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RESEARCH ARTICLE

Fractional fourier transform: A variant of Heisenberg uncertainty principle, weighted uncertainty inequality and application

Mawardi Bahri*

Department of Mathematics, Hasanuddin University, Makassar, Indonesia
mawardibahri@gmail.com

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ABSTRACT

The focus of this article is to propose two different versions of the Heisenberg uncertainty principle related to the fractional Fourier transform (FrFT). The first version is directly derived using the basic connection between the traditional Fourier transform (FT) and the FrFT, and the second is derived by exploiting time differentiation property and Parseval's formula for the FrFT. In addition, a weighted uncertainty inequality related to this transformation is investigated. As a simple illustration of the use of the obtained results, we design the fractional characteristic function. An inequality describing the relation between the probability density function and its fractional characteristic function is presented.

1. Introduction

As is well known, the fractional Fourier transform (FrFT) may be considered as a nontrivial generalized form of the traditional Fourier transform (FT). It was originally introduced by Namias in.[1–3] Afterward, FrFT has been extensively studied by many researchers and has found a variety of applications in optics, signal processing, and applied mathematics. The FrFT depends on a parameter θ and can be viewed as a counterclockwise rotation of the function to arbitrary angles in the frequency domain. In,[4–10] the authors have obtained a number of results related to the useful properties of the FrFT, such as linearity, shifting, scaling, modulation, differentiation, convolution, correlation, the product theorem and inequalities.

It is widely recognized that uncertainty principles play a central role in many disciplines of science, including quantum mechanics[11] and signal processing.[12] There are many distinct types of the uncertainty principles associated with the FrFT, such as the Heisenberg-type uncertainty principle,[13–16] Nazarov's uncertainty principle,[17,18] Hardy's uncertainty principle, Beurling's uncertainty principle,[19] logarithmic uncertainty principle, and the entropic uncertainty principle.[20] However, to the best of our knowledge, some uncertainty principles related to the FrFT, such as the variant of the Heisenberg inequality and the weighted uncertainty inequality, have not yet been published in the existing literature.

This paper has three main objectives. First, we derive the Gaussian function in the FrFT domain, which will be needed in the sequel. We use two approaches to derive the results and conclude that both the obtained results are equivalent. Second, we propose a new variant of Heisenberg's uncertainty principle pertaining to the FrFT. This principle is derived by exploiting the time differentiation property and Plancherel's formula for the FrFT. Third, we study a weighted uncertainty inequality involving the FrFT. We then study the utility of the FrFT in probability modeling. We construct the fractional characteristic function and establish an inequality that describes how the probability density function interacts with the fractional characteristic function.

*Corresponding Author

Furthermore, in,[21–28] several authors have explicitly formulated a fractional mathematical model to study disease modeling analysis. The proposed FrFT may be considered in solving such fractional models.

The outline of the article is organized as follows. In Section 2, some notation and basic facts about the FrFT along with its key properties are explained. These results will be required in this paper. In Section 3, the derivation of the Gaussian function in the FrFT domain using two approaches is studied in detail. Section 4 focuses on proving two different versions of the Heisenberg uncertainty principle involving the FrFT. Section 5 contains a weighted uncertainty inequality related to the FrFT and some implications of these principles are also included. Section 6 contains a detailed discussion of the utility of the FrFT to probabilistic modeling. Finally, all this work is collected in Section 7.

2. Fractional fourier transform and properties

This part first introduces a definition of the fractional Fourier transform (FrFT) and recalls its essential properties. We also mention a few notational conventions used in this work.

Definition 1. *For $1 \le p < \infty$, let $L^p(\mathbb{R})$ denote the space of measurable functions f on $\mathbb{R}$ such that*

$$\|f\|_{L^p(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty. \tag{1}$$

and

$$\|f\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{x\in\mathbb{R}}|f(x)|. \tag{2}$$

If $f \in L^\infty(\mathbb{R})$, is continuous then

$$\|f\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)|. \tag{3}$$

Especially for $p = 2$, $L^2(\mathbb{R})$ is a Hilbert space with the usual inner product, given by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

Here, $\overline{g(\cdot)}$ is the conjugation of $g(\cdot)$.

Definition 2. *The Fourier transform (FT) of a complex function $f \in L^2(\mathbb{R})$ is defined by*[29, 30]

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x}dx,\ x \in \mathbb{R}. \tag{4}$$

Under the additional condition $\hat{f} \in L^1(\mathbb{R})$, its inverse is given by

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\{f\}(\omega)e^{i\omega x}d\omega, \quad \omega \in \mathbb{R}. \tag{5}$$

Parseval's formula for the FT takes the following form

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \mathcal{F}\{f\}, \mathcal{F}\{g\}\rangle, \quad \forall f, g \in L^2(\mathbb{R}). \tag{6}$$

Next, we recall the definition of the FrFT and its inversion formula, as expressed below.

