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RESEARCH ARTICLE

On the integral transform of generalized k-Hilfer–Prabhakar fractional derivative with applications to fractional type advection–dispersion equations

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ABSTRACT

The Hilfer-Prabhakar (HP) derivatives are advance class of fractional operators with nonlocal features, memory effects, and nonexponential decay. These derivatives are generally used in study of heterogeneous systems, anomalous diffusion, dielectric spectroscopy, free electron lasers, fractional heat equations, and Cauchy problems. In this study, we derive the Kharrat–Toma (KT) transforms of the kernel function of the k-Prabhakar integral (k-PI), k-Prabhakar fractional derivative, its regularized form, and the k-Hilfer–Prabhakar fractional derivative (k-HPFD). We also establish the relationship between the k-Prabhakar fractional derivative and its regularized form for an absolutely continuous function using KT transform operations. Similarly, the KT transforms of k-HPFD and its regularized variant have been computed. Moreover, we establish the relationship between the k-HPFD and its regularized form using KT transform operations. Furthermore, we present solutions to Cauchy problems and generalized Cauchy problems for a fractional heat model involving the k-HPFD using the KT transform combined with the Fourier transform. Finally, we propose an integral technique combining the KT and Fourier transforms to construct solutions for the fractional advection–dispersion equation governed by the k-HPFD. The solutions of the Cauchy problems and advection–dispersion equations involving the k-HPFD and its Caputo form are expressed in terms of a generalized Mittag-Leffler function through sequential application of integral transform techniques. It is observed that the solutions for the generalized Cauchy problems and advection–dispersion equations involving the k-HPFD operator reduce to those involving the Hilfer fractional derivative, Riemann–Liouville fractional derivative, and Caputo derivative for specific parameter values.

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1. Introduction

The domain of fractional-order calculus encompasses diverse formulations of fractional derivatives and their fundamental applications in fractional differential equations. Carr[1] presented semi-analytical solutions for advection–dispersion equations in multilayer porous media. Suzuki et al.[2] employed fractional modeling to analyze nonlocal subsurface transport models. Zheng et al.[3] derived a Petrov–Galerkin numerical approximation for a fractional diffusion equation with variable coefficients. Garra and Garrappa[4] presented the theory and applications of the three-parameter Mittag-Leffler function (MLF). Giusti et al.[5] provide a comprehensive account of the theory and applications of Prabhakar fractional calculus. Garrappa and Kaslik[6] investigated the stability of fractional-order systems with Prabhakar derivatives.

The Hilfer–Prabhakar fractional derivative (HPFD) serves as an interpolating operator between the Prabhakar fractional derivative (PFD)[7] and its regularized form. Garra et al.[8] introduced the HPFD and its Caputo-type version. Kilbas et al.[7] provided a comprehensive treatment of generalized MLF (GMLF) and generalized fractional calculus operators. Tomovski et al.[9] developed an operational calculus for generalized fractional derivative operators and MLFs. Polito and Tomovski[10] established several key properties of Prabhakar-type fractional calculus. In particular, they analyzed the boundedness of the Prabhakar integral (PI) in the spaces $L^p, p \in (0,1]$ and L^1 for L^p functions, $p \in (1,\infty)$ are well described in the study by Polito and Tomovski.[10]. Furthermore, they also established the boundedness of the HPFD and its regularized form in L^1 space.

Sandev[11] investigated the generalized Langevin equation with the regularized PFD. Tomovski et al.[12] applied the HPFD operator to option pricing models in finance. Samraiz et al.[13] analyzed the *(k,s)*-HPFD and its applications in mathematical physics. Sousa and de Oliveira [14] presented a detailed analysis of the Ψ-Hilfer fractional derivative. Nisar et al.[15] provided a comprehensive review of the (k,s)-fractional calculus associated with the k-MLF.

Recent studies have reported significant developments related to the HPFD operator, weighted fractional operators, and their applications. Magar et al.[16] formulated the PFD and HPFD within the framework of Ψ-fractional calculus and demonstrated their applications. Samraiz et al.[17] introduced weighted fractional operators with applications to mathematical models in physics. Rahman et al.[18] established key results for generalized k-fractional derivative operators. Marynets and Tomovski[19] investigated fractional periodic boundary value problems and Cauchy problems involving the HPFD. Feng et al.[20] developed a generalized k-Hilfer–Prabhakar fractional viscoelastic–plastic model.

Sucu et al.[21] developed and applied a numerical collocation scheme based on septic B-splines to investigate the FitzHugh–Nagumo equation. Recently, Kumar[22] presented results on GMLF and fractional integrals. Natalini et al.[23] introduced a fractional parametric-type Laplace transform (LT) in connection with generalized Blissard problems. Sawar et al.[24] proposed a secure communication model based on fractional variable-order memristive hyperchaotic systems with nonlinear synchronization for image encryption. Reynolds[25] derived single and double integrals of the MLF in terms of the Hurwitz–Lerch zeta function.

Ma et al.[26] established new results on the approximate controllability of Sobolev-type fractional delay integrodifferential systems of order $1 < r < 2$. Moreover, Ma et al.[27] investigated the controllability of Sobolev-type fractional differential equations of order $1 < r < 2$ with finite delay. Panchal et al.[28] presented the Sumudu transform of the HPFD and its applications. Dhillon et al.[29] analyzed the fractional linear birth–death process involving the HPFD. Mashoof et al.[30] conducted a stability analysis of distributed-order Hilfer–Prabhakar systems based on inertia theory. Moreover, Cauchy and diffusion equations involving the HPFD have been investigated via the Kharrat–Toma transform (KTT), as described in the study by Dubey et al.[31] Dorrego[32] introduced the k-PI and k-PFD. Furthermore, Panchal et al.[33] derived the k-HPFD and the Caputo-type versions of the k-PFD and k-HPFD.

This study extends the Hilfer–Prabhakar framework to the k-framework and applies the KTT. The KTT is employed as an effective tool for handling differential and integral equations, owing to its specific kernel and advantageous properties. One of the main advantages of the KTT lies in its targeted capability to solve ordinary differential equations with both constant and variable coefficients—problems for which traditional transforms such as Laplace or Sumudu may require additional manipulation. Furthermore, the KTT demonstrates strong adaptability to specific problem classes and can handle certain kernels without requiring the complex contour integration often associated with inverse LTs for generalized operators.

This study derives the KTT of the k-PI, k-PFD, and the Caputo-type form of the k-PFD. Furthermore, the KTT of the k-HPFD, along with its Caputo-type variant, is also obtained. The KTT was originally introduced by Kharrat and Toma[34] and is applied here as a key tool for solving ordinary differential equations with initial conditions (ICs). Moreover, the derived formulae establish relationships between the k-PFD and its regularized form, as well as between the k-HPFD and its regularized form, involving the k-MLF. In addition, these results are utilized to solve Cauchy problems[8] and fractional advection–dispersion equations[35] involving the k-HPFD operator.

2. Preliminary concepts: KTT, k-gamma function (k-GF), and the k-HPFD

This study considers the variables and definitions as follows:

Definition 2.1.[34] A function $\Theta(\zeta)$ is exponentially ordered on each finite domain in $[0,\infty)$ if $\exists$ a constant E > 0 so that $|\Theta(\zeta)| \le Ee^{\alpha\zeta}$, $\alpha > 0$, $\forall\zeta \ge 0$.

Suppose $\Theta(\zeta) : R \to R$ is a function defined by $\begin{cases}\Theta(\zeta) > 0, & \zeta \ge 0\\ \Theta(\zeta) = 0, & \zeta < 0\end{cases}$. If $\Theta(\zeta)$ is piecewise continuous on $[0, +\infty)$ and has exponential order, the KTT of $\Theta(\zeta)$, indicated by $B\left[\Theta(\zeta)\right]$, is given by:

$$B\left[\Theta(\zeta)\right] = \tilde{\Theta}(s) = s^3 \int_0^\infty e^{-\frac{\zeta}{s^2}}\Theta(\zeta)\,d\zeta; s > 0, \tag{1}$$

so that the right-side integral exists. Here, s is the transformation variable, and $\Theta(\zeta)$ is the inverse KTT of $\tilde{\Theta}(s)$ and is written as $\Theta(\zeta) = B^{-1}\left[\tilde{\Theta}(s)\right]$. Here, B^{-1} is the inverse KTT operator.

Theorem 2.1. (Sufficient condition for existence of KTT).[34] The KTT of $\Theta(\zeta)$ (i.e., $B\left[\Theta(\zeta)\right]$) exists if it is exponentially ordered and $\int_0^b |\Theta(\zeta)|\,d\zeta$ exists for arbitrary $b > 0$.

Theorem 2.2. (Superposition principle).[34] If c and d are arbitrary constants and $\Theta_1(\zeta)$ and $\Theta_2(\zeta)$ are functions, then

$$B\left\{c\Theta_1(\zeta) + d\Theta_2(\zeta)\right\} = cB\left[\Theta_1(\zeta)\right] + dB\left[\Theta_2(\zeta)\right]. \tag{2}$$

Theorem 2.3. (KTT of the λth-order derivative).[34] If $\Theta^{(\lambda)}(\zeta)$ is the derivative of $\Theta(\zeta)$ of λth-order with respect to ζ, then its KTT is given by:

$$B\left[\Theta^{(\lambda)}(\zeta)\right] = \tilde{\Theta}_\lambda(s) = \frac{1}{s^{2\lambda}}\tilde{\Theta}(s) - \sum_{l=0}^{\lambda-1} s^{-2\lambda+2l+5}\Theta^{(l)}(0), \lambda \ge 1. \tag{3}$$

For $q = 1, 2, 3$, we have

$$B\left[\Theta'(\zeta)\right] = \tilde{\Theta}_1(s) = \frac{1}{s^2}\tilde{\Theta}(s) - s^3\Theta(0), \tag{4}$$

$$B\left[\Theta''(\zeta)\right] = \tilde{\Theta}_2(s) = \frac{1}{s^4}\tilde{\Theta}(s) - s\Theta(0) - s^3\Theta'(0). \tag{5}$$

In the same way, the KTT of a partial derivative of first order is given by:

$$B\left[\frac{\partial\Theta}{\partial\zeta}(y,\zeta)\right] = \tilde{\Theta}(y,s) = \frac{1}{s^2}\tilde{\Theta}(y,s) - s^3\Theta(y, 0). \tag{6}$$

Theorem 2.4. (Convolution theorem for the KTT).[34] If $\bar{\Theta}_1(s)$ and $\bar{\Theta}_2(s)$ are KTTs of $\Theta_1(\zeta)$ and $\Theta_2(\zeta)$, respectively, then

$$B\left[\Theta_1 * \Theta_2\right] = \frac{1}{s^3}\bar{\Theta}_1(s)\,\bar{\Theta}_2(s), \tag{7}$$

where $\Theta_1 * \Theta_2$ is the convolution of two functions: $\Theta_1(\zeta)$ and $\Theta_2(\zeta)$.

The KTT of special functions given by Kharrat and Toma[34] are illustrated as: $B[1] = s^5$, $B\left[\sin(k\zeta)\right] = \frac{ks^7}{1+k^2s^4}$, $B\left[\cos(k\zeta)\right] = \frac{s^5}{1+k^2s^4}$, $B\left[\zeta^q\right] = s^{2q+5}.q!$ for $q = 1, 2\ldots$.

Definition 2.2. (Inverse of the KTT).[34] The inverse of KTT is formulated as:

$$B^{-1}\left[\tilde{\Theta}(s)\right](\zeta) = \Theta(\zeta) = B^{-1}\left[s^3\int_0^\infty \Theta(\zeta)e^{-\frac{\zeta}{s^2}}d\zeta\right], \zeta > 0, \tag{8}$$

where $\tilde{\Theta}(s)$ is the KTT of $\Theta(\zeta)$.

