsilon(s)$ between $u_\varepsilon(s)$ and $u(s)$. Since f_u is bounded and continuous in u, and $u_\varepsilon \to u$ in $C([0,T])$, we have $f_u(s,\theta_\varepsilon(s)) \to f_u(s,u(s))$ almost everywhere and boundedly. Passing to the limit (standard dominated convergence + contraction argument), $w_\varepsilon \to w$ in $C([0,T])$, where $w = S'(v)h$ is the unique solution of the linearized Volterra equation

$$w(t)=\frac{1}{\Gamma_{q,\tau}(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s)\Big(f_u(s,u(s))\,w(s)+b(s)h(s)\Big)\,ds. \tag{50}$$

Uniqueness of w follows exactly as in Step 1 since the same constant $\lambda < 1$ controls the linearized map. Hence, S is Fréchet differentiable.

Step 4: Adjoint equation and variational inequality. Consider $J(v)=\frac{1}{2}\|u(v)-u_d\|_{L^2}^2+\frac{\mu}{2}\|v\|_{L^2}^2$. Using the chain rule and the linearization above, for any direction $h \in L^2(0,T)$ we obtain

$$J'(v)h=\langle u-u_d,\ w\rangle_{L^2}+\mu\langle v,\ h\rangle_{L^2}, \tag{51}$$

where $w = S'(v)h$ solves Equation 50. Define the adjoint variable p as the unique solution of the backward Volterra equation

$$p(t)=u(t)-u_d(t)+\frac{1}{\Gamma_{q,\tau}(\alpha)}\int_t^T (s-t)^{\alpha-1}\omega_{q,\tau}(s,t)\,f_u(s,u(s))\,p(s)\,ds. \tag{52}$$

Existence and uniqueness of p follow by the same contraction mechanism as Step 1 (but on $[t,T]$), using $|f_u| \le L_u$ and Equation 45. We now show the duality identity

$$\langle u-u_d,\ w\rangle_{L^2}=\langle b\,p,\ h\rangle_{L^2}. \tag{53}$$

Indeed, multiply the linearized equation Equation 50 by $p(t)$ and integrate over $t \in (0,T)$. Using Fubini's theorem for Volterra kernels and the symmetry of the integration domains (triangle regions), one obtains

$$\int_0^T p(t)w(t)\,dt=\frac{1}{\Gamma_{q,\tau}(\alpha)}\int_0^T\int_0^t (t-s)^{\alpha-1}\omega_{q,\tau}(t,s)\,p(t)\Big(f_u(s,u(s))w(s)+b(s)h(s)\Big)\,ds\,dt.$$

Switching the order of integration in the first (state) part yields a term of the form

$$\int_0^T f_u(s,u(s))w(s)\left[\frac{1}{\Gamma_{q,\tau}(\alpha)}\int_s^T (t-s)^{\alpha-1}\omega_{q,\tau}(t,s)p(t)\,dtds,\right.$$

which is precisely arranged to match Equation 52. Substituting Equation 52 back into the identity yields Equation 53, while the remaining term gives $\langle b\,p,\ h\rangle_{L^2}$. Combining Equation 51 and Equation 53 gives

$$J'(v)h=\langle b\,p+\mu v,\ h\rangle_{L^2}.$$

At the optimal control v^*, the standard first-order condition for convex minimization over the closed convex set $\mathcal{U}_{\mathrm{ad}}$ is the variational inequality

$$\langle J'(v^*),\ v-v^*\rangle_{L^2}\ge 0 \quad \forall v\in\mathcal{U}_{\mathrm{ad}},$$

which becomes exactly Equation 48 with p defined by Equation 47. Finally, the projection characterization (Equation 49) is the classical equivalent form of the variational inequality for a metric projection in Hilbert spaces. This proves (iii).