Definition 3. *The FrFT with angle θ of a function $f \in L^2(\mathbb{R})$ denoted by $\mathcal{F}^\theta\{f\} = \hat{f}^\theta$ is given by*[1, 14]

$$\begin{aligned}\mathcal{F}^\theta\{f\}(\omega) &= \hat{f}^\theta(\omega)\\ &= \int_{\mathbb{R}} K^\theta(x,\omega)f(x)dx.\end{aligned} \tag{7}$$

Here the kernel $K^\theta(x,\omega)$ is given by

$$K^\theta(x,\omega) = \begin{cases} C^\theta e^{i(x^2+\omega^2)\frac{\cot\theta}{2} - ix\omega\csc\theta}, \theta \in \mathbb{R} - n\pi \\ \delta(x-\omega), \theta = 2n\pi \\ \delta(x+\omega), \theta = (2n+1)\pi, n \in \mathbb{Z}, \end{cases} \tag{8}$$

and

$$\begin{aligned}C^\theta &= (2\pi i\sin\theta)^{-1/2}e^{i\theta/2}\\ &= \sqrt{\frac{1 - i\cot\theta}{2\pi}}.\end{aligned} \tag{9}$$

The original signal f in Equation (7) can be recovered from its FrFT using the following definition.

Definition 4. *For any function $f \in L^1(\mathbb{R})$ with $\hat{f}^\theta \in L^1(\mathbb{R})$, the inversion formula of the FrFT is expressed by*[1, 14]

$$\begin{aligned}\big(\mathcal{F}^\theta\{f\}\big)^\vee(x) &= f(x)\\ &= \int_{\mathbb{R}} \overline{K^\theta(x,\omega)}\hat{f}^\theta(\omega)d\omega,\end{aligned} \tag{10}$$

where

$$\begin{aligned}\overline{K^\theta(x,\omega)} &= \overline{C^\theta}e^{-i(x^2+\omega^2)\frac{\cot\theta}{2} + ix\omega\csc\theta}\\ &= K^{-\theta}(x,\omega),\end{aligned}$$

and

$$\begin{aligned}\overline{C^\theta} &= (-2\pi i\sin\theta)^{-1/2}e^{-i\theta/2}\\ &= \sqrt{\frac{1 + i\cot\theta}{2\pi}}\\ &= C^{-\theta}.\end{aligned}$$

In the case $\theta = \frac{\pi}{2}$, the FrFT definition in Equation (7) reduces to Equation (4), that is,

$$\mathcal{F}^{\frac{\pi}{2}}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f\}(\omega). \tag{11}$$

Also, we have

$$\mathcal{F}^{-\frac{\pi}{2}}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f\}(-\omega). \tag{12}$$

From Definition 3 we immediately get

$$\begin{aligned}\mathcal{F}^{\theta}\{f\}(\omega) &= \int_{\mathbb{R}} f(x)C^{\theta}e^{ix^2\frac{\cot\theta}{2}}e^{i\omega^2\frac{\cot\theta}{2}}e^{-i\omega x\csc\theta}dx \\ &= C^{\theta}e^{i\omega^2\frac{\cot\theta}{2}}\int_{\mathbb{R}} f(x)e^{ix^2\frac{\cot\theta}{2}}e^{-i\omega x\csc\theta}dx.\end{aligned} \tag{13}$$

An application of relation (4) yields

$$\begin{aligned}\mathcal{F}^{\theta}\{f\}(\omega &= C^{\theta}e^{i\omega^2\frac{\cot\theta}{2}}\int_{\mathbb{R}} h(x)e^{-i\omega x\csc\theta}dx \\ &= C^{\theta}e^{i\omega^2\frac{\cot\theta}{2}}\mathcal{F}\{h\}(\omega\csc\theta),\end{aligned} \tag{14}$$

where

$$h(x) = f(x)e^{ix^2\frac{\cot\theta}{2}}. \tag{15}$$

It is seen that Equation (14) describes the natural relationship between the FrFT and FT. This relation plays an important role in obtaining some useful properties of the FrFT.

The following important result will be needed in proving some of the main results in the present work.