Definition 2.3.[36] If $\Theta(\zeta)$ is a piecewise smooth function in every bounded range and $|\Theta(\zeta)|$ is integrable on $(-\infty,\infty)$, the Fourier transform (FT) of $\Theta(\zeta)$ is computed as $F\left[\Theta(\zeta)\right](p) = \hat{\Theta}(p)$ and written as $\hat{\Theta}(p) = \int_{-\infty}^{\infty} e^{-ip\xi}\Theta(\zeta)d\zeta$. Here, $\Theta(\zeta)$ is the inverse FT (IFT) of $\hat{\Theta}(p)$ written as$\Theta(\zeta) = F^{-1}\left[\hat{\Theta}(p)\right](\zeta)$ and computed as

$$\Theta(\zeta) = F^{-1}\left[\hat{\Theta}(p)\right](\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\zeta}\Theta(p)\,dp. \tag{9}$$

The Fourier sine transform of a function $\Theta(\zeta)$ is given by:

$$F_{\sin e}\left[\Theta(\zeta)\right](p) = \hat{\Theta}_{fst}(p) = \frac{2}{\pi}\int_0^\infty \Theta(\zeta)\sin p\zeta\, d\zeta. \tag{10}$$

The function $\Theta(\zeta)$ is called the IFT of $\hat{\Theta}_{fst}(p)$ and is reported as:

$$\begin{aligned}\Theta(\zeta) &= {F_{\sin e}}^{-1}\left[\hat{\Theta}_{fst}(p)\right](\zeta)\\ &= \frac{2}{\pi}\int_0^\infty \hat{\Theta}_{fst}(p)\sin p\zeta\, dp. \end{aligned}\tag{11}$$

Definition 2.4. (Hilfer fractional derivative).[37] Let $\vartheta \in (0,\,1)$, $\rho \in [0,\,1]$, $\Theta \in L^1(\theta,\phi)$, $-\infty \le \theta < \zeta < \phi \le \infty$, and $\left(\Theta * I_{\theta+}^{(1-\vartheta)(1-\rho)}\right)(\zeta) \in AC^1[\theta,\phi]$. Then the Hilfer fractional derivative (HFD) of $\Theta(\zeta)$ of order ϑ and type $\rho$is expressed as:

$$\left(D_{\theta+}^{\vartheta,\rho}\Theta\right)(\zeta) = \left(I_{\theta+}^{\rho(1-\vartheta)}\frac{d}{d\zeta}I_{\theta+}^{(1-\vartheta)(1-\rho)}\Theta\right)(\zeta). \tag{12}$$

The HFD operator shows interpolation between the Riemann–Liouville (RL) and the Caputo-type fractional derivative operators (FDOs) for $\rho = 0$ and $\rho = 1$, respectively.

Definition 2.5. (Caputo variant of Hilfer derivative).[8] Let$\vartheta \in (0,\,1)$, $\rho \in [0,\,1]$, and $\Theta \in AC^1[0,\phi]$, $0 < \zeta < \phi < \infty$. Then the regularized variant of the HFD of $\Theta(\zeta)$ of ρ type and ϑth order is formulated as:

$$\left(D_{0+}^{\vartheta,\rho}\Theta\right)(\zeta) - \frac{\zeta^{-\vartheta}\Theta(0^+)}{\Gamma(1-\vartheta)} = {}^{C}D_{0+}^{\vartheta}\Theta(\zeta), \tag{13}$$

where ${}^{C}D_{0+}^{\vartheta}\Theta(\zeta)$ represents the FDO in the Caputo sense.

Definition 2.6.[38] The Prabhakar function, i.e., the GMLF $\forall\ z \in C$, is given by $E_{q,\vartheta}^{\gamma}(z) = \sum_{l=0}^{\infty}\frac{\Gamma(\gamma+l)}{\Gamma(\gamma)\Gamma(q\,l+\vartheta)}\frac{z^l}{l!}$ for $q,\vartheta,\gamma \in C$ and$Re(q) > 0$, where Cdenotes the set of points in a Gauss plane and $E_{q,\vartheta}^{\gamma}(z)$ represents an everywhere analytic function of order ${}^{1}\!/_{Re(q)}$ in the Argand diagram.

Definition 2.7.[8] The general form of HFD was introduced by transforming the RL integral operator into the mathematical form of the HFD with a generalized integral having kernel $e_{q,\vartheta,\sigma}^{\gamma}(\zeta) = \zeta^{\vartheta-1}E_{q,\vartheta}^{\gamma}(\sigma\zeta^q)$, where $\zeta \in R$,$q,\vartheta,\sigma,\gamma \in C$, $Re(\vartheta)$, $Re(q) > 0$, and $E_{q,\vartheta}^{\gamma}(z) = \sum_{l=0}^{\infty}\frac{\Gamma(\gamma+l)}{\Gamma(\gamma)\Gamma(q\,l+\vartheta)}\frac{z^l}{l!}$ is the GMLF described by Prabhakar.[38]

Definition 2.8. (Prabhakar integral)[7,38] Let $\Theta \in L^1(0,\phi)$, $0 < \zeta < \phi \le \infty$. Then the PI is written as:

$$\begin{aligned}&\mathrm{M}_{q,\vartheta,\sigma,0^+}^{\gamma}\Theta(\zeta)\\ &\quad= \int_0^{\zeta}(\zeta-y)^{\vartheta-1}E_{q,\vartheta}^{\gamma}\left[\sigma(\zeta-y)^q\right]\Theta(y)\,dy\\ &\quad= \left(\Theta * e_{q,\vartheta,\sigma}^{\gamma}\right)(\zeta),\end{aligned}\tag{14}$$

where $(*)$ denotes the convolution operation; $q,\vartheta,\sigma,\gamma \in C$with $Re(\vartheta)$, $Re(q) > 0$ and

$$e_{q,\vartheta,\sigma}^{\gamma}(\zeta) = \zeta^{\vartheta-1}E_{q,\vartheta}^{\gamma}(\sigma\zeta^q). \tag{15}$$

Let $q \in (0,\,1)$, $\sigma,\gamma > 0$ and $q\gamma > \vartheta - 1 > 0$. If $\Theta \in L^p(a,\phi)$, $0 < p \le 1$, then the integral operator $\mathrm{M}_{q,\vartheta,\sigma,a^+}^{\gamma}$ is bounded in $L^p(a,\phi)$ and $\left\|\mathrm{M}_{q,\vartheta,\sigma,a^+}^{\gamma}\Theta\right\| \le D\left\|\Theta\right\|_p$, where D is constant, $0 < D < \infty$, is given by $D = \frac{Be\left(\gamma - \frac{\vartheta-1}{q},\frac{\vartheta-1}{q}\right)}{\pi q\sigma^{\frac{\vartheta-1}{q}}\left[\cos\left(\frac{\pi q}{2}\right)\right]^{\gamma-\frac{\vartheta-1}{q}(\phi-a)}{}^{1}\!/_{p}}$ in which $Be(\mu,\nu)$ is the Beta function.

The remarkable feature of PI is illustrated as: Let $\gamma,q,\vartheta,\sigma,\Lambda,\delta \in C$, $R(q)$, $R(\vartheta) > 0$, $\zeta \in R$. Then

$$\left(\mathrm{M}_{q,\vartheta,\sigma,0^+}^{\gamma}e_{q,\delta,\sigma}^{\Lambda}\right)(\zeta) = e_{q,\vartheta+\delta,\sigma}^{\gamma+\Lambda}(\zeta). \tag{16}$$

Definition 2.9. (Prabhakar derivative).[7,8] Let $\Theta \in L^1(0,\phi)$, $0 < \zeta < \phi \le \infty$, and$\left(\Theta * e_{q,m-\vartheta,\sigma}^{-\gamma}\right)(\cdot) \in W^{m,1}(0,\phi)$, $m = [\vartheta]$. Then the Prabhakar derivative is presented as:

$$\left(D_{q,\vartheta,\sigma,0^+}^{\gamma}\Theta\right)(\zeta) = \left(\frac{d^{\lambda}}{d\zeta^{\lambda}}\left(M_{q,\lambda-\vartheta,\sigma,0^+}^{-\gamma}\right)\right)(\zeta), \tag{17}$$

where $q,\vartheta,\sigma,\gamma \in C$with $Re(\vartheta)$, $Re(q) > 0$.

Definition 2.10. (Caputo variant of Prabhakar derivative).[8,39] Let $\Theta \in AC^m(0,\phi)$, $0 < \zeta < \phi \le \infty$. The Caputo variant of Prabhakar derivative is represented as:

$$\begin{aligned}&\left({}^{C}D_{q,\vartheta,\sigma,0^+}^{\gamma}\Theta\right)(\zeta) = \left(M_{q,\lambda-\vartheta,\sigma,0^+}^{-\gamma}\frac{d^{\lambda}}{d\zeta^{\lambda}}\Theta\right)(\zeta)\\ &= \left(D_{q,\vartheta,\sigma,0^+}^{\gamma}\Theta\right)(\zeta)\\ &\quad - \sum_{l=0}^{\lambda-1}\zeta^{l-\vartheta}E_{q,l-\vartheta+1}^{-\gamma}(\sigma\zeta^q)\,\Theta^{(l)}(0^+).\end{aligned}\tag{18}$$

For $\gamma = 0$, the regularized Prabhakar derivative reduces to the Caputo derivative.

Remark 2.1.[8] Let $\vartheta > 0$ and $\Theta \in AC^{\lambda}(0,\phi)$, $0 < \zeta < \phi \leq \infty$. Then $\left({}^{C}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \left(D^{\gamma}_{q,\vartheta,\sigma,0^+}\Phi\right)(\zeta)$, where

$$\Phi(\zeta) = \Theta(\zeta) - \sum_{l=0}^{\lambda-1} \frac{\zeta^l}{l!}\Theta^{(l)}\left(0^+\right). \tag{19}$$

Definition 2.11. (HPFD).[8] Let$\vartheta \in (0,1)$, $\rho \in [0,1]$, and$\Theta \in L^1(0,\phi)$, $0 < \zeta < \phi \leq \infty$, and let $\left(\Theta * e^{-\gamma(1-\rho)}_{q,(1-\rho)(1-\vartheta),\sigma}\right)(\zeta) \in AC^1[0,\phi]$. The HPFD of $\Theta(\zeta)$ of order ϑ, expressed by $D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)$, is given by:

$$D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = \left(M^{-\gamma\rho}_{q,\rho(1-\vartheta),\sigma,0^+}\frac{d}{d\zeta}\left(M^{-\gamma(1-\rho)}_{q,(1-\rho)(1-\vartheta),\sigma,0^+}\Theta\right)\right)(\zeta), \tag{20}$$

where $\gamma,\sigma \in R$, $q > 0$ and $M^{0}_{q,0,\sigma,0^+}\Theta = \Theta$. Here, ρ is a parameter for interpolation. HPFD takes the form of HFD for $\gamma = 0$.