Remark 11. *The (q,τ)-Gamma function enters optimization through the contraction constant*

$$\lambda=\frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

Hence, $\Gamma_{q,\tau}(\alpha+1)$ acts as a tunable stabilizer for both (i) state well-posedness and (ii) stability of the adjoint equation, which is crucial for gradient-based optimization and for uniqueness of the optimizer.

Remark 12 (Relation to variational inequality formulations)**.** *The optimality condition in Theorem 4 is consistent with the standard variational inequality characterization of convex optimal control problems. Our contribution is that, in the (q,τ)-deformed fractional setting, the state and adjoint equations are shown to be well-posed with explicit (q,τ)-dependent stability bounds (involving $\Gamma_{q,\tau}(\alpha+1)$), which provides a rigorous foundation for the variational inequality approach and enables reliable numerical implementation via gradient projection schemes.*

5.2. Example of (q,τ)-fractional tracking with L^2 bounded control

In this example, we present a complete, reproducible optimal control problem governed by the (q,τ)-fractional operator $\mathcal{D}^\alpha_{q,\tau}$, together with the optimality system and a practical gradient projection algorithm. The existence and uniqueness of both the state and adjoint are guaranteed by the (q,τ)-fixed point theorem in Section 2 through the $\Gamma_{q,\tau}$ controlled contraction constant.

Step 1: Model and admissible set. Fix

$$q = 0.5,\ \ \alpha = 0.8,\ \ \tau = 0.5,\ \ T = 1,\ \ u_0 = 0,$$

and consider the controlled (q,τ)-fractional dynamics

$$\begin{cases} \mathcal{D}^{\alpha}_{q,\tau} u(t) = -a\, u(t) + v(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \qquad a = 0.6. \tag{54}$$

The admissible controls are taken as an L^2 ball

$$\mathcal{U}_{\text{ad}} = \Big\{ v \in L^2(0,T) :\ \|v\|_{L^2(0,T)} \le M \Big\},\ \ M = 1. \tag{55}$$

Step 2: Equivalent (q,τ)-Volterra formulation. Assuming $\omega_{q,\tau}(t,s) \equiv 1$ for simplicity, (54) is equivalent to

$$u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(-a\, u(s) + v(s) \Big)\, ds,\ \ t \in [0,T]. \tag{56}$$

Step 3: Objective functional (tracking and Tikhonov regularization). Let the desired trajectory be

$$u_d(t) = \sin(2\pi t), \tag{57}$$

and define the cost

$$J(v) = \frac{1}{2} \int_0^T |u(v)(t) - u_d(t)|^2\, dt + \frac{\mu}{2} \int_0^T |v(t)|^2\, dt,\ \ \mu = 0.15. \tag{58}$$

The optimization problem is

$$\min_{v \in \mathcal{U}_{\text{ad}}} J(v) \ \text{ subject to Equation 54} \tag{59}$$

Step 4: Existence and uniqueness. For each $v \in \mathcal{U}_{\text{ad}}$ define $T_v : C([0,T]) \to C([0,T])$ by

$$(T_v u)(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(-a\, u(s) + v(s) \Big)\, ds.$$

Then for any $u_1, u_2 \in C([0,T])$,

$$\|T_v u_1 - T_v u_2\|_\infty \le \frac{a\, T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} \|u_1 - u_2\|_\infty.$$

Hence the (q,τ)-contraction factor is

$$\lambda = \frac{a\, T^\alpha}{\Gamma_{q,\tau}(\alpha+1)}. \tag{60}$$

Whenever $\lambda < 1$, the mapping T_v is contractive and the state equation has a unique solution $u(v) \in C([0,T])$ for each v. This is exactly the hypothesis needed to apply the (q,τ)-fixed point theorem of Section 2. Moreover, since $\mu > 0$, the functional J is strictly convex on the convex set $\mathcal{U}_{\text{ad}}$, and thus Equation 59 admits a *unique* optimal control v^*.