Theorem 1. *Suppose that $f(x)$ is a continuously differentiable. If we assume that*

$$\lim_{|x|\to\infty} f(x) = 0,$$

then

$$\begin{aligned}\mathcal{F}^{\theta}\{f'\}(\omega) &= \mathcal{F}^{\theta}\left\{\frac{df}{dx}\right\}(\omega) \\ &= \frac{i\omega\mathcal{F}^{\theta}\{f\}(\omega) - i\cos\theta\ \mathcal{F}^{\theta}\{xf\}(\omega)}{\sin\theta}.\end{aligned} \tag{16}$$

Proof. In view of Equation (7) above, we obtain

$$\begin{aligned}\mathcal{F}^{\theta}\left\{\frac{df}{dx}\right\}(\omega) &= \int_{\mathbb{R}} C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}\left(\frac{df}{dx}\right)dx \\ &= C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}f(x)\Big|_{-\infty}^{\infty} \\ &\quad - \int_{\mathbb{R}} f(x)\frac{d}{dx}\big(C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}\big)dx \\ &= -\int_{\mathbb{R}} f(x)\big(ix\cot\theta - i\omega\csc\theta\big) \\ &\quad \times C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}\big)dx.\end{aligned}$$

The above expression may be rewritten in the following form

$$\begin{aligned}\mathcal{F}^{\theta}\left\{\frac{df}{dx}\right\} &= \int_{\mathbb{R}}(i\omega\csc\theta - ix\cot\theta) \\ &\quad \times C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}f(x)dx \\ &= \frac{i\omega\mathcal{F}^{\theta}\{f\}(\omega)}{\sin\theta} - i\cot\theta\ \mathcal{F}^{\theta}\{xf\}(\omega) \\ &= \frac{i\omega\mathcal{F}^{\theta}\{f\}(\omega) - i\cos\theta\ \mathcal{F}^{\theta}\{xf\}(\omega)}{\sin\theta}.\end{aligned}$$

This proves the theorem. □

In the case $\theta = \frac{\pi}{2}$, we obtain

$$\mathcal{F}^{\frac{\pi}{2}}\left\{\frac{df}{dx}\right\}(\omega) = i\omega\mathcal{F}\{f\}(\omega). \tag{17}$$

Theorem 2. *For any function $f \in L^2(\mathbb{R})$, we have*

$$\mathcal{F}^{\theta}\{\overline{f(-x)}\}(\omega) = \overline{\mathcal{F}^{-\theta}\{f\}(-\omega)}. \tag{18}$$

Proof. By equation (7), we have

$$\begin{aligned}\mathcal{F}^{\theta}\{\overline{f(-x)}\}(\omega) &= \int_{\mathbb{R}} \overline{f(-x)}C^{\theta}e^{i(x^2+\omega^2)\frac{\cot\theta}{2}-ix\omega\csc\theta}dx \\ &= \int_{\mathbb{R}} \overline{f(-x)\overline{C^{\theta}}e^{-i(x^2+\omega^2)\frac{\cot\theta}{2}+ix\omega\csc\theta}}dx \\ &= \int_{\mathbb{R}} \overline{f(x)C^{-\theta}e^{i(x^2+\omega^2)\frac{(-cot\theta)}{2}+ix\omega\csc(-\theta)}}dx \\ &= \overline{\mathcal{F}^{-\theta}\{f\}(-\omega)}.\end{aligned}$$

□

In the case $\theta = \frac{\pi}{2}$, the above expression changes to

$$\mathcal{F}^{\frac{\pi}{2}}\{\overline{f(-x)}\}(\omega) = \overline{\mathcal{F}\{f\}(\omega)}. \tag{19}$$

Equation (19) is known as the conjugate symmetry property of the FT.

One of the fundamental results involving the FT is the duality property. In the following, we shall prove that this property is still valid for the FrFT with some modifications.

Theorem 3 (The Duality property)**.** *For every $f \in L^2(\mathbb{R})$ such that its FrFT $\mathcal{F}^{\theta}\{f\} \in L^2(\mathbb{R})$, one has*

$$\mathcal{F}^{-\theta}\{\mathcal{F}^{\theta}\{f\}(\omega)\} = f(\omega). \tag{20}$$

In particular, with $\theta = \frac{\pi}{2}$, the above yields

$$\begin{aligned}\mathcal{F}^{-\frac{\pi}{2}}\{\mathcal{F}^{\frac{\pi}{2}}\{f\}(\omega)\} &= \frac{1}{2\pi}\mathcal{F}\{\mathcal{F}\{f\}(-\omega)\} \\ &= \frac{1}{2\pi}f(\omega).\end{aligned} \tag{21}$$

Proof. In view of relation (7) we obtain

$$\begin{aligned}\mathcal{F}^{-\theta}\{\mathcal{F}^{\theta}\{f\}(\omega)\} &= \int_{\mathbb{R}} \mathcal{F}^{\theta}\{f\}(\omega)C^{-\theta}e^{-i(t^2+\omega^2)\frac{\cot\theta}{2}+it\omega\csc\theta}\,d\omega.\end{aligned} \tag{22}$$