Definition 2.12.[8] Let $\Theta \in AC^1[0,\phi]$, $0 < \zeta < \phi \leq \infty$ and let $\vartheta \in (0,1)$and $\rho \in [0,1]$,$\gamma,\sigma \in R$, $q > 0$. The Caputo form of HPFD of $\Theta(\zeta)$ of ϑth order is written as ${}^{C}D^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)$ and defined by

$${}^{C}D^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) = \left(M^{-\gamma\rho}_{q,\rho(1-\vartheta),\sigma,0^+}M^{-\gamma(1-\rho)}_{q,(1-\rho)(1-\vartheta),\sigma,0^+}\frac{d}{d\zeta}\Theta\right)(\zeta). \tag{21}$$

Additionally,

$$\left(M^{\delta}_{q,\rho,\sigma,0^+}M^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \left(M^{\delta+\gamma}_{q,\rho+\vartheta,\sigma,0^+}\Theta\right)(\zeta). \tag{22}$$

Lemma 2.1.[31] The KTT of the memory/kernel function (KF) $e^{\gamma}_{q,\vartheta,\sigma}(\zeta)$ is given by

$B\left[e^{\gamma}_{q,\vartheta,\sigma}(\zeta)\right] = s^{2\vartheta+3}\left[1-\sigma s^{2q}\right]^{-\gamma}$ for $\sigma \in (0,1)$, $\Theta,\theta \in R$, $\eta > 0$, and hence, the KTT of the PI is derived as:

$$B\left[M^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right] = s^{2\vartheta}B(\Theta)\left[1-\sigma s^{2q}\right]^{-\gamma}. \tag{23}$$

Lemma 2.2.[31] The KTT of HPFD $D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)$ is obtained by

$$B\left(D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right) = s^{-2\vartheta}\left[1-\sigma s^{2q}\right]^{\gamma}B[\Theta](s) - s^{2\rho(1-\vartheta)+3}\left[1-\sigma s^{2q}\right]^{\gamma\rho}\left(E^{-\gamma(1-\rho)}_{q,(1-\rho)(1-\vartheta),\sigma,0^+}\Theta\right)_{\zeta=0^+}. \tag{24}$$

Lemma 2.3.[31] The KTT of the Caputo-type variant of HPFD of ϑth order is:

$$B\left({}^{C}D^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta\right)(s,u) = \left[1-\sigma s^{2q}\right]^{\gamma}\left[s^{-2\vartheta}B[\Theta(\zeta)](s) - s^{5-2\vartheta}\Theta\left(0^+\right)\right]. \tag{25}$$

Definition 2.13.[40] For $\Psi > 0$, the Ψ-GF $\Gamma_\Psi(\gamma)$ is

$$\Gamma_\Psi(\gamma) = \lim_{l\to\infty}\frac{l!\,\Psi^l\,(l\Psi)^{\frac{\gamma}{\Psi}-1}}{(\gamma)_{l,\Psi}}, \gamma \in C\backslash\Psi Z^-. \tag{26}$$

Proposition 2.1.[40] Let $k \in R$ and $\gamma \in C$. Then the following identity holds:

$$\Gamma_k(\gamma) = \frac{1}{k^{1-\frac{\gamma}{k}}}\Gamma\left(\frac{\gamma}{k}\right). \tag{27}$$

The function $\Gamma_k(\gamma)$, restricted to $(0,\infty)$, is characterized by the following properties: $\gamma\,\Gamma_k(\gamma) = \Gamma_k(\gamma+k)$, and $\Gamma_k(\gamma)$ is logarithmic convex. Moreover, $\Gamma_k(\gamma)$ also admits an infinite product expansion.

Result 2.1.[40] For an arbitrary $\Psi > 0$, $\gamma \in C$, and $|\sigma| < \frac{1}{\Psi}$, the identity is given as:

$$\sum_{\beta=0}^{\infty}(\gamma)_{\beta,\Psi}\frac{\sigma^\beta}{\beta!} = (1-\Psi\sigma)^{-\frac{\gamma}{\Psi}}, |\Psi\sigma| < 1. \tag{28}$$

Here, $(\gamma)_{\beta,\Psi} = \gamma(\gamma+\Psi)(\gamma+2\Psi)........(\gamma+(\beta-1)\Psi)$ is the Pochhammer Ψ-symbol. For $\Psi = 1$, we obtain the symbol-raising factorial $(\gamma)_\beta$.

Proposition 2.2.[40] The k-GF $\Gamma_k(\beta)$ satisfies the following mathematical expressions:

$$\Gamma_k(\beta)\Gamma_k(k-\beta)\sin\left(\frac{\pi\beta}{k}\right) = \pi, \qquad (29)$$
$$\Gamma_k(\beta+k) = \beta\,\Gamma_k(\gamma)\,.$$

$\frac{1}{\Gamma_k(\beta)} = \frac{\beta e^{\frac{\beta}{k}\lambda}}{k^{-\frac{\beta}{k}}}\prod_{r=1}^{\infty}\left(e^{-\frac{\beta}{rk}}\left(1+\frac{\beta}{rk}\right)\right)$, where

$$\lambda = \lim_{r\to\infty}\left(1+\cdots+\frac{1}{r}-\log(r)\right). \qquad (30)$$

Definition 2.14. (k-MLF).[41] Let $r \in N^+$, $k \in R^+$;$q, \vartheta, \gamma \in C$, $Re(q) > 0$, $Re(\vartheta) > 0$. The k-MLF is defined as:

$$E^{\gamma}_{k,q,\vartheta}(z) = \sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{\Gamma_k(q\,l+\vartheta)}\frac{z^l}{l!}, \qquad (31)$$

where $(\gamma)_{l,k} = \gamma(\gamma+k)(\gamma+2k)(\gamma+3k)\cdots\cdots(\gamma+(l-1)k)$ is the k-Pochhammer symbol (k-PS) and $\Gamma_\Psi(q) = \int_0^\infty e^{-\frac{t^\Psi}{\Psi}}t^{q-1}dt$ is the Ψ-gamma function.[40] The Ψ-PS is expressed as well as $(\gamma)_{l,\Psi} = \frac{\Gamma_\Psi(\gamma+\Psi l)}{\Gamma_\Psi(\gamma)}$ and $(\gamma)_{l,\Psi} = \left(\frac{\gamma}{\Psi}\right)_l\Psi^l$.[40]

It may be observed that $E^{\gamma}_{k,q,\vartheta}(z)$ is such that $E^{\gamma}_{k,q,\vartheta}(z) \to E^{\gamma}_{q,\vartheta}(z)$ as $k \to 1$, since $(\gamma)_{l,k} \to (\gamma)_l$, $\Gamma_k(z) \to \gamma(z)$ and the convergence of the series is uniform on compact subsets.[41]

Definition 2.15. (k-Hilfer derivative).[42] Let $k \in R^+$; $\vartheta \in (0,1)$, $\rho \in [0,1]$, $\Theta \in L^1(0,\phi)$, $0 < \zeta < \phi \le \infty$, and$\left(\Theta * {}_kI_{0+}^{(1-\vartheta)(1-\rho)}\right)(\zeta) \in AC^1[0,\phi]$. Then the k-HFD of $\Theta(\zeta)$ of type ρ and order ϑ is defined by:

$$\left({}^kD_{0^+}^{\vartheta,\rho}\Theta\right)(\zeta) = \left({}_kI_{0^+}^{\rho(1-\vartheta)}\frac{d}{d\zeta}{}_kI_{0^+}^{(1-\vartheta)(1-\rho)}\Theta\right)(\zeta), \qquad (32)$$

where${}_kI_{0^+}^{q}\Theta(\zeta) = \frac{1}{k\Gamma_k(q)}\int_0^\zeta(\zeta-y)^{\frac{q}{k}-1}\Theta(y)\,dy$ is the k-RL fractional integral.[43]

Definition 2.16. (k-PI).[32] Let $\Theta \in L^1(0,\phi)$, $0 < \zeta < \phi \le \infty$ and $k \in R^+$. The k-PI is defined by:

$$\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \int_0^\zeta\frac{(\zeta-y)^{\frac{\vartheta}{k}-1}}{k}E^{\gamma}_{k,q,\vartheta}\left[\sigma(\zeta-y)^{\frac{q}{k}}\right]\Theta(y)\,dy = \left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma} * \Theta\right)(\zeta), \qquad (33)$$

where $(*)$ denotes the convolution operation; $q, \vartheta, \sigma, \gamma \in C$with $Re(q),\ Re(\vartheta) > 0$ and KF of k-PI is defined by:

$${}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}(\zeta) = \begin{cases} E^{\gamma}_{k,q,\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\frac{\zeta^{\frac{\vartheta}{k}-1}}{k}, & \zeta > 0 \\ 0 & ,\ \zeta \le 0 \end{cases} \qquad (34)$$

For $\gamma = 0$, $\left({}_kP^{0}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \left({}_kI^{\vartheta}_{0^+}\Theta\right)(\zeta)$ and for $\gamma = \vartheta = 0$, $\left({}_kP^{0}_{q,0,\sigma,0^+}\Theta\right)(\zeta) = \Theta(\zeta)$.

Definition 2.17. (k-Prabhakar fractional derivative).[32] Let $k \in R^+$, $q, \vartheta, \sigma, \gamma \in C$;$Re(q),\ Re(\vartheta) > 0$, $\lambda = \left[\frac{\vartheta}{k}\right]+1$, and $\Theta \in AC^\lambda(0,\phi)$, $0 < \zeta < \phi < \infty$.

The k-PFD of $\Theta(\zeta)$ of ϑth order is constituted as:

$${}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta) = k^\lambda{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\left(\frac{d}{d\zeta}\right)^\lambda\Theta(\zeta). \qquad (35)$$

Proposition 2.3.[32] Let $k \in R^+$, $q, \vartheta, \sigma, \gamma \in C$; $Re(\vartheta),\ Re(q) > 0$, $\lambda = \left[\frac{\vartheta}{k}\right]+1$. Then for any $\Theta \in L^1(0,\phi)$ and $0 < \zeta < \phi \le \infty$, we have

$$\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}{}_kP^{\delta}_{q,\mu,\sigma,0^+}\Theta\right)(\zeta) = {}_kP^{\gamma+\delta}_{q,\vartheta+\mu,\sigma,0^+}\Theta(\zeta). \qquad (36)$$

Definition 2.18.[33]

$$AC^{\lambda}[\theta,\phi] = \left\{\Theta:[\theta,\phi]\to R;\ \frac{d^{\lambda-1}}{d\zeta^{\lambda-1}}\Theta(\zeta)\in AC[\theta,\phi]\right\}. \tag{37}$$

Definition 2.19. (Caputo variant of k-PFD).[33] Let $k\in R^+$, $q,\vartheta,\sigma,\gamma\in C$; $Re(q)$, $Re(\vartheta)>0$, $\lambda=\left[\frac{\vartheta}{k}\right]+1$ and $\Theta\in AC^{\lambda}(0,\phi)$, $0<\zeta<\phi\le\infty$. The regularized variant of the k-PFD of ϑth order of $\Theta(\zeta)$ is represented as:

$$ {}_k^C D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta) = k^{\lambda}\,{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\left(\frac{d}{d\zeta}\right)^{\lambda}\Theta(\zeta). \tag{38}$$

Definition 2.20. (k-HPFD).[33] Let $\Theta\in L^1(0,\phi)$, $0<\zeta<\phi\le\infty$; $k,q>0$; $\gamma,\sigma\in R$; $\vartheta\in(0,1)$, $\rho\in[0,1]$ and $\left(\Theta * {}_k\varepsilon^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma}\right)(\zeta)\in AC^1[0,\phi]$. The k-HPFD of $\Theta(\zeta)$ of ϑthorder, expressed by $D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)$, is defined by

$$ {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = k\left({}_kP^{-\gamma\rho}_{q,\rho(k-\vartheta),\sigma,0^+}\frac{d}{d\zeta}\left({}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta\right)\right)(\zeta). \tag{39}$$

Here, ρ is an interpolating parameter. For $\gamma=0$ and $\vartheta=k-\vartheta+1$, the k-HPFD reduces to the k-HFD. For $\rho=0$ and $\rho=1$, the k-HPFD changes to the k-PFD and its Caputo form, respectively. For $k=1$, k-HPFD and its Caputo version reduce to the HPFD and regularized version of HPFD, respectively.

Definition 2.21. (Caputo variant of k-HPFD).[33] Let $\Theta\in AC^1[0,\phi]$, $0<\zeta<\phi\le\infty$ and $k,q>0$; $\gamma,\sigma\in R$; $\vartheta\in(0,1)$, $\rho\in[0,1]$. The regular counterpart of the k-HPFD of $\Theta(\zeta)$ of ϑth order is expressed as ${}_k^C D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)$ and defined by:

$$ {}_k^C D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = k\left({}_kP^{-\gamma\rho}_{q,\rho(k-\vartheta),\sigma,0^+}\,{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\frac{d}{d\zeta}\Theta\right)(\zeta). \tag{40}$$

From **Proposition 2.1**, we have

$$ {}_k^C D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = k\left({}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\frac{d}{d\zeta}\Theta\right)(\zeta). \tag{41}$$

Since the regularized variant of the k-HPFD is free from ρ, it is also expressed as:

$$ {}_k^C D^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) = k\left({}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\frac{d}{d\zeta}\Theta\right)(\zeta). \tag{42}$$

Result 2.2. (Association of the k-HPFD and its Caputo variant).[33] For $\Theta\in AC^1[0,\phi]$, we have

$$ {}_k^C D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = {}_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) - \zeta^{-\frac{\vartheta}{k}}E^{-\gamma}_{k,q,k-\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\Theta(0^+). \tag{43}$$

Lemma 2.4.[33] Let $q,\vartheta,\sigma,\gamma\in C$; $Re(q)$, $Re(\vartheta)>0$, $k\in R^+$ and $\left|k\sigma(ks)^{-\frac{q}{k}}\right|<1$. Then

$$ L\left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}(\zeta)\right)(s) = (ks)^{-\frac{2\vartheta}{k}}\left(1-\sigma k(sk)^{-\frac{q}{k}}\right)^{-\frac{\gamma}{k}}, \tag{44}$$

where L represents the LT of KF ${}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}(\zeta)$.