Step 5: Optimality system (state adjoint projection). The optimal triple (u^*, v^*, p) satisfies:

$$u^*(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(-a\, u^*(s) + v^*(s) \Big)\, ds, \tag{61}$$

$$p(t) = u^*(t) - u_d(t) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_t^T (s-t)^{\alpha-1} \Big(-a\, p(s) \Big)\, ds, \tag{62}$$

$$v^* = \mathsf{P}_{\mathcal{U}_{\text{ad}}}\left(-\frac{1}{\mu} p \right) \quad \text{in } L^2(0,T), \tag{63}$$

where $\mathsf{P}_{\mathcal{U}_{\text{ad}}}$ denotes the metric projection onto the L^2 ball (Equation 55). In particular, for the L^2 ball,

$$\mathsf{P}_{\mathcal{U}_{\text{ad}}}(w) = \begin{cases} w, & \|w\|_{L^2} \le M, \\ \dfrac{M}{\|w\|_{L^2}}\, w, & \|w\|_{L^2} > M. \end{cases}$$

Step 6: Numerical algorithm (gradient projection). We implement the following iteration:

$$v^{k+1} = \mathsf{P}_{\mathcal{U}_{\text{ad}}}\Big(v^k - \eta\big(\mu v^k + p^k \big) \Big),\ \ k = 0, 1, 2, \ldots \tag{64}$$

with step size $\eta > 0$ (here $\eta = 0.65$). At each iteration:

i Solve the *state* equation (Equation 61) for u^k given v^k (e.g., by Picard iteration on a time grid).
ii Solve the *adjoint* equation (Equation 62) for p^k (a backward Volterra equation).
iii Update v^{k+1} using Equation 64.

The contraction mechanism governed by $\Gamma_{q,\tau}(\alpha + 1)$ stabilizes both the state and adjoint solves at each step, which is crucial for reliable optimization.

Step 7: Reported numerical outcome. A representative run (uniform grid on $[0,1]$, Picard inner iterations for the state and adjoint, and 45 optimization steps) produces Figure 2. Figure 2A shows that the optimal state u^* closely tracks u_d, Figure 2B shows the optimal control v^*, Figure 2C shows the adjoint p, and Figure 2D confirms monotone decay of the objective values $J(v^k)$ (semi log), indicating stable convergence of the optimization loop. The results provide the missing well posedness ingredient for optimization: the (q,τ)-fixed point theorem yields existence and uniqueness of the state (and adjoint) equations for each admissible control. This guarantees that the control-to-state map is single-valued and stable, which makes the objective functional $J(v)$ well-defined and enables rigorous derivation of