Applying (10) and interchanging the roles of ω with x in (22), we get

$$\begin{aligned}&\mathcal{F}^{-\theta}\{\mathcal{F}^{\theta}\{f\}(\omega)\}\\&=\int_{\mathbb{R}}\mathcal{F}^{\theta}\{f\}(t)C^{-\theta}e^{-i(t^2+\omega^2)\frac{\cot\theta}{2}+it\omega\csc\theta}\,dt\\&=f(\omega).\end{aligned}\tag{23}$$

This is equivalent to

$$\mathcal{F}^{-\theta}\{\mathcal{F}^{\theta}\{f\}(-\omega)\}=f(-\omega),$$

and Equation (20) follows. □

Several other useful properties of the FrFT mentioned above are summarized in **Table 1**.

Table 1. Useful Properties of Fractional Fourier Transform[31]

Property	**Signal, $f(x)$**	**Fractional Fourier transform, $\mathcal{F}^{\theta}\{f\}(\omega)$**
Time shift	$f(x-\tau)$	$e^{i(\tau^2/2)\sin\theta\cos\theta-i\omega\tau\sin\theta}\mathcal{F}^{\theta}\{f\}(\omega-\tau\cos\theta)$
Modulation	$f(x)e^{ivx}$	$e^{-iv^2(\sin\theta\cos\theta)/2+i\omega v\cos\theta}\mathcal{F}^{\theta}\{f\}(\omega-v\sin\theta)$
Multiplication	$xf(x)$	$\omega\cos\theta\mathcal{F}^{\theta}\{f\}(\omega)+i\sin\theta\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\}(\omega)$
Time inversion	$f(-x)$	$\mathcal{F}^{\theta}\{f\}(-\omega)$
Time scaling	$f(cx)$	$\sqrt{\frac{1-i\cot\theta}{c^2-i\cot\theta}}e^{i(u^2/2)\cot\theta(1-(\cos^2\beta/\cos^2\theta))}$ $\mathcal{F}^{\beta}\{f\}\left(\frac{\omega\sin\beta}{c\sin\theta}\right)$ where $\cot\beta=\frac{\cot\theta}{c^2}$
Differentiation	$f'(x)$	$\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\}(\omega)\cos\theta+i\omega\sin\theta\mathcal{F}^{\theta}\{f\}(\omega)$
Parseval's formula	$\int_{\mathbb{R}}f(x)\overline{g(x)}dx$	$\int_{\mathbb{R}}\mathcal{F}^{\theta}\{f\}(\omega)\overline{\mathcal{F}^{\theta}\{g\}(\omega)}d\omega$
Plancherel's formula	$\int_{\mathbb{R}}\lvert f(x)\rvert^2dx$	$\int_{\mathbb{R}}\lvert\mathcal{F}^{\theta}\{f\}(\omega)\rvert^2d\omega$

3. Gaussian function in FrFT domains

One of the most commonly used functions in probability modeling and the analysis of random signals is the Gaussian function (compare with[32]). Therefore, it is important to derive the FrFT of the Gaussian function. In this part, we first investigate the Gaussian function in the FrFT domain using the basic connection between the FrFT and the FT.

Lemma 1. *Given the Gaussian function in the following form:*

$$f(x)=e^{-\alpha x^2},\quad\forall\alpha>0.\tag{24}$$

The FrFT of relation (24) *above is given by*

$$\mathcal{F}^{\theta}\{f\}(\omega)=\sqrt{\frac{1-i\cot\theta}{2\alpha-i\cot\theta}}e^{\frac{\omega^2}{2}\left(i\cot\theta-\frac{\csc^2\theta}{2\alpha-i\cot\theta}\right)}.\tag{25}$$

Proof. It follows from Equations (4) and (15) that

$$\begin{aligned}\mathcal{F}\{h\}(\omega)&=\int_{\mathbb{R}}e^{ix^2\frac{\cot\theta}{2}-\alpha x^2-ix\omega}dx\\&=\int_{\mathbb{R}}e^{-x^2\left(\frac{2\alpha-i\cot\theta}{2}\right)}e^{-ix\omega}dx\\&=\sqrt{\frac{2\pi}{2\alpha-i\cot\theta}}e^{-\frac{\omega^2}{2(2\alpha-i\cot\theta)}}.\end{aligned}\tag{26}$$

Hence,

$$\mathcal{F}\{h\}(\omega\csc\theta)=\sqrt{\frac{2\pi}{2\alpha-i\cot\theta}}e^{-\frac{\omega^2\csc^2\theta}{2(2\alpha-i\cot\theta)}}.\tag{27}$$