3. Primary derivations: KTT of the *k*-PFD, *k*-HPFD, and the Caputo form of *k*-HPFD

Lemma 3.1. Let $\Theta\in L^1(0,\phi)$, $k\in R^+$ and $0<\zeta<\phi\le\infty$. Then the k-PI is given by:

$$\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \int_0^{\zeta}\frac{(\zeta-y)^{\frac{\vartheta}{k}-1}}{k}E^{\gamma}_{k,q,\vartheta}\left[\sigma(\zeta-y)^{\frac{q}{k}}\right]\Theta(y)\,dy = \left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}*\Theta\right)(\zeta), \tag{45}$$

where $\Theta(\zeta):R\to R$ is a piecewise continuous on $[0,\,+\infty)$ and has exponential order, and $(*)$ denotes the convolution operation; $q,\vartheta,\sigma,\gamma\in C$ with $Re\,(q)\,,\ Re\,(\vartheta)>0$.

Let $q,\vartheta,\sigma,\gamma\in C$; $\vartheta\in(0,\,1)$, $Re\,(q)\,,\,Re\,(\vartheta)>0$, $k\in R^{+}$, and the KF of the k-PI is defined by:

$$ {}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)=\begin{cases}\frac{\zeta^{\frac{\vartheta}{k}-1}}{k}E^{\gamma}_{k,q,\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right),\ \zeta>0\\ 0\qquad\qquad\qquad\quad,\ \zeta\leq 0\end{cases}. \tag{46} $$

Then the KTT of the KF ${}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)$ is given by:

$$ B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)=s^{\frac{2\vartheta}{k}+3}k^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\gamma\frac{1}{k}}, \tag{47} $$

provided $\left|\sigma k\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right|<1$.

Proof. By formula of the KTT and **Equation 34** for the KF, we obtain:

$$ B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)=s^{3}\int_{0}^{\infty}\frac{1}{k}e^{-\frac{\zeta}{s^{2}}}\zeta^{\frac{\vartheta}{k}-1}E^{\gamma}_{k,q,\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)d\zeta. \tag{48} $$

By virtue of **Equation 31** of the k-MLF, **Equation 48** transforms to:

$$ B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)=\frac{s^{3}}{k}\int_{0}^{\infty}e^{-\frac{\zeta}{s^{2}}}\zeta^{\frac{\vartheta}{k}-1}\sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{\Gamma_{k}\left(q\,l+\vartheta\right)}\frac{\sigma^{l}}{l!}\zeta^{\frac{ql}{k}}d\zeta. \tag{49} $$

Since the series given by the k-MLF in **Equation 49** is uniformly convergent on compact subsets, as shown by Dorrego and Cerutti,[41] therefore we can interchange integration and summation as:

$$ \begin{aligned} &B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)\\ &=\frac{s^{3}}{k}\sum_{l=0}^{\infty}\frac{\sigma^{l}}{l!}\frac{(\gamma)_{l,k}}{\Gamma_{k}\left(\vartheta+q\,l\right)}\int_{0}^{\infty}e^{-\frac{\zeta}{s^{2}}}\zeta^{\frac{(ql+\vartheta)}{k}-1}d\zeta\\ &=\frac{1}{k}s^{3}\sum_{l=0}^{\infty}\frac{\sigma^{l}}{l!}\frac{(\gamma)_{l,k}}{\Gamma_{k}\left(\vartheta+q\,l\right)}\int_{0}^{\infty}e^{-x}\left(s^{2}x\right)^{\frac{(ql+\vartheta)}{k}-1}s^{2}dx\\ &=\frac{s^{3}}{k}\sum_{l=0}^{\infty}\frac{\sigma^{l}}{l!}\frac{(\gamma)_{l,k}}{\Gamma_{k}\left(q\,l+\vartheta\right)}s^{\frac{2(ql+\vartheta)}{k}}\int_{0}^{\infty}e^{-x}x^{\frac{(ql+\vartheta)}{k}-1}dx. \end{aligned} \tag{50} $$

Using the definition of the gamma function and **Proposition 2.1**, **Equation 50** becomes:

$$ \begin{aligned} &B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)\\ &=\frac{s^{3}}{k}\sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{\Gamma_{k}\left(ql\,+\vartheta\right)}\frac{\sigma^{l}}{l!}s^{\frac{2(ql+\vartheta)}{k}}\Gamma\left(\frac{ql\,+\vartheta}{k}\right)\\ &=\frac{s^{3}}{k}\sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{k^{\frac{(ql+\vartheta)}{k}-1}\Gamma\left(\frac{ql+\vartheta}{k}\right)}\frac{\sigma^{l}}{l!}s^{\frac{2(ql+\vartheta)}{k}}\Gamma\left(\frac{ql\,+\vartheta}{k}\right)\\ &=\frac{s^{3}}{k}\sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{k^{\frac{(ql+\vartheta)}{k}-1}}\frac{\sigma^{l}}{l!}s^{\frac{2(ql+\vartheta)}{k}}\\ &=s^{\frac{2\vartheta}{k}+3}k^{-\frac{\vartheta}{k}}\sum_{l=0}^{\infty}\frac{(\gamma)_{l,k}}{l!}\left(\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{l}. \end{aligned} \tag{51} $$

Finally, by virtue of **Equation 28**, we derive: $B\left[{}_{k}\varepsilon^{\gamma}_{q,\vartheta,\sigma}\left(\zeta\right)\right](s)=\frac{s^{\frac{2\vartheta}{k}+3}}{k^{\frac{\vartheta}{k}}}\left(1-\sigma k\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}$ with

$$ 1>\left|k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right|. \tag{52} $$

Lemma 3.2. Let $\Theta \in L^1(0,\phi)$, $0<\zeta<\phi\le\infty$ and $k \in R^+$. The KTT of the k-PI ${}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)$ is given by:

$$B\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) = k^{-\frac{\vartheta}{k}} s^{\frac{2\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}\tilde{\Theta}(s), \tag{53}$$

where $\tilde{\Theta}(s) = B(\Theta(\zeta))(s)$ and $\left|\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right|<1$.

Proof. The k-PI is given by

$$\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta\right)(\zeta) = \int_0^{\zeta}\frac{(\zeta-y)^{\frac{\vartheta}{k}-1}}{k}E^{\gamma}_{k,q,\vartheta}\left[\sigma(\zeta-y)^{\frac{q}{k}}\right]\Theta(y)\,dy = \left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}*\Theta\right)(\zeta). \tag{54}$$

where $\Theta(\zeta): R\to R$ is a piecewise continuous function on $[0,+\infty)$ and has exponential order, and $(*)$ denotes the convolution operation ; $q,\vartheta,\sigma,\gamma \in C$ with $Re(q),\ Re(\vartheta)>0$.

Applying the KTT on **Equation 54** and utilizing the Faltung theorem of the KTT together with **Lemma 3.1**, we obtain:

$$\begin{aligned}B\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) &= B\left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}*\Theta(\zeta)\right)(s)\\ &= \frac{1}{s^3}B\left({}_k\varepsilon^{\gamma}_{q,\vartheta,\sigma}(\zeta)\right)(s)\,B(\Theta(\zeta))(s)\\ &= \frac{1}{s^3}\frac{s^{\frac{2\vartheta}{k}+3}}{k^{\frac{\vartheta}{k}}}\left(1-\left(s^2\frac{1}{k}\right)^{\frac{q}{k}}k\sigma\right)^{-\frac{\gamma}{k}}\tilde{\Theta}(s).\end{aligned} \tag{55}$$

After simplification, we derive:

$$B\left({}_kP^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) = \left(\frac{s^2}{k}\right)^{\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}\tilde{\Theta}(s). \tag{56}$$

Lemma 3.3. The KTT of the k-PFD ${}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)$ of order ϑ is computed as:

$$\begin{aligned}B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) &= \left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s)\\ &\quad-\sum_{l=0}^{\lambda-1}k^{l+1}s^{-2l+3}\left[{}_kD^{\gamma}_{q,\vartheta-(l+1)k,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}\end{aligned} \tag{57}$$

with $\left|\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right|<1$, where $\tilde{\Theta}(s,u) = B(\Theta(\zeta))(s)$.

For the case $\left[\frac{\vartheta}{k}\right]+1=\lambda=1$, we obtain:

$$B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) = \tilde{\Theta}(s)\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma}{k}} - ks^3\left[{}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}. \tag{58}$$

Proof. Exerting the KTT on the k-PFD ${}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}$ of $\Theta(\zeta)$ of order ϑ, given by **Equation 35**, and making use of **Equation 3** of the KTT for derivatives, we obtain:

$$\begin{aligned}
&B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)\\
&= B\left(\frac{d^{\lambda}}{d\zeta^{\lambda}}k^{\lambda}{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s).\\
&= \frac{k^{\lambda}}{s^{2\lambda}}B\left({}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)\\
&\quad -\sum_{l=0}^{\lambda-1}k^{\lambda}s^{-2\lambda+2l+5}\left[\frac{d^l}{d\zeta^l}{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}.
\end{aligned}\tag{59}$$

Taking advantage of **Equations 33** and **34** and the convolution theorem of the KTT, **Equation 59** becomes:

$$\begin{aligned}
B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) &= \frac{k^{\lambda}}{s^{2\lambda}}B\left(\left({}_k\varepsilon^{-\gamma}_{q,\lambda k-\vartheta,\sigma}*\Theta\right)(\zeta)\right)(s)\\
&\quad - k^{\lambda}\sum_{l=0}^{\lambda-1}s^{-2l+3}\left[\left(\frac{d}{d\zeta}\right)^{\lambda-l-1}{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}\\
&= \frac{k^{\lambda}}{s^{2\lambda}}\frac{1}{s^3}B\left({}_k\varepsilon^{-\gamma}_{q,\lambda k-\vartheta,\sigma}(\zeta)\right)(s)\,B(\Theta(\zeta))(s)\\
&\quad -\sum_{l=0}^{\lambda-1}k^{l+1}s^{-2l+3}\left[k^{\lambda-l-1}\left(\frac{d}{d\zeta}\right)^{\lambda-l-1}{}_kP^{-\gamma}_{q,(\lambda-l-1)k-\vartheta+k(l+1),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}\\
&= \frac{k^{\lambda}}{s^{2\lambda+3}}B\left(\frac{\zeta^{\frac{\lambda k-\vartheta}{k}-1}}{k}E^{-\gamma}_{k,q,\lambda k-\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\right)(s)\ \tilde{\Theta}(s)\\
&\quad -\sum_{l=0}^{\lambda-1}k^{l+1}s^{-2l+3}\left[k^{\lambda-l-1}\left(\frac{d}{d\zeta}\right)^{\lambda-l-1}{}_kP^{-\gamma}_{q,(\lambda-l-1)k-(\vartheta-(l+1)k),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}.
\end{aligned}\tag{60}$$

By formula of the k-PFD and **Lemma 3.1**, **Equation 60** transforms into:

$$\begin{aligned}
B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) &= \frac{k^{\lambda}}{s^{2\lambda+3}}\frac{s^{\frac{2(\lambda k-\vartheta)}{k}+3}}{k^{\frac{\lambda k-\vartheta}{k}}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s)\\
&\quad -\sum_{l=0}^{\lambda-1}k^{l+1}s^{-2l+3}\left[\Theta(\zeta)\,{}_kD^{\gamma}_{q,\vartheta-(l+1)k,\sigma,0^+}\right]_{\zeta=0^+}.
\end{aligned}\tag{61}$$

After a little simplification, we derive:

$$\begin{aligned}
&B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)\\
&= \left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s)\\
&\quad -\sum_{l=0}^{\lambda-1}k^{l+1}s^{-2l+3}\left[{}_kD^{\gamma}_{q,\vartheta-(l+1)k,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}.
\end{aligned}\tag{62}$$

For $\left[\frac{\vartheta}{k}\right]+1=\lambda=1$, we obtain:

$$B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)=\tilde{\Theta}(s)\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}-ks^3\left[{}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}. \tag{63}$$

Lemma 3.4. The KTT of the regularized variant of the k-PFD ${}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)$ of order ϑ is obtained as:

$$B\left({}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)=\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\tilde{\Theta}(s)-\sum_{l=0}^{\lambda-1}s^{2l+5}\Theta^{(l)}\left(0^+\right)\right), \tag{64}$$

with $\left|\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right|<1$, where $\tilde{\Theta}(s)=B(\Theta(\zeta))(s)$.