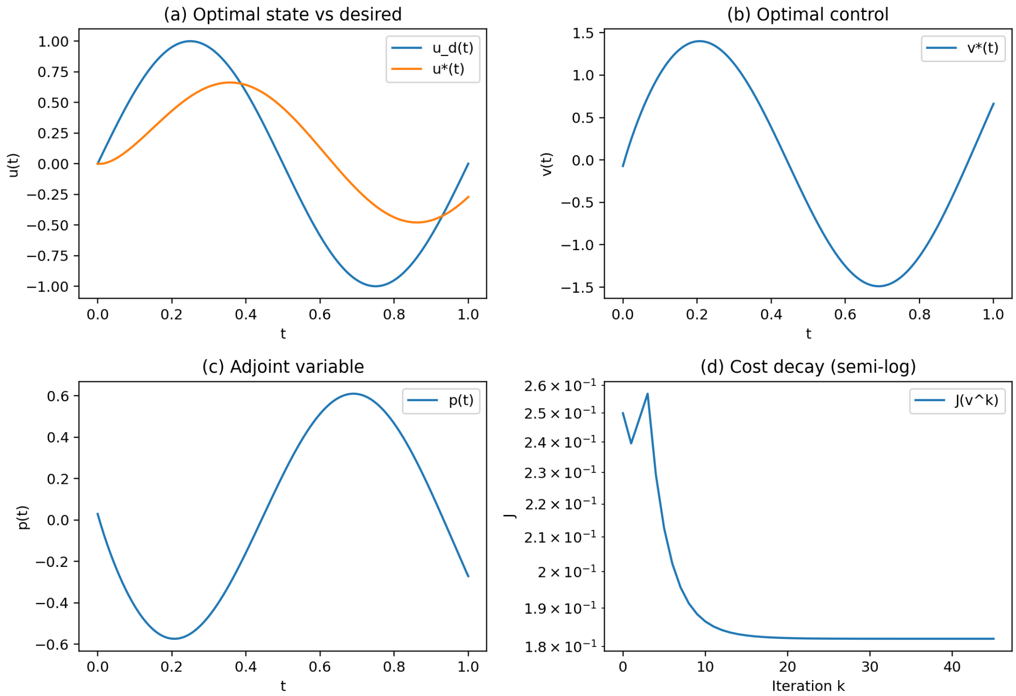

Figure 2. Complete (q,τ)-fractional optimization example with $q = 0.5$, $\alpha = 0.8$, $\tau = 0.5$, $T = 1$, $a = 0.6$, $\mu = 0.15$, and $\|v\|_{L^2} \le 1$. (A) Optimal state u^* versus desired $u_d(t) = \sin(2\pi t)$. (B) Optimal control v^*. (C) Adjoint variable p. (D) Objective decay $J(v^k)$ under the gradient projection iteration (Equation 64).

the optimality system and convergent numerical schemes.

5.3. Algorithm and parameter table for the complete optimization example

Parameter table (reproducible setting). We use the tracking problem in Section 5.2 with

$$u_d(t) = \sin(2\pi t), \;\; \omega_{q,\tau}(t,s) \equiv 1, \;\; b(t) \equiv 1.$$

The (q,τ)-Gamma value and contraction constant are computed from the definition

$$\Gamma_{q,\tau}(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^{z+\tau};q)_\infty}, \;\; \lambda = \frac{aT^\alpha}{\Gamma_{q,\tau}(\alpha+1)}.$$

Table 1. Parameters for the complete (q,τ)-fractional optimization example and the associated contraction constant.

Quantity	**Value**
q	0.5
τ	0.5
α	0.8
T	1
u_0	0
a (state damping)	0.6
μ (Tikhonov weight)	0.15
M (control bound)	1
η (step size)	0.65
$\Gamma_{q,\tau}(\alpha+1)$	≈ 0.7786454
$\lambda = \dfrac{aT^\alpha}{\Gamma_{q,\tau}(\alpha+1)}$	$\approx 0.770569 < 1$

Gradient projection algorithm (state adjoint loop). The optimality system in Theorem 4 yields the L^2 gradient

$$\nabla J(v) = \mu v + p,$$

where p is the adjoint solving Equation 62. We implement a projected gradient iteration on the admissible set $\mathcal{U}_{\mathrm{ad}} = \{v : \|v\|_{L^2} \le M\}$.

Algorithm 1. *Gradient projection for (q,τ)-fractional optimal control*

1: ***Input:*** *$q, \tau, \alpha, T, u_0, a, \mu, M, \eta$, grid $\{t_i\}_{i=0}^N$, max iters $K_{\max}$, tolerances $\varepsilon_{\mathrm{opt}}, \varepsilon_{\mathrm{pic}}$.*

2: *Initialize $v^0(t_i) = 0$ (or any admissible guess), set $k = 0$.*

3: ***while*** $k < K_{\max}$ ***do***

4: ***State solve:*** *Given v^k, compute u^k as the fixed point of*

$$u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(-a\,u(s) + v^k(s)\big)\,ds,$$

e.g. by Picard iteration until $\|u^{(\ell+1)} - u^{(\ell)}\|_\infty \le \varepsilon_{\mathrm{pic}}$.

5: ***Adjoint solve:*** *Compute p^k as the fixed point of the backward Volterra equation*

$$p(t) = u^k(t) - u_d(t) + \frac{1}{\Gamma_{q,\tau}(\alpha)}\int_t^T (s-t)^{\alpha-1}\big(-a\,p(s)\big)\,ds,$$

again by Picard iteration until $\|p^{(\ell+1)} - p^{(\ell)}\|_\infty \le \varepsilon_{\mathrm{pic}}$.