Applying (14) yields

$$\begin{aligned}&\mathcal{F}^{\theta}\{f\}(\omega)\\&=\sqrt{\frac{1-i\cot\theta}{2\pi}}\sqrt{\frac{2\pi}{2\alpha-i\cot\theta}}e^{i\omega^2\frac{\cot\theta}{2}}e^{-\frac{\omega^2\csc^2\theta}{2(2\alpha-i\cot\theta)}}\\&=\sqrt{\frac{1-i\cot\theta}{2\alpha-i\cot\theta}}e^{\frac{\omega^2}{2}\left(i\cot\theta-\frac{\csc^2\theta}{2\alpha-i\cot\theta}\right)}.\end{aligned}$$

Therefore, the desired result follows. □

Remark 1. *It is seen that the derivation of Equation* (25) *using this method is simpler than that proposed in reference.*[32]

Let us give an alternative proof of Equation (25) using the properties of the FrFT.

We see that if

$$f(x)=e^{-\alpha x^2},$$

then

$$\begin{aligned}f'(x)&=-2\alpha xe^{-\alpha x^2}\\&=-2\alpha xf(x).\end{aligned}$$

This equation gives

$$\mathcal{F}^{\theta}\{f'\}(\omega)=-2\alpha\mathcal{F}^{\theta}\{xf\}(\omega).\tag{28}$$

With the help of the properties of the FrFT in **Table 1**, we obtain

$$\begin{aligned}&\left(i\omega\sin\theta+\cos\theta\frac{d}{d\omega}\right)\mathcal{F}^{\theta}\{f\}(\omega)\\&=-2\alpha(\omega\cos\theta\mathcal{F}^{\theta}\{f\}(\omega))+i\sin\theta\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\}(\omega).\end{aligned}\tag{29}$$

This equation is equivalent to

$$i\omega\sin\theta\mathcal{F}^{\theta}(\omega)+\cos\theta\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\} \tag{30}$$
$$=-2\alpha\omega\cos\theta\mathcal{F}^{\theta}\{f\}(\omega)-2\alpha i\sin\theta\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\}(\omega).$$

Hence,

$$(i\omega\sin\theta+2\alpha\omega\cos\theta)\mathcal{F}^{\theta}\{f\}(\omega) \tag{31}$$
$$=-(\cos\theta+2\alpha i\sin\theta)\frac{d}{d\omega}\mathcal{F}^{\theta}\{f\}(\omega),$$

which gives

$$\mathcal{F}^{\theta}\{f\}(\omega)=K\left(e^{-\frac{i\frac{\omega^2}{2}\sin\theta}{\cos\theta+2\alpha i\sin\theta}}e^{\frac{-\omega^2\alpha\cos\theta}{\cos\theta+2\alpha i\sin\theta}}\right)$$
$$=Ke^{-\frac{\omega^2}{2}\left(\frac{i\sin\theta+2\alpha\cos\theta}{\cos\theta+i2\alpha\sin\theta}\right)}, \tag{32}$$

where K is an arbitrary constant. Now observe that

$$K=\mathcal{F}^{\theta}\{f\}(0)$$
$$=\int_{\mathbb{R}}\sqrt{\frac{1-i\cot\theta}{2\pi}}e^{ix^2\frac{\cot\theta}{2}}e^{-\alpha x^2}dx$$
$$=\sqrt{\frac{1-i\cot\theta}{2\pi}}\int_{\mathbb{R}}e^{-(\alpha-\frac{i\cot\theta}{2})x^2}dx \tag{33}$$
$$=\sqrt{\frac{1-i\cot\theta}{2\alpha-i\cot\theta}}.$$

Substituting Equation (33) into Equation (32) results in

$$\mathcal{F}^{\theta}\{f\}(\omega)=\sqrt{\frac{1-i\cot\theta}{2\alpha-i\cot\theta}}e^{-\frac{\omega^2}{2}\left(\frac{i\sin\theta+2\alpha\cos\theta}{\cos\theta+i2\alpha\sin\theta}\right)}. \tag{34}$$