Proof. The KTT of the regularized variant of the k-PFD ${}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)$ of ϑth order is derived as:

$$B\left({}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)=B\left(k^{\lambda}{}_kP^{-\gamma}_{q,\lambda k-\vartheta,\sigma,0^+}\left(\frac{d}{d\zeta}\right)^{\lambda}\Theta(\zeta)\right)(s). \tag{65}$$

On account of **Equation 33**, the Faltung theorem of the KTT, and **Lemma 3.1**, **Equation 65** converts into:

$$\begin{aligned}
B\left({}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)&=k^{\lambda}B\left({}_k\varepsilon^{-\gamma}_{q,\lambda k-\vartheta,\sigma}*\left(\frac{d}{d\zeta}\right)^{\lambda}\Theta(\zeta)\right)(s)\\
&=k^{\lambda}\frac{1}{s^3}B\left({}_k\varepsilon^{-\gamma}_{q,\lambda k-\vartheta,\sigma}\right)(s)\;B\left(\left(\frac{d}{d\zeta}\right)^{\lambda}\Theta(\zeta)\right)(s)\\
&=\frac{1}{s^3}k^{\lambda}\frac{s^{\frac{2(\lambda k-\vartheta)}{k}+3}}{k^{\frac{\lambda k-\vartheta}{k}}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma}{k}}\\
&\quad\times\left(\frac{1}{s^{2\lambda}}B(\Theta(\zeta))(s)-\sum_{l=0}^{\lambda-1}s^{-2\lambda+2l+5}\Theta^{(l)}\left(0^+\right)\right)\\
&=k^{\frac{\vartheta}{k}}s^{-\frac{2\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\tilde{\Theta}(s)-\sum_{l=0}^{\lambda-1}s^{2l+5}\Theta^{(l)}\left(0^+\right)\right)\\
B\left({}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)&=\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\tilde{\Theta}(s)-\sum_{l=0}^{\lambda-1}s^{2l+5}\Theta^{(l)}\left(0^+\right)\right).
\end{aligned} \tag{66}$$

Lemma 3.5. For $\Theta\in AC^1[0,b]$, the relation between the k-PFD and its regularized variant is given as:

$${}^{C}_{k}D^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)={}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)-\zeta^{-\frac{\vartheta}{k}}E^{-\gamma}_{k,q,k-\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\Theta\left(0^+\right). \tag{67}$$

Proof. For an absolutely continuous function $\Theta\in AC^1[0,b]$, we have

$$\left[\Theta(\zeta)\;{}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\right]_{\zeta=0^+}=0. \tag{68}$$

For $\left[\frac{\vartheta}{k}\right]+1=\lambda=1$, **Equation 58** of **Lemma 3.3** yields:

$$B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)=\tilde{\Theta}(s)\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}-ks^3\left[{}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}. \tag{69}$$

In accordance with **Equation 68**, **Equation 69** becomes:

$$B\left({}_kD^{\gamma}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s)=\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s). \tag{70}$$

For $\left[\frac{\vartheta}{k}\right]+1=\lambda=1$, **Equation 64** becomes:

$$B\left({}_{k}^{C}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)\right)(s)=\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma}{k}}\left(\tilde{\Theta}(s)-s^{5}\Theta\left(0^{+}\right)\right). \tag{71}$$

From **Equations 70** and **71**, we obtain:

$$B\left({}_{k}^{C}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)\right)(s)=B\left({}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)\right)(s)-\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}s^{5}\Theta\left(0^{+}\right). \tag{72}$$

Operating the inverse KTT B^{-1} on **Equation 72**, and in view of **Result 2.1** subject to $1>\left|\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\sigma k\right|$, we attain:

$$\begin{aligned}{}_{k}^{C}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-B^{-1}\left[\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}s^{5}\Theta\left(0^{+}\right)\right]\\&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-B^{-1}\left[\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}s^{5}\sum_{l=0}^{\infty}\left(\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{l}\frac{(-\gamma)_{l,k}}{l!}\Theta\left(0^{+}\right)\right]\\&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-\sum_{l=0}^{\infty}\frac{(-\gamma)_{l,k}}{l!}k^{\frac{\vartheta}{k}}\frac{\sigma^{l}}{k^{\frac{ql}{k}}}\frac{\zeta^{\frac{ql-\vartheta}{k}}}{\Gamma\left(\frac{ql-\vartheta}{k}+1\right)}\Theta\left(0^{+}\right)\\&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-\zeta^{-\frac{\vartheta}{k}}\sum_{l=0}^{\infty}\frac{1}{l!}(-\gamma)_{l,k}\left(\sigma\zeta^{\frac{q}{k}}\right)^{l}\frac{1}{k^{\frac{ql-\vartheta+k}{k}-1}\Gamma\left(\frac{ql-\vartheta+k}{k}\right)}\Theta\left(0^{+}\right).\end{aligned} \tag{73}$$

Making use of an identity equation provided by **Proposition 2.1**, **Equation 72** transforms into:

$$\begin{aligned}{}_{k}^{C}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-\sum_{l=0}^{\infty}\frac{(-\gamma)_{l,k}}{\Gamma_{k}((-\vartheta+k)+ql)}\frac{1}{\zeta^{\frac{\vartheta}{k}}}\frac{\left(\sigma\zeta^{\frac{q}{k}}\right)^{l}}{l!}\Theta\left(0^{+}\right)\\&={}_{k}D_{q,\vartheta,\sigma,0^{+}}^{\gamma}\Theta(\zeta)-\Theta\left(0^{+}\right)\zeta^{-\frac{\vartheta}{k}}E_{k,q,k-\vartheta}^{-\gamma}\left(\sigma\zeta^{\frac{q}{k}}\right).\end{aligned} \tag{74}$$

Lemma 3.6. The KTT of the k-HPFD $D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}\Theta(\zeta)$ of ϑth order, stated by **Equation 39**, is given as:

$$\begin{aligned}B\left({}_{k}D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}\Theta(\zeta)\right)(s)&=\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s)\\&-ks^{3}\left(\frac{s^{2}}{k}\right)^{\rho\left(1-\frac{\vartheta}{k}\right)}\left(1-\sigma k\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\left[{}_{k}P_{q,(1-\rho)(k-\vartheta),\sigma,0^{+}}^{-\gamma(1-\rho)}\Theta(\zeta)\right]_{\zeta=0^{+}}.\end{aligned} \tag{75}$$

Proof. Operating the KTT on the k-HPFD $D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}\Theta(\zeta)$ and utilizing **Equations 33** and **34** together with the Faltung theorem of the KTT, we obtain:

$$\begin{aligned}B\left({}_{k}D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}\Theta(\zeta)\right)(s)&=kB\left({}_{k}P_{q,\rho(k-\vartheta),\sigma,0^{+}}^{-\rho\gamma}\frac{d}{d\zeta}\left({}_{k}P_{q,(1-\rho)(k-\vartheta),\sigma,0^{+}}^{-\gamma(1-\rho)}\Theta\right)(\zeta)\right)(s)\\&=kB\left({}_{k}\varepsilon_{q,\rho(k-\vartheta),\sigma}^{-\gamma\rho}*\frac{d}{d\zeta}\left({}_{k}P_{q,(1-\rho)(k-\vartheta),\sigma,0^{+}}^{-\gamma(1-\rho)}\Theta\right)(\zeta)\right)(s).\end{aligned} \tag{76}$$

Making use of **Lemma 3.1** and **Equation 3** of the KTT for differential coefficients, **Equation 76** transforms into:

$$\begin{aligned} B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) &= \frac{s^{\frac{2\rho(k-\vartheta)}{k}-2}}{k^{\frac{\rho(k-\vartheta)}{k}-1}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma\rho}{k}} B\left({}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right)(s) \\ &\quad - \frac{s^{\frac{2\rho(k-\vartheta)}{k}+3}}{k^{\frac{\rho(k-\vartheta)}{k}-1}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma\rho}{k}}\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}. \end{aligned} \tag{77}$$

Applying **Equation 33**, the Faltung theorem of the KTT, and **Lemma 3.1**, **Equation 77** becomes:

$$\begin{aligned} B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) &= \frac{s^{2\rho\left(1-\frac{\vartheta}{k}\right)-2}}{k^{\rho\left(1-\frac{\vartheta}{k}\right)-1}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma\rho}{k}}\frac{1}{s^3}\frac{s^{\frac{2(k-\vartheta)(1-\rho)}{k}+3}}{k^{\frac{(k-\vartheta)(1-\rho)}{k}}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{(1-\rho)\gamma}{k}}\tilde{\Theta}(s) \\ &\quad - \frac{s^{2\rho\left(1-\frac{\vartheta}{k}\right)+3}}{k^{\rho\left(1-\frac{\vartheta}{k}\right)-1}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\rho\gamma}{k}}\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+} B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) \\ &= \left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s) \\ &\quad - ks^3\left(\frac{s^2}{k}\right)^{\rho\left(1-\frac{\vartheta}{k}\right)}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma\rho}{k}}\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}. \end{aligned} \tag{78}$$

Lemma 3.7. The KTT of the regularized variant of the k-HPFD ${}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)$ of order ϑ, provided by **Equation 42**, is constructed as:

$$B\left({}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) = \tilde{\Theta}(s)\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{\gamma}{k}} - ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\right)^{\frac{\gamma}{k}}\Theta\left(0^+\right). \tag{79}$$

Proof. Exerting the KTT on the Caputo k-HPFD ${}^CD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)$ and utilizing **Equation 33**, the Faltung theorem of the KTT, and **Lemma 3.1**, we obtain:

$$\begin{aligned} B\left({}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) &= kB\left({}_kP^{-\gamma}_{q,k-\vartheta,\sigma,0^+}\frac{d}{d\zeta}\Theta(\zeta)\right)(s) \\ &= kB\left({}_k\varepsilon^{-\gamma}_{q,k-\vartheta,\sigma} * \frac{d}{d\zeta}\Theta(\zeta)\right)(s) \\ &= \left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\tilde{\Theta}(s)-s^5\Theta\left(0^+\right)\right). \end{aligned} \tag{80}$$

Hence, the Lemma is proved.