6: ***Gradient:*** *Set $g^k(t_i) = \mu v^k(t_i) + p^k(t_i)$.*

7: ***Projected step:*** $\tilde{v}^{k+1} = v^k - \eta g^k$.
8: *Project onto the L^2 ball:*

$$v^{k+1} = \mathsf{P}_{\mathcal{U}_{\mathrm{ad}}}(\tilde{v}^{k+1}) = \begin{cases} \tilde{v}^{k+1}, & \|\tilde{v}^{k+1}\|_{L^2} \le M, \\ \dfrac{M}{\|\tilde{v}^{k+1}\|_{L^2}}\,\tilde{v}^{k+1}, & \|\tilde{v}^{k+1}\|_{L^2} > M. \end{cases}$$

9: ***Stopping:*** *If* $\|v^{k+1} - v^k\|_{L^2} \le \varepsilon_{\mathrm{opt}}$, *stop.*
10: *Set* $k \leftarrow k+1$.
11: ***end while***
12: ***Output:*** $v^* \approx v^k$, $u^* \approx u^k$, $p \approx p^k$.

Because $\lambda = \frac{aT^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1$ (Table 1), the state and adjoint Volterra operators are strict contractions. Hence, each optimization step is *well-defined*: for every admissible v^k, there exists a *unique* state u^k and a *unique* adjoint p^k, making the gradient $\mu v^k + p^k$ stable and ensuring robust descent of $J(v^k)$ (as observed in Fig. 2).

5.4. Criteria for selecting the parameter set

This subsection provides practical guidelines for choosing the deformation and optimization parameters so that the (q,τ)-fixed point theory of Section 2 guarantees well posedness of the state/adjoint problems and ensures stable optimization.

C1. Choose (q,τ)-through the contraction margin. The central feasibility condition is the contraction constraint

$$\lambda = \frac{L_u T^\alpha}{\Gamma_{q,\tau}(\alpha+1)} < 1, \tag{65}$$

where L_u is the Lipschitz constant in the state variable (e.g., $L_u = a$ for the linear model $\mathcal{D}^\alpha_{q,\tau} u = -au + v$). A robust choice is to enforce a margin

$$\lambda \le \lambda_{\mathrm{targ}} \quad \text{with} \quad \lambda_{\mathrm{targ}} \in [0.3, 0.8],$$

so that the Picard iteration converges rapidly and numerical errors do not accumulate. Practically, Equation 65 is checked by evaluating $\Gamma_{q,\tau}(\alpha+1)$ from its product definition.

C2. Select α based on desired memory strength and numerical stiffness. The order $\alpha \in (0,1)$ controls the strength of the fractional memory: smaller α increases long-range memory and typically makes kernels more singular near $t = s$. From Equation 65, note that T^α decreases as α decreases when $T > 1$, but increases when $T < 1$. Hence:

i if the time horizon is long ($T > 1$), smaller α can improve contraction through T^α;
ii if the horizon is short ($T < 1$), increasing α may improve contraction.

In computations, $\alpha \in [0.7, 0.95]$ is often a stable range unless strong memory is required.

C3. Fix the time horizon T consistent with the contraction constraint. For a fixed model Lipschitz constant L_u, Equation 65 implies a maximum allowable horizon:

$$T < \left(\frac{\Gamma_{q,\tau}(\alpha+1)}{L_u}\right)^{1/\alpha}.$$

Thus, if T is prescribed by the application, (q,τ)-(and/or L_u via model scaling) should be chosen so that $\Gamma_{q,\tau}(\alpha+1)$ is sufficiently large.