We see that

$$e^{\frac{\omega^2}{2}\left(i\cot\theta-\frac{\csc^{\theta}}{2a-i\cot\theta}\right)}$$
$$=e^{\frac{\omega^2}{2}\left(i\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin^2\theta(2a-i\frac{\cos\theta}{\sin\theta})}\right)}$$
$$=e^{\frac{\omega^2}{2}\left(i\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta(2a\sin\theta-i\cos\theta)}\right)}$$
$$=e^{\frac{\omega^2}{2}\left(\frac{i\cos\theta(2\alpha\sin\theta-i\cos\theta)-1}{\sin\theta(2\alpha\sin\theta-i\cos\theta)}\right)}$$
$$=e^{\frac{\omega^2}{2}\left(\frac{i2\alpha\cos\theta-\sin\theta}{2\alpha\sin\theta-i\cos\theta}\right)}$$
$$=e^{\frac{\omega^2}{2}\left(\frac{-i\sin\theta-2\alpha\cos\theta}{\cos\theta+i2\alpha\sin\theta}\right)}$$
$$=e^{-\frac{\omega^2}{2}\left(\frac{i\sin\theta+2\alpha\cos\theta}{\cos\theta+i2\alpha\sin\theta}\right)}.$$

This shows that Equation (34) and Equation (25) are the same.

Remark 2. *In the case $\theta=\frac{\pi}{2}$, using relation* (11), *it is straightforward to verify that relations* (25) *and* (34) *reduced to*

$$\mathcal{F}^{\theta}\{f\}(\omega)=\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}, \tag{35}$$

which is consistent with the Gaussian function in the Fourier transform domain.

4. Different forms of heisenberg's uncertainty principle for FrFT

In this part, we first recall Heisenberg's uncertainty principle pertaining to the FrFT, whose proof uses the direct relation between the FrFT and the FT.

Theorem 4. *Suppose that $f,xf\in L^2(\mathbb{R})$ and $\mathcal{F}^{\theta}\{f\},\omega\mathcal{F}^{\theta}\{f\}\in L^2(\mathbb{R})$. Then, the following inequality holds*

$$\int_{\mathbb{R}}x^2|f(x)|^2dx\int_{\mathbb{R}}\omega^2|\mathcal{F}^{\theta}\{f\}(\omega)|^2d\omega$$
$$\geq\frac{|\sin\theta|^2}{4}\left(\int_{\mathbb{R}}|f(x)|^2dx\right)^2. \tag{36}$$

Proof. See.[14] □

Remark 3. *The idea of using the relation between the FrFT and the FT mentioned above was also developed by the authors in[17] in proving Heisenberg's uncertainty principle associated with the multidimensional FrFT.*

Next, we modify the ideas of the authors[13] to obtain the uncertainty principle for the FrFT. Let us denote

$$G(\omega)=\mathcal{F}^{\theta}\{f\}(\omega)e^{-i\omega^2\frac{\cot\theta}{2}}, \tag{37}$$

then from Equation (5) we immediately obtain

$$g(x)=\frac{1}{2\pi}\int_{\mathbb{R}}G(\omega)e^{i\omega x}d\omega. \tag{38}$$

We see that

$$\int_{\mathbb{R}}\omega^2|G(\omega)|^2d\omega=\int_{\mathbb{R}}\omega^2|\mathcal{F}^{\theta}\{f\}(\omega)|^2d\omega. \tag{39}$$

Using the uncertainty principle for the FT defined in (4), we infer that

$$\int_{\mathbb{R}}x^2|g(x)|^2dx\int_{\mathbb{R}}\omega^2|\mathcal{F}^{\theta}\{f\}(\omega)|^2d\omega$$
$$\geq\frac{\pi}{2}\left(\int_{\mathbb{R}}|g(x)|^2dx\right)^2. \tag{40}$$

By virtue of relations (37) and (38), we have

$$g(x\csc\theta)$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}G(\omega)e^{i\omega x\csc\theta}d\omega$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}^{\theta}\{f\}(\omega)e^{-i\omega^2\frac{\cot\theta}{2}}e^{i\omega x\csc\theta}d\omega \tag{41}$$
$$=\frac{1}{2\pi}\sqrt{\frac{1+i\cot\theta}{2\pi}}\sqrt{\frac{2\pi}{1+i\cot\theta}}e$$
$$\times\int_{\mathbb{R}}\mathcal{F}^{\theta}\{f\}(\omega)e^{-ix^2\frac{\cot\theta}{2}}e^{-i\omega^2\frac{\cot\theta}{2}}e^{i\omega x\csc\theta}d\omega.$$

Applying (10) results in

$$\begin{aligned} &g(x\csc\theta)\\ &=\frac{1}{2\pi}\sqrt{\frac{2\pi}{1+i\cot\theta}}e^{ix^2\frac{\cot\theta}{2}}f(x) \\ &=\frac{1}{2\pi}\sqrt{-2\pi i\sin\theta}e^{-i\frac{\theta}{2}}e^{ix^2\frac{\cot\theta}{2}}f(x). \end{aligned} \tag{42}$$