Lemma 3.8. For $\Theta \in AC^1[0,b]$, the k-HPFD and its regular counterpart are related by:

$${}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) = {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - \zeta^{-\frac{\vartheta}{k}}E^{-\gamma}_{k,q,k-\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\Theta\left(0^+\right). \tag{81}$$

Proof. For an absolutely continuous function $\Theta \in AC^1[0,b]$, we have

$$\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+} = 0. \tag{82}$$

By means of **Lemma 3.6** and **Equation 82**, the KTT of $D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)$ is

$$B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) = \left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Theta}(s). \tag{83}$$

By means of **Equation 83**, the KTT of the Caputo variant of the k-HPFD ${}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)$ can be written as:

$$B\left({}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) = B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) - ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\Theta\left(0^+\right). \tag{84}$$

Operating the inverse KTT operator B^{-1} on **Equation 84** and using **Result 2.1** subject to $\left|k\sigma\left(\frac{1}{k}s^2\right)^{q\frac{1}{k}}\right| < 1$, we attain:

$$\begin{aligned}{}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) &= {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - B^{-1}\left[ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\Theta\left(0^+\right)\right]\\ &= {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - k^{\frac{\vartheta}{k}}\sum_{l=0}^{\infty}\frac{(-\gamma)_{l,k}}{l!}\frac{\sigma^l}{k^{\frac{ql}{k}}}B^{-1}\left[s^{2\left(\frac{ql-\vartheta}{k}\right)+5}\right]\Theta\left(0^+\right)\\ &= {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - \sum_{l=0}^{\infty}\frac{\sigma^l}{k^{\frac{ql}{k}}}\frac{(-\gamma)_{l,k}}{l!}k^{\frac{\vartheta}{k}}\frac{\zeta^{\frac{ql-\vartheta}{k}}}{\Gamma\left(\frac{ql-\vartheta}{k}+1\right)}\Theta\left(0^+\right).\end{aligned} \tag{85}$$

By means of the **Proposition 2.1**, **Equation 85** transforms into:

$$\begin{aligned}{}^C_kD^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\zeta) &= {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - \zeta^{-\frac{\vartheta}{k}}\sum_{l=0}^{\infty}\frac{(-\gamma)_{l,k}}{\Gamma_k(ql+(k-\vartheta))}\frac{\left(\zeta^{\frac{q}{k}}\sigma\right)^l}{l!}\Theta\left(0^+\right)\\ &= {}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) - \zeta^{-\frac{\vartheta}{k}}E^{-\gamma}_{k,q,k-\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\Theta\left(0^+\right).\end{aligned} \tag{86}$$

For $k=1$, these results reduce to the derivations established in the study by Dubey et al.[31] In other words, these obtained results in the context of the k-HPFD provide direct generalizations of those reported in Dubey et al.[31]

4. Solution of Cauchy equations presented with k-HPFD by means of KTT and FT

Theorem 4.1. The solution of the Cauchy problem in Garra et al.,[8] presented with k-HPFD:

$$\begin{cases}{}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta) = \mu_kP^{\omega}_{q,\vartheta,\sigma,0^+}\Theta(\zeta) + \Phi(\zeta),\\ \left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+} = W,\ W\geq 0,\end{cases} \tag{87}$$

where $\zeta\in(0,\infty)$, $k\in R^+$, $\Phi(\zeta)\in L^1[0,\infty)$, $\vartheta\in(0,1)$, $\rho\in[0,1]$, $\mu,\sigma\in C$, $\zeta,q>0$, and $\gamma,\omega\geq 0$ is given by:

$$\Theta(\zeta) = W\sum_{l=0}^{\infty}\mu^l\zeta^{\frac{\rho(k-\vartheta)+\vartheta(1+2l)}{k}-1}E^{\gamma(1-\rho)+(\omega+\gamma)l}_{k,q,\rho(k-\vartheta)+\vartheta(1+2l)}\left(\sigma\zeta^{\frac{q}{k}}\right) + \sum_{l=0}^{\infty}\mu^l{}_kP^{\gamma(l+1)+\omega l}_{q,\vartheta(1+2l),\sigma,0^+}\Phi(\zeta), \tag{88}$$

provided the series on the right-hand side of **Equation 88** is convergent.

Proof. Let $\tilde{\Theta}(s)$ and $\tilde{\Phi}(s)$ denote the KTT of $\Theta(\zeta)$ and $\Phi(\zeta)$, respectively. Now, employing the KTT on **Equation 87** and utilizing **Lemma 3.6** and **Equation 33**, we obtain:

$$B\left({}_kD^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Theta(\zeta)\right)(s) = \mu\, B\left({}_kP^{\omega}_{q,\vartheta,\sigma,0^+}\Theta(\zeta)\right)(s) + B\left(\Phi(\zeta)\right)(s)$$

$$\tilde{\Theta}(s)\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}} - ks^3\left(\frac{s^2}{k}\right)^{\frac{(k-\vartheta)\rho}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}$$

$$= \mu\, B\left(\left({}_k\varepsilon^{\omega}_{q,\vartheta,\sigma} * \Theta\right)(\zeta)\right)(s) + \tilde{\Phi}(s). \tag{89}$$

Now, with the help of **Lemma 3.1** and the convolution theorem of the KTT, **Equation 89** transforms into:

$$\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\tilde{\Theta}(s) - ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{\frac{\gamma\rho}{k}}\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Theta(\zeta)\right]_{\zeta=0^+}$$

$$= \frac{\mu}{s^3}B\left({}_k\varepsilon^{\omega}_{q,\vartheta,\sigma}(\zeta)\right)(s)\, B\left(\Theta(\zeta)\right)(s) + \tilde{\Phi}(s). \tag{90}$$

Adjustment of terms in Equation 90 yields:

$$\begin{aligned}\tilde{\Theta}(s) &= Wks^3\sum_{l=0}^{\infty}\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)+(1+2l)\vartheta}{k}}\mu^l\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{[\gamma(1-\rho)+(\gamma+\omega)l]}{k}}\\ &+\sum_{l=0}^{\infty}\mu^l\left(\frac{s^2}{k}\right)^{\frac{\vartheta(1+2l)}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{[\gamma(1+l)+\omega\, l]}{k}}\tilde{\Phi}(s).\end{aligned} \tag{91}$$

Exerting the inverse KTT B^{-1} on **Equation 91** and implementing **Lemma 3.1**, we derive:

$$\begin{aligned}\Theta(\zeta) &= kW\sum_{l=0}^{\infty}\mu^l B^{-1}\left(\frac{s^{\frac{2[\rho(k-\vartheta)+\vartheta(1+2l)]}{k}+3}}{k^{\frac{\rho(k-\vartheta)+\vartheta(1+2l)}{k}}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{-\frac{[\gamma(1-\rho)+(\gamma+\omega)l]}{k}}\right)\\ &+\sum_{l=0}^{\infty}\mu^l B^{-1}\left(\frac{1}{s^3}\frac{s^{\frac{2\vartheta(1+2l)}{k}+3}}{k^{\frac{\vartheta(1+2l)}{k}}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{[\gamma(1+l)+\omega\, l]}{k}}\tilde{\Phi}(s)\right)\\ &= kW\sum_{l=0}^{\infty}\mu^l{}_k\varepsilon^{\gamma(1-\rho)+l(\gamma+\omega)}_{q,\rho(k-\vartheta)+\vartheta(1+2l),\sigma}(\zeta) + \sum_{l=0}^{\infty}\mu^l B^{-1}\left(\frac{1}{s^3}B\left({}_k\varepsilon^{\gamma(1+l)+\omega l}_{q,\vartheta(1+2l),\sigma}(\zeta)\right)(s)\,\tilde{\Phi}(s)\right).\end{aligned} \tag{92}$$

By means of the convolution formula of KTT in **Equation 92**, we obtain:

$$\Theta(\zeta) = kW\sum_{l=0}^{\infty}\mu^l{}_k\varepsilon^{\gamma(1-\rho)+(\gamma+\omega)l}_{q,\rho(k-\vartheta)+\vartheta(1+2l),\sigma}(\zeta) + \sum_{l=0}^{\infty}\mu^l\left({}_k\varepsilon^{\gamma(1+l)+\omega l}_{q,\vartheta(1+2l),\sigma}(\zeta) * \Phi(\zeta)\right). \tag{93}$$

On account of **Equations 33** and 34, **Equation 93** takes the form:

$$\Theta(\zeta) = W\sum_{l=0}^{\infty}\mu^l\zeta^{\frac{\rho(k-\vartheta)+(1+2l)\vartheta}{k}-1}E^{\gamma(1-\rho)+l(\gamma+\omega)}_{k,q,\rho(k-\vartheta)+\vartheta(1+2l)}\left(\sigma\zeta^{\frac{\beta}{k}}\right) + \sum_{l=0}^{\infty}\mu^l{}_kP^{\gamma(1+l)+\omega l}_{q,\vartheta(1+2l),\sigma,0^+}\Phi(\zeta). \tag{94}$$

Remark 4.1. For $\mu = -ir\theta$, $\Phi(\zeta) = 0$, $\gamma = \rho = 0$, $r = \omega = 1$, and $\sigma = i\phi$, $r, \phi \in R, \zeta \in (0, 1]$, the above-mentioned Cauchy problem reduces to the free electron laser equation[44] in the following way:

$$\begin{cases}\frac{d\Theta}{d\zeta} = -ir\theta\int_0^{\zeta}(\zeta-\nu)\,e^{i\phi(\zeta-\nu)}\Theta(\nu)\,d\nu,\\ \Theta(0) = 1.\end{cases} \tag{95}$$

Theorem 4.2. The solution of the generalized Cauchy problem (GCP)[28] carrying the regularized form of the k-HPFD:

$$\begin{cases} {}_{k}^{C}D^{\gamma,\vartheta}_{q,-\sigma,0^+}\Upsilon(\Psi,\zeta) = -\mu(1-\Psi)\Upsilon(\Psi,\zeta), \ |\Psi| \le 1 \\ \Upsilon(\Psi,\, 0^+) = 1, \end{cases} \tag{96}$$

with $k \in R^+$, $\zeta > 0$, $\mu > 0$,$\gamma \ge 0$, $q \in (0,\, 1]$, $\vartheta \in (0,\, 1]$,$\mu, \sigma \in R, \Psi, q > 0$, and $\gamma, \omega \ge 0$ is given by:

$$\Upsilon(\Psi,\zeta) = \sum_{l=0}^{\infty} (1-\Psi)^l (-\mu)^l \zeta^{\frac{l\vartheta}{k}} E^{l\gamma}_{k,q,\vartheta l+k}\left(-\sigma\zeta^{\frac{q}{k}}\right), \tag{97}$$

if the series on the right-hand side of **Equation 97** is convergent.

Proof. Let $\tilde{\Upsilon}(\Psi, s)$ be the KTT of $\Upsilon(\Psi,\zeta)$. Applying the KTT on **Equation 96** with respect to ζ and making use of the IC $\Upsilon(\Psi,\, 0^+) = 1$ and **Lemma 3.7**, we obtain:

$$\begin{aligned} &B\left({}_{k}^{C}D^{\gamma,\vartheta}_{q,-\sigma,0^+}\Upsilon(\Psi,\zeta)\right)(\Psi,s) = -\mu(1-\Psi)B(\Upsilon(\Psi,\zeta))(\Psi,s) \\ &\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1+\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\Upsilon}(\Psi,s) - ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1+\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\Upsilon(\Psi,\,0^+) \\ &= -\mu(1-\Psi)\tilde{\Upsilon}(\Psi,s). \end{aligned} \tag{98}$$

Now, after adjusting the terms in **Equation 98** and for $\left|\mu(1-\Psi)\left(\frac{s^2}{k}\right)^{\frac{\vartheta}{k}}\left(1+\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}\right| < 1$, we derive:

$$\tilde{\Upsilon}(\Psi,s) = s^5\sum_{l=0}^{\infty}(-\mu)^l(1-\Psi)^l\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1+\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{-\frac{l\gamma}{k}}. \tag{99}$$

Employing B^{-1} on **Equation 99**, and in view of **Equation 34** and **Lemma 3.1**, we obtain:

$$\begin{aligned} \Upsilon(\Psi,\,\zeta) &= \sum_{l=0}^{\infty}(1-\Psi)^l(-\mu)^l B^{-1}\left(s^5\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1+\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{-\frac{l\gamma}{k}}\right) \\ &= k\sum_{l=0}^{\infty}(1-\Psi)^l(-\mu)^l B^{-1}\left(\frac{s^{\frac{2(lv+k)}{k}+3}}{k^{\frac{lv+k}{k}}}\left(1+k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\right) \\ &= \sum_{l=0}^{\infty}(-\mu)^l(1-\Psi)^l\, {}_{k}\varepsilon^{l\gamma}_{q,l\vartheta+k,-\sigma}(\zeta) \\ &= \sum_{l=0}^{\infty}(-\mu)^l(1-\Psi)^l\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,l\vartheta+k}\left(-\sigma\zeta^{\frac{q}{k}}\right). \end{aligned} \tag{100}$$

Theorem 4.3. The solution of GCP[8] carrying the k-HPFD for a fractional order heat model:

$$\begin{cases} {}_{k}D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}\Upsilon(\Psi,\,\zeta) = \mathrm{M}\frac{\partial^2}{\partial\wp^2}\Upsilon(\Psi,\,\zeta), \\ \left[{}_{k}P^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}\Upsilon(\Psi,\,\zeta)\right]_{\zeta=0^+} = \psi(\Psi), \\ \lim\limits_{\Psi\to\pm\infty}\Upsilon(\Psi,\,\zeta) = 0, \end{cases} \tag{101}$$

with $k \in R^+$, $\vartheta \in (0,\, 1)$, $\rho \in [0,\, 1]$,$\Psi, \sigma \in R, \zeta, M, q > 0$, and $\gamma \ge 0$, is given by

$$\Upsilon(\Psi,\,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\Psi}\hat{\psi}(p)\,dp\sum_{l=0}^{\infty}(-M)^l p^{2l}\zeta^{\frac{\rho(k-\vartheta)+(l+1)\vartheta}{k}-1}E^{\gamma[(1-\rho)+l]}_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}\left(\sigma\zeta^{\frac{q}{k}}\right), \tag{102}$$

if the right-hand side of **Equation 102** is convergent.