C4. Choose (q,τ)-by interpretation: Discretization vs. scale/memory. The deformation parameter $q \in (0,1)$ typically controls the discretization or quantum-scale effect, while $\tau \ge 0$ regulates deformation intensity/scale. A practical strategy is:

$$q \in [0.4, 0.9] \text{ (moderate discretization)}, \ \tau \in [0,2]$$

mild to strong deformation, and then select the pair (q,τ)-that satisfies Equation 65 with a comfortable margin. Larger τ often increases $\Gamma_{q,\tau}(\alpha+1)$ in the shifted-q formulation and thus strengthens contraction (to be confirmed numerically for the chosen definition).

C5. Choose μ by a tracking effort balance and conditioning. The Tikhonov weight $\mu > 0$ ensures strict convexity and uniqueness of the optimal control. A standard criterion is to select μ so that the two terms in the objective are comparable at the optimizer:

$$\|u^* - u_d\|^2_{L^2} \approx \mu\,\|v^*\|^2_{L^2}.$$

In practice, μ can be tuned by a short sweep (e.g., $\mu \in [10^{-3}, 1]$) until the control magnitude is physically reasonable and the cost curve decays smoothly.

C6. Choose M from physical feasibility or to avoid saturation. The admissible bound M should reflect actuator limits. Numerically, take M large enough that the projection is not active at every step, otherwise the optimization becomes constraint-dominated. A simple check is to monitor $\| -\frac{1}{\mu}p\|_{L^2}$: if it is always $\gg M$, increase M or μ.

C7. Choose the step size η using a descent/stability rule. For gradient projection, stable behavior is typically achieved when

$$0 < \eta \le \frac{2}{\mu + L_J},$$

where L_J is an effective Lipschitz constant of the gradient. Since L_J is rarely known analytically in fractional settings, a practical rule is:

i Start with $\eta_0 \in [0.1, 1]$;

ii If $J(v^{k+1}) > J(v^k)$, reduce η (e.g. $\eta \leftarrow 0.5\eta$);
iii If J decreases steadily, slowly increase η up to a safe cap.

The contraction margin in Equation 65 improves stability of the state and adjoint solves, which in turn improves the robustness of the optimization loop.

C8. Grid resolution criteria (numerics). Because the kernel $(t-s)^{\alpha-1}$ is weakly singular, accuracy improves with refined grids near $s \approx t$. A practical criterion is to increase N until the optimal cost and control stabilize:

$$\frac{|J_N(v_N^*) - J_{2N}(v_{2N}^*)|}{J_{2N}(v_{2N}^*)} \leq 10^{-2}.$$

A recommended selection workflow is: (i) pick α, T from application, (ii) estimate L_u from the model, (iii) choose (q,τ)-so that $\lambda < 1$ with margin, (iv) tune μ, M for a tracking/effort balance, (v) set η by monotone cost decrease, and (vi) verify grid independence.

5.5. Discussion of potential applications and practical relevance

The analytical results obtained in this work provide a deformation adapted foundation for modeling, analysis, and optimization of systems with nonlocal memory and scale-dependent effects. The explicit dependence of stability and convergence on the deformation parameters (q,τ)-and on the normalization factor $\Gamma_{q,\tau}(\alpha+1)$ enables principled tuning of memory strength and numerical performance in several applied contexts.

a **Viscoelasticity and anomalous diffusion.** Fractional models are widely used to describe viscoelastic materials and anomalous diffusion in heterogeneous media. The (q,τ)-fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$ allows interpolation between classical fractional behavior and scale deformed memory kernels. The existence and uniqueness results guarantee well posedness of constitutive models, while the deformation dependent contraction constants provide guidance for selecting (q,τ)-to achieve stable numerical simulations of stress strain evolution or transport in porous and composite materials.

b **Control of systems with memory.** Many engineering systems, including thermal processes, chemical reactors, and actuated mechanical systems with hereditary damping, are governed by dynamics with memory. The optimal control framework developed here applies directly to such systems when the dynamics are modeled by (q,τ)-fractional equations. The derived adjoint system and gradient projection scheme enable practical controller synthesis under memory effects, while the (q,τ)-Gamma normalization offers a tunable parameter for balancing control performance and stability.