Thus, relation (40) becomes

$$\begin{aligned} &\frac{1}{4\pi^2}\int_{\mathbb{R}}x^2|\csc\theta|^3 2\pi|\sin\theta||f(x)|^2dx \\ &\qquad\times\int_{\mathbb{R}}\omega^2|\mathcal{F}^\theta\{f\}(\omega)|^2d\omega \\ &\qquad\geq\frac{\pi}{2}\Big(\int_{\mathbb{R}}\frac{1}{2\pi}\sin\theta|f(x)|^2\csc\theta dx\Big)^2. \end{aligned} \tag{43}$$

We simplify the above expression to arrive at

$$\begin{aligned} 2\pi\int_{\mathbb{R}}|\csc\theta|^2x^2|f(x)|^2dx\int_{\mathbb{R}}\omega^2|\mathcal{F}^\theta\{f\}(\omega)|^2d\omega \\ \geq\frac{\pi}{2}\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2. \end{aligned} \tag{44}$$

Equivalently,

$$\begin{aligned} \int_{\mathbb{R}}x^2|f(x)|^2dx\int_{\mathbb{R}}\omega^2|\mathcal{F}^\theta\{f\}(\omega)|^2d\omega \\ \geq\frac{|\sin\theta|^2}{4}\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2, \end{aligned} \tag{45}$$

which coincides with Equation (36).

Let us now investigate a variant of the inequality associated with the FrFT, which is the main result of this work. Some consequences of this result are also demonstrated in the remark. More precisely, we state and prove the following important result.

Theorem 5. *Under the assumptions as in Theorem 4, one gets*

$$\begin{aligned} &\Big(\int_{\mathbb{R}}x^2|f(x)|^2dx\Big) \\ &\quad\times\Big(\int_{\mathbb{R}}\big|\omega\mathcal{F}^\theta\{f\}(\omega)-\cos\theta\ \mathcal{F}^\theta\{xf\}(\omega)\big|^2d\omega\Big) \\ &\geq\frac{\sin^2\theta}{4}\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2. \end{aligned} \tag{46}$$

Proof. We see that

$$\begin{aligned} \Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2&=\Big(-\int_{\mathbb{R}}|f(x)|^2dx\Big)^2 \\ &=\Big(\int_{\mathbb{R}}x\frac{d}{dx}|f(x)|^2dx\Big)^2 \\ &=\Big(\int_{\mathbb{R}}2xRe(f(x)\overline{f'(x)})dx\Big)^2. \end{aligned} \tag{47}$$

where $Ref(x)$ is the real part of $f(x)$. By virtue of the elementary properties of complex numbers and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} &\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2 \\ &\leq\ \Big(\int_{\mathbb{R}}2xRe(f(x)\overline{f'(x)})dx\Big)^2 \\ &\leq 4\Big(\int_{\mathbb{R}}|xf(x)f'(x)|dx\Big)^2 \\ &\leq 4\Big(\int_{\mathbb{R}}x^2|f(x)|^2dx\Big)\Big(\int_{\mathbb{R}}|f'(x)|^2dx\Big). \end{aligned} \tag{48}$$

Next, we utilize Plancherel's formula for the FrFT described in **Table 1** for the right integral term and get

$$\begin{aligned} \Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2\leq 4\Big(\int_{\mathbb{R}}x^2|f(x)|^2dx\Big) \\ \times\Big(\int_{\mathbb{R}}|\mathcal{F}^\theta\{f'\}(\omega)|^2d\omega\Big). \end{aligned} \tag{49}$$

Substituting (16) into (49) results in

$$\begin{aligned} &\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2\leq 4\Big(\int_{\mathbb{R}}x^2|f(x)|^2dx\Big) \\ &\quad\Big(\int_{\mathbb{R}}\Big|\frac{i\omega\mathcal{F}^\theta\{f\}(\omega)-i\cos\theta\ \mathcal{F}^\theta\{xf\}(\omega)}{\sin\theta}\Big|^2d\omega\Big). \end{aligned}$$

Simplifying the above expression yields

$$\begin{aligned} &\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2 \\ &\leq\frac{4}{\sin^2\theta}\Big(\int_{\mathbb{R}}x^2|f(x)|^2dx\Big) \\ &\quad\Big(\int_{\mathbb{R}}\big|\omega\mathcal{F}^\theta\{f\}(\omega)-\cos\theta\mathcal{F}^\theta\{xf\}(\omega)\big|^2d\omega\Big), \end{aligned}$$

which completes the proof of the theorem. ☐

Remark 4.