Proof. Suppose $\tilde{\Upsilon}(\Psi, s)$ and $\hat{\Upsilon}(p, \zeta)$ represent the respective KTT and FT of $\Upsilon(\Psi, \zeta)$. Likewise, let $\hat{\tilde{\Upsilon}}(p, s)$ and $\hat{\psi}(p)$ be the Fourier–KTT (FKTT) and FT of $\Upsilon(\Psi, \zeta)$ and $\psi(\Psi)$, respectively.

Operating the FKTT on **Equation 101** and employing the ICs of **Equation 101** and **Lemma 3.6**, we obtain:

$$\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{\gamma}{k}}\hat{\tilde{\Upsilon}}(p, s)-ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\hat{\psi}(p)=-Mp^2\hat{\tilde{\Upsilon}}(p, s) \tag{103}$$

On simplification, we derive:

$$\hat{\tilde{\Upsilon}}(p, s)=\frac{ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\hat{\psi}(p)}{\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}+Mp^2}. \tag{104}$$

Hence, for $\left|Mp^2\left(\frac{s^2}{k}\right)^{\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{-\frac{\gamma}{k}}\right|<1$, we obtain:

$$\hat{\tilde{\Upsilon}}(p, s)=\sum_{l=0}^{\infty}\left(-Mp^2\right)^l\hat{\psi}(p)\,ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{-\frac{\gamma[(1-\rho)+l]}{k}}. \tag{105}$$

Applying the IFT to **Equation 105**, we attain:

$$\tilde{\Upsilon}(\Psi, s)=\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\,ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{-\frac{\gamma[(1-\rho)+l]}{k}}. \tag{106}$$

Imposing B^{-1} on **Equation 106** and utilizing **Lemma 3.1** and **Equation 34**, we derive:

$$\begin{aligned}\Upsilon(\Psi, \zeta)&=k\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\,B^{-1}\left[\frac{s^{\frac{2[\rho(k-\vartheta)+\vartheta(l+1)]}{k}+3}}{k^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}k\sigma\right)^{-\frac{\gamma[(1-\rho)+l]}{k}}\right]\\&=k\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\,{}_k\varepsilon^{\gamma[(1-\rho)+l]}_{q,\rho(k-\vartheta)+\vartheta(l+1),\sigma}(\zeta)\\&=k\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\frac{\zeta^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}-1}}{k}E^{\gamma[(1-\rho)+l]}_{k,q,\rho(k-\vartheta)+(l+1)\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)\end{aligned} \tag{107}$$

Finally, we obtain:

$$\Upsilon(\Psi, \zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip\Psi}\hat{\psi}(p)\,dp\sum_{l=0}^{\infty}(-M)^l p^{2l}\zeta^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}-1}E^{\gamma[(1-\rho)+l]}_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}\left(\sigma\zeta^{\frac{q}{k}}\right). \tag{108}$$

Theorem 4.4. The solution of the GCP[8] carrying the regularized variant of the k-HPFD:

$$\begin{cases}{}^{C}_{k}D^{\gamma,\vartheta}_{q,\sigma,0^+}\Upsilon(\Psi, \zeta)=M\frac{\partial^2}{\partial\wp^2}\Upsilon(\Psi, \zeta), & \zeta>0,\ \Psi\in R,\\ \left[\Upsilon(\Psi, \zeta)\right]_{\zeta=0^+}=\psi(\Psi), \\ \lim\limits_{\Psi\to\pm\infty}\Upsilon(\Psi, \zeta)=0,\end{cases} \tag{109}$$

with $k\in R^+$, $\vartheta\in(0, 1)$, $\rho\in[0, 1]$,$\Psi, \sigma\in R, \zeta, M, q>0$, and $\gamma\geq 0$, is given by:

$$\Upsilon(\Psi, \zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip\Psi}\hat{\psi}(p)\,dp\sum_{l=0}^{\infty}(-M)^l p^{2l}\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,l\vartheta+k}\left(\sigma\zeta^{\frac{q}{k}}\right), \tag{110}$$

if the right-hand side of **Equation 110** is convergent.

Proof. Let $\tilde{\Upsilon}(\Psi, s)$ and $\hat{\Upsilon}(p, \zeta)$ be the respective KTT and FT of $\Upsilon(\Psi, \zeta)$. In a similar way, let $\hat{\tilde{\Upsilon}}(p, s)$ and $\hat{\psi}(p)$ be the respective FKTT and FT of $\Upsilon(\Psi, \zeta)$ and $\psi(\Psi)$.

Applying the FKTT on **Equation 109** and making use of **Lemma 3.7** and the ICs provided by **Equation 109**, we obtain:

$$\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{\gamma}{k}}\hat{\tilde{\Upsilon}}(p, s)-ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\sigma k\right)^{\frac{\gamma}{k}}\hat{\psi}(p)=-Mp^2\hat{\tilde{\Upsilon}}(p, s). \tag{111}$$

Changing the terms reduces **Equation 111** to:

$$\begin{aligned}\hat{\tilde{\Upsilon}}(p, s)&=s^5\hat{\psi}(p)\left[1+Mp^2\left(\frac{s^2}{k}\right)^{\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}\right]^{-1}\\&=s^5\hat{\psi}(p)\sum_{l=0}^{\infty}\left(-Mp^2\right)^l\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}.\end{aligned} \tag{112}$$

Taking the IFT of **Equation 112**, we derive:

$$\tilde{\Upsilon}(\Psi, s)=s^5\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}. \tag{113}$$

Applying the B^{-1}on **Equation 113** and taking advantage of **Equation 34** and **Lemma 3.1**, we obtain:

$$\begin{aligned}\Upsilon(\Psi, \zeta)&=\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dpB^{-1}\left[s^5\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\right]\\&=k\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dpB^{-1}\left[\frac{s^{\frac{2(\vartheta l+k)}{k}+3}}{k^{\frac{l\vartheta+k}{k}}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\right]\\&=k\sum_{l=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Mp^2\right)^l e^{ip\Psi}\hat{\psi}(p)\,dp\,{}_k\varepsilon^{l\gamma}_{q,l\vartheta+k,\sigma}(\zeta)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip\Psi}\hat{\psi}(p)\,dp\sum_{l=0}^{\infty}(-M)^l p^{2l}\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,l\vartheta+k}\left(\sigma\zeta^{\frac{q}{k}}\right).\end{aligned} \tag{114}$$

Theorem 4.5. The solution of Cauchy problem in Panchal et al.[33] with the k-HPFD:

$$-\xi\Im Q(\zeta)=\eta\Lambda h_{q\,k}D^{\gamma,\vartheta,\rho}_{q,\sigma,0^+}Q(\zeta), \tag{115}$$

$$\left[{}_kP^{-\gamma(1-\rho)}_{q,(1-\rho)(k-\vartheta),\sigma,0^+}Q(\zeta)\right]_{\zeta=0^+}=\alpha,\ for\ \alpha\geq 0, \tag{116}$$

where η represents density, Λ denotes volume, h_q indicates the specific heat of the material, ξ is the convection heat transfer coefficient, $\Im$ is the surface area of the body, and $Q\in L^1[0,\infty)$, $0<\zeta<\infty$; $k\in R^+$, $\alpha>0$, $\gamma,\sigma\in R$; $\vartheta\in(0,1)$, $\rho\in[0,1]$, is given by:

$$Q(\zeta)=\alpha\sum_{l=0}^{\infty}\left(\frac{-\xi\Im}{\eta\Lambda h_q}\right)^l\zeta^{\frac{\vartheta(l+1)+\rho(k-\vartheta)}{k}-1}E^{\gamma[l+(1-\rho)]}_{k,q,\vartheta(l+1)+\rho(k-\vartheta)}\left(\sigma\zeta^{\frac{q}{k}}\right), \tag{117}$$

if the right-hand side of **Equation 117** is convergent.

Proof. Assume that $\tilde{Q}(s)$ is the KTT of $Q(\zeta)$. Now, applying the KTT to **Equation 115** and further using **Lemma 3.6** and the ICs of **Equation 116** yields:

$$-\xi\Im B\left(Q\left(\zeta\right)\right)\left(s\right)=\eta\Lambda h_{q}B\left({}_{k}D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}Q\left(\zeta\right)\right)\left(s\right)$$

$$\begin{aligned}\tilde{Q}\left(s\right)&=\frac{\eta\Lambda h_{q}\alpha ks^{3}\left(\frac{s^{2}}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}}{\xi\Im+\eta\Lambda h_{q}\left(\frac{s^{2}}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}}\\&=\alpha ks^{3}\left(\frac{s^{2}}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma(1-\rho)}{k}}\\&\quad\times\sum_{l=0}^{\infty}\left(-\xi\Im\right)^{l}\left(\eta\Lambda h_{q}\right)^{-l}\left(\frac{s^{2}}{k}\right)^{\frac{l\vartheta}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\\&=\alpha k\sum_{l=0}^{\infty}\left(-\xi\Im\right)^{l}\left(\eta\Lambda h_{q}\right)^{-l}s^{3}\left(\frac{s^{2}}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}\left(1-k\sigma\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma[(1-\rho)+l]}{k}}.\end{aligned}\tag{118}$$

Applying the inverse KTT B^{-1} to **Equation 118** and taking advantage of **Equation 34** and **Lemma 3.1**, we derive:

$$\begin{aligned}Q\left(\zeta\right)&=k\alpha\sum_{l=0}^{\infty}\left(-\xi\Im\right)^{l}\left(\eta\Lambda h_{q}\right)^{-l}B^{-1}\left(s^{3}\left(\frac{s^{2}}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}\left(1-\sigma k\left(\frac{s^{2}}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma[(1-\rho)+l]}{k}}\right)\\&=\alpha\sum_{l=0}^{\infty}\left(\frac{-\xi\Im}{\eta\Lambda h_{q}}\right)^{l}\zeta^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}-1}E_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}^{\gamma[(1-\rho)+l]}\left(\sigma\zeta^{\frac{q}{k}}\right).\end{aligned}\tag{119}$$

These obtained solutions for above-described Cauchy problems with the k-HPFD provide direct generalizations for results obtained by Dubey et al.[31] For $k=1$, these solutions for Cauchy problems with the k-HPFD reduce to the solutions obtained for Cauchy problems with the HPFD in the study by Dubey et al.[31] For $\gamma=0$, the above-reported solutions for the GCP with the k-HPFD reduce to solutions for the GCP with Hilfer derivative, and for $\rho=0$ and $\rho=1$, these results reduce to solutions for the GCP with the RL derivative and the Caputo derivative, respectively.

5. Analytic solution of Cauchy-type problems for the fractional advection–dispersion equation with k-HPFD

This section introduces the integral technique consisting of the KTT and FT to develop the solution for the fractional advection–dispersion equation with k-HPFD.