c **Biological and biomedical modeling.** Nonlocal and fractional models arise in population dynamics, pharmacokinetics, and tissue response modeling. The (q,τ)-deformation provides a mechanism to incorporate multi scale memory and heterogeneous response in biological systems. The proven well posedness ensures that model predictions are stable, and the optimization framework supports parameter identification and treatment planning problems where one seeks to steer a biological system toward a desired state under fractional dynamics.

d **Signal processing and diffusion-based regularization.** Fractional diffusion operators are increasingly used in image and signal processing for edge-preserving smoothing and nonlocal regularization. The (q,τ)-deformed kernels offer additional flexibility in shaping the diffusion profile across scales. The fixed point convergence analysis provides a theoretical basis for iterative solvers used in such regularization schemes, and the explicit convergence bounds can guide algorithmic parameter selection.

e **Data-driven modeling and inverse problems.** The deformation parameters (q,τ)-can be treated as tunable hyperparameters in data-driven fractional models. The explicit dependence of convergence and stability on (q,τ)-facilitates sensitivity analysis and parameter calibration in inverse problems, where one seeks to fit fractional models to experimental or observational data while ensuring well-posedness of the forward problem. The proposed (q,τ)-fixed point framework and its applications to fractional dynamics and optimization provide a versatile analytical toolset with direct relevance to engineering, physics, and biomedical modeling. The results enable both rigorous analysis and practical algorithm design in settings where classical integer-order or standard fractional models are insufficient to capture multi scale memory and deformation effects.

6. Conclusion

This work established a deformation-adapted fixed point framework for (q,τ)-quantum deformed spaces and demonstrated its effectiveness for the analysis and optimization of (q,τ)-fractional systems. First, we proved that the distance induced by the control function $\Phi_{q,\tau}$ defines a complete metric whenever the underlying space is complete. This establishes a rigorous geometric foundation for applying contraction arguments in a deformation-dependent setting. Based on this framework, a Banach type (q,τ)-fixed point theorem was derived, together with explicit convergence estimates for Picard iterations. The obtained bounds show that the contraction constant depends explicitly on the deformation parameters (q,τ)-and, in fractional applications, on the normalization factor $\Gamma_{q,\tau}(\alpha+1)$. This result clarifies how quantum deformation modifies convergence speed and stability in comparison with the classical case.

The fixed point theory was then applied to nonlinear fractional equations governed by the (q,τ)-fractional operator $\mathcal{D}^{\alpha}_{q,\tau}$. Under explicit deformation-dependent conditions, existence and uniqueness of solutions were established. The analysis shows that the (q,τ)-Gamma function plays a stabilizing role by regulating the effective memory strength of the fractional kernel and ensuring contractivity of the associated Volterra operator. Furthermore, we formulated and analyzed optimal control problems for (q,τ)-fractional systems. The new fixed point results guarantee the well posedness of both the state and adjoint equations, which enabled the derivation of a complete first order optimality system. A gradient projection scheme was implemented, and numerical simulations confirmed monotone decay of the objective functional and convergence of the optimization iterations, in agreement with the theoretical stability bounds. The results demonstrate that the proposed (q,τ)-fixed point framework provides a unified and practically effective tool for analyzing and optimizing quantum deformed fractional models. The explicit dependence of stability and convergence on (q,τ)-and $\Gamma_{q,\tau}$ offers a tunable mechanism for controlling memory effects and numerical performance. Future studies should focus on extending the present analysis to variable order and distributed order (q,τ)-fractional operators, as well as to stochastic deformation kernels and large scale networked optimization problems.

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Author contributions

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Formal analysis: All authors
Investigation: All authors
Methodology: Rabha W. Ibrahim
Writing – original draft: All authors
Writing – review & editing: All authors

Availability of data

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AI tools statement

AI tools is used to check the language of the paper.

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