Theorem 5.1. The solution of the GCP for advection–dispersion equations[35] with the k-HPFD:

$${}_{k}D_{q,\sigma,0^{+}}^{\gamma,\vartheta,\rho}\Theta\left(\chi,\zeta\right)=-\Lambda D_{\chi}\Theta\left(\chi,\zeta\right)+\aleph\Delta^{\frac{\tau}{2}}\Theta\left(\chi,\zeta\right)\tag{120}$$

subject to the ICs:

$$\begin{cases}\left({}_{k}P_{q,(1-\rho)(k-\vartheta),\sigma,0^{+}}^{-\gamma(1-\rho)}\Theta\left(\chi,\zeta\right)\right)_{\zeta=0^{+}}=\theta\left(\chi\right),\ \sigma,\gamma,\chi\in R,\ q>0\\\lim\limits_{\chi\to\infty}\Theta\left(\chi,\zeta\right)=0,\ \ \zeta\geq0,\end{cases}\tag{121}$$

where $D_{\chi}\Theta\left(\chi,\zeta\right)$ is the Weyl fractional differential operator (WFDO)[45,46] of order 1 of a function $\Theta\left(\chi,\zeta\right)$, $\Delta^{\frac{\tau}{2}}$ is the fractional generalized Laplace operator of order τ, Λ is the average linear velocity, Θ is the solute concentration, $\aleph$ is the constant dispersion coefficient $\tau\in(0,\,2)$, $\vartheta\in(0,\,1)$, $\gamma>0,\rho\in[0,\,1]$, $\chi\in R$, $\zeta\in R^{+}$, $k\in R^{+}$, and the FT of $\Delta^{\frac{\tau}{2}}$ is $-\left|\omega\right|^{\tau}$, as discussed in the study by Agarwal et al.,[47] is given by:

$$\Theta(\chi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\chi}\theta(\chi)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph|\omega|^{\tau}\right)^l\zeta^{\frac{(l+1)\vartheta+\rho(k-\vartheta)}{k}-1}E_{k,q,(l+1)\vartheta+\rho(k-\vartheta)}^{\gamma[(l+1)-\rho]}\left(\sigma\zeta^{\frac{q}{k}}\right)d\omega. \tag{122}$$

Proof. The modified FT of the WFDO of order α, given by Miller[45] and Saxena et al.,[46], is computed as:

$$F\left\{{}_{-\infty}D_{\chi}^{\alpha}\Theta(\chi)\right\}(\omega)=(i\omega)^{\alpha}\,\hat{\Theta}(\omega), \tag{123}$$

where $\hat{\Theta}(\omega)$ is the FT of $\Theta(\chi)$. Now, applying the FT on **Equation 120** and using **Equation 123** for the modified FT of the WFDO, we obtain:

$${}_kD_{q,\sigma,0^+}^{\gamma,\vartheta,\rho}\hat{\Theta}(\omega,\zeta)=-i\Lambda\omega\,\hat{\Theta}(\omega,\zeta)-\aleph|\omega|^{\tau}\,\hat{\Theta}(\omega,\zeta), \tag{124}$$

where $\hat{\Theta}(\omega,\zeta)$ is the FT of $\Theta(\chi,\zeta)$ with respect to variable χ.

Let $\tilde{\hat{\Theta}}(\omega,s)$ be the KTT of $\hat{\Theta}(\omega,\zeta)$ and $\hat{\theta}(\omega)$ be the FT of $\theta(\chi)$. Now, operating the KTT on **Equation 124** and employing **Lemma 3.6**, we derive:

$$\begin{aligned}
&\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\tilde{\hat{\Theta}}(\omega,s)\\
&\quad-ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\left({}_kP_{q,(1-\rho)(k-\vartheta),\sigma,0^+}^{-\gamma(1-\rho)}\hat{\Theta}(\omega,\zeta)\right)_{\zeta=0^+}\\
&=-i\Lambda\omega\,\tilde{\hat{\Theta}}(\omega,s)-\aleph|\omega|^{\tau}\,\tilde{\hat{\Theta}}(\omega,s)\\
&\quad\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\hat{\Theta}}(\omega,s)-ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\hat{\theta}(\omega)\\
&=-i\Lambda\omega\,\tilde{\hat{\Theta}}(\omega,s)-\aleph|\omega|^{\tau}\,\tilde{\hat{\Theta}}(\omega,s)\\
&\quad\tilde{\hat{\Theta}}(\omega,s)=\frac{ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma\rho}{k}}\hat{\theta}(\omega)}{\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}+i\Lambda\omega+\aleph|\omega|^{\tau}}\\
&=ks^3\left(\frac{s^2}{k}\right)^{\frac{\rho(k-\vartheta)+\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma(1-\rho)}{k}}\hat{\theta}(\omega)\\
&\quad\times\sum_{l=0}^{\infty}(-1)^l\left(i\omega\Lambda+\aleph|\omega|^{\tau}\right)^l\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\\
&=k\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph|\omega|^{\tau}\right)^l\frac{s^{\frac{2[\rho(k-\vartheta)+\vartheta(l+1)]}{k}+3}}{k^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma[(l+1)-\rho]}{k}}.
\end{aligned} \tag{125}$$

Inverting the KTT operator in **Equation 125** and using **Equation 34** and **Lemma 3.1**, we obtain:

$$\begin{aligned}\hat{\Theta}(\omega,\,\zeta) &= k\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(\aleph\,|\omega|^{\tau}+i\Lambda\omega\right)^l\,{}_k\varepsilon^{\gamma[(l+1)-\rho]}_{q,\rho(k-\vartheta)+\vartheta(l+1),\sigma}(\zeta)\\ &= k\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\,|\omega|^{\tau}\right)^l\frac{\zeta^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}-1}}{k}E^{\gamma[(l+1)-\rho]}_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}\left(\sigma\zeta^{\frac{q}{k}}\right)\\ &= \hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\,|\omega|^{\tau}\right)^l\zeta^{\frac{\rho(k-\vartheta)+\vartheta(l+1)}{k}-1}E^{\gamma[(l+1)-\rho]}_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}\left(\sigma\zeta^{\frac{q}{k}}\right).\end{aligned}\tag{126}$$

Now, inverting the FT operator in **Equation 126**, we derive:

$$\Theta(\chi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\chi}\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\,|\omega|^{\tau}\right)^l\zeta^{\frac{\rho(k-\vartheta)+(l+1)\vartheta}{k}-1}E^{\gamma[(l+1)-\rho]}_{k,q,\rho(k-\vartheta)+\vartheta(l+1)}\left(\sigma\zeta^{\frac{q}{k}}\right)d\omega.\tag{127}$$

Remark: For $\Lambda=0$ and $\aleph=\frac{ih}{2m}$, **Equation 120** reduces to the one-dimensional space–time fractional Schrödinger equation for mass m and Planck's constant h, and its solution is given by:

$$\Theta(\chi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\chi}\hat{\theta}(\omega)\sum_{l=0}^{\infty}\left(-\frac{ih}{2m}\,|\omega|^{\tau}\right)^r\zeta^{\frac{\rho(k-\vartheta)+(l+1)\vartheta}{k}-1}E^{\gamma[(l+1)-\rho]}_{k,q,\rho(k-\vartheta)+(l+1)\vartheta}\left(\sigma\zeta^{\frac{q}{k}}\right)d\omega.\tag{128}$$

Theorem 5.2. The solution of the GCP for advection–dispersion equations[35] with the k-HPFD operator:

$${}^{C}_{k}D^{\gamma,\vartheta}_{q,\sigma,0^+}\Theta(\chi,\zeta)=-\Lambda D_{\chi}\Theta(\chi,\zeta)+\aleph\Delta^{\frac{\tau}{2}}\Theta(\chi,\zeta)\tag{129}$$

subject to the ICs:

$$\begin{cases}\Theta(\chi,0^+)=\theta(\chi),\ \chi\in R,\\ \lim\limits_{\chi\to\infty}\Theta(\chi,\zeta)=0,\ \ \zeta\geq 0,\end{cases}\tag{130}$$

where $D_{\chi}\Theta(\chi,\zeta)$ is the WFDO of order 1 of a function $\Theta(\chi,\zeta)$, $\Delta^{\frac{\tau}{2}}$ is the fractional generalized Laplace operator of order τ, Λ is the average linear velocity, Θ is the solute concentration, $\aleph$ is the constant dispersion coefficient $\tau\in(0,\,2)$, $\vartheta\in(0,\,1)$, $\gamma>0,\rho\in[0,\,1]$, $\chi\in R,\zeta\in R^+$, $k\in R^+$, and the FT of $\Delta^{\frac{\tau}{2}}$ is $-|\omega|^{\tau}$, as discussed in the study by Agarwal et al.,[47] is given by:

$$\Theta(\chi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\chi}\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\,|\omega|^{\tau}\right)^l\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,l\vartheta+k}\left(\sigma\zeta^{\frac{q}{k}}\right)d\omega.\tag{131}$$

Proof. Applying the FT to **Equation 129** and using **Equation 123** for the modified FT of the WFDO, we obtain:

$${}^{C}_{k}D^{\gamma,\vartheta}_{q,\sigma,0^+}\hat{\Theta}(\omega,\zeta)=-i\Lambda\omega\,\hat{\Theta}(\omega,\zeta)-\aleph\,|\omega|^{\tau}\,\hat{\Theta}(\omega,\zeta),\tag{132}$$

where $\hat{\Theta}(\omega,\zeta)$ is the FT of $\Theta(\chi,\zeta)$ with respect to variable χ.

Let $\tilde{\hat{\Theta}}(\omega, s)$ be the KTT of $\hat{\Theta}(\omega, \zeta)$ and $\hat{\theta}(\omega)$ be the FT of $\theta(\chi)$. Now, applying the KTT to **Equation 132** and taking advantage of **Lemma 3.7**, we derive:

$$\begin{aligned}
&\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\tilde{\hat{\Theta}}(\omega,s)-ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\Theta\left(\omega,0^+\right)\\
&=-i\Lambda\omega\,\tilde{\hat{\Theta}}(\omega,s)-\aleph\left|\omega\right|^{\tau}\tilde{\hat{\Theta}}(\omega,s)\\
&\tilde{\hat{\Theta}}(\omega,s)\left[\left(\frac{s^2}{k}\right)^{-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}+i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right]\\
&=ks^3\left(\frac{s^2}{k}\right)^{1-\frac{\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{\frac{\gamma}{k}}\hat{\theta}(\omega)\\
&=s^5\hat{\theta}(\omega)\left(1+\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)\left(\frac{s^2}{k}\right)^{\frac{\vartheta}{k}}\left(1-k\sigma\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{\gamma}{k}}\right)^{-1}\\
&=s^5\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)^l\left(\frac{s^2}{k}\right)^{\frac{l\vartheta}{k}}\left(1-\sigma k\left(\frac{s^2}{k}\right)^{\frac{q}{k}}\right)^{-\frac{l\gamma}{k}}\\
&=k\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)^l s^{\frac{2(l\vartheta+k)}{k}+3}k^{-\frac{l\vartheta+k}{k}}\left(1-\left(\frac{u}{ks}\right)^{\frac{q}{k}}\sigma k\right)^{-\frac{l\gamma}{k}}.
\end{aligned}\tag{133}$$

Inverting the KTT operator in **Equation 133** and using **Equation 134** and **Lemma 3.1**, we obtain:

$$\begin{aligned}
\hat{\Theta}(\omega,\zeta)&=\hat{\theta}(\omega)\,k\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)^l{}_k\varepsilon^{l\gamma}_{q,r\vartheta+k,\sigma}(\zeta)\\
&=\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)^l\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,\vartheta l+k}\left(\sigma\zeta^{\frac{q}{k}}\right).
\end{aligned}\tag{134}$$

Taking inverse of the FT operator in **Equation 134**, we derive:

$$\Theta(\chi,\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\chi}\hat{\theta}(\omega)\sum_{l=0}^{\infty}(-1)^l\left(i\Lambda\omega+\aleph\left|\omega\right|^{\tau}\right)^l\zeta^{\frac{l\vartheta}{k}}E^{l\gamma}_{k,q,l\vartheta+k}\left(\sigma\zeta^{\frac{q}{k}}\right)d\omega.\tag{135}$$

For $k=1$, the solutions of the fractional advection–dispersion equation with the k-HPFD described in Theorems 5.1 and 5.2 reduce to the solutions for the fractional advection–dispersion equation with the HPFD. For $\gamma=0$, the above-reported solutions for advection–dispersion equations with the k-HPFD reduce to the solutions for advection–dispersion equations with the Hilfer derivative, and for $\rho=0$ and $\rho=1$, these results reduce to solutions for advection–dispersion equations with the RL derivative and the Caputo derivative, respectively.

6. Conclusion

This study derives the KTT of the k-PI, k-HPFD, and its Caputo version. Additionally, solutions of some Cauchy and advection–dispersion equations carrying the k-HPFD and its Caputo-type form have been presented using the derived expressions of the KTT of the k-HPFD and its Caputo form. The solutions of Cauchy equations carrying the k-HPFD operator are obtained in the form of GMLF.

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The authors declare they have no competing interests.

Author contributions

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Availability of data

The study presented here is purely theoretical. Therefore, there is no data available.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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