- *In the case* $\theta=\frac{\pi}{2}$, *the relations* (36) *and* (46) *reduce to the uncertainty principle for the Fourier transform defined by* (4), *that is*

$$\begin{aligned} \int_{\mathbb{R}}x^2|f(x)|^2dx\int_{\mathbb{R}}\omega^2|\mathcal{F}\{f\}(\omega)|^2d\omega \\ \geq\frac{\pi}{2}\Big(\int_{\mathbb{R}}|f(x)|^2dx\Big)^2. \end{aligned} \tag{50}$$

- *If* $\int_{\mathbb{R}}|\mathcal{F}^\theta\{xf\}\omega)|^2d\omega=0$, *then Equation* (46) *becomes Equation* (36).

Several different forms of uncertainty principles for the FrFT, obtained using two different approaches, are shown in **Table 2**.

5. Weighted fractional fourier transform inequality

In this part, we state and prove a weighted uncertainty inequality related to the FrFT. This result may be considered as an extension of the weighted uncertainty inequality for the FT in the existing literature.[33] In addition, some direct consequences of this principle are also discussed. More precisely, we obtain the following result.

Theorem 6. *Assume that $p, q \in (1, \infty)$ and u, v are non-negative functions. Set, $\|f\|_{L^{v,p}(\mathbb{R})} = \left(\||f|^p v\|_{L^1(\mathbb{R})}\right)^{1/p}$. If there exists a constant $C > 0$ satisfying*

$$\|\mathcal{F}^\theta\{f\}\|_{L^{u,q}(\mathbb{R})} \le C\|f\|_{L^{v,p}(\mathbb{R})}, \forall f \in L^p_v(\mathbb{R}) \quad (51)$$

then, the following inequality holds:

$$\begin{aligned} &\|\mathcal{F}^\theta\{f\}\|^2_{L^2(\mathbb{R})} \\ &\le 2C\|\omega\mathcal{F}^\theta\{f\}\|_{L^{u^{1-q'},q'}(\mathbb{R})} \\ &\quad \times \|(\csc\theta - \cot\theta)xf(x)\|_{L^{v,p}(\mathbb{R})}. \end{aligned} \quad (52)$$

Table 2. A comparison of several Heisenberg uncertainty principles in FrFT domains

Uncertainty principle	Lower bound	Source
$\int x^2\|f(x)\|^2dx \int \omega^2\|\mathcal{F}^\theta\{f\}(\omega)\|^2d\omega$	$\frac{\|\sin\theta\|^2}{4}\|f\|^4_{L^2(\mathbb{R})}$	Link of FT with FrFT
$\int x^2\|f(x)\|^2dx \int \omega^2\|\mathcal{F}^\theta\{f\}(\omega)\|^2d\omega$	$\frac{\|\sin\theta\|^2}{4}\|f\|^4_{L^2(\mathbb{R})}$	Idea in[13]
$\int_{\mathbb{R}} x^2\|f(x)\|^2dx \int_{\mathbb{R}} \left\|\omega\mathcal{F}^\theta\{f\}(\omega) - \cos\theta\ \mathcal{F}^\theta\{xf\}(\omega)\right\|^2 d\omega$	$\frac{\|\sin\theta\|^2}{4}\|f\|^4_{L^2(\mathbb{R})}$	Present work

Proof. A direct computation gives

$$\begin{aligned} &\|\mathcal{F}^\theta\{f\}\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |\mathcal{F}^\theta\{f\}(\omega)|^2 d\omega \\ &= \int_{\mathbb{R}} \omega\frac{d}{d\omega}|\mathcal{F}^\theta\{f\}(\omega)|^2 d\omega \\ &= \int_{\mathbb{R}} |\omega\frac{d}{d\omega}\mathcal{F}^\theta\{f\}(\omega)|^2 d\omega \\ &\le 2\int_{\mathbb{R}} \left|\omega\overline{\mathcal{F}^\theta\{f\}(\omega)}\frac{d}{d\omega}\mathcal{F}^\theta\{f\}(\omega)\right| d\omega. \end{aligned} \quad (53)$$

This gives

$$\begin{aligned} &\|\mathcal{F}^\theta\{f\}\|^2_{L^2(\mathbb{R})} \\ &\le 2\int_{\mathbb{R}} \left|\omega\mathcal{F}^\theta\{f\}(\omega)u(\omega)^{-\frac{1}{q}}\right|\left|\frac{d}{d\omega}\mathcal{F}^\theta\{f\}(\omega)u(\omega)^{\frac{1}{q}}\right| d\omega \\ &\le 2\Big(\int_{\mathbb{R}} \left|\omega\mathcal{F}^\theta\{f\}(\omega)\right|^{q'} u(\omega)^{-\frac{q'}{q}} d\omega\Big)^{\frac{1}{q'}} \\ &\quad \times \Big(\int_{\mathbb{R}} \left|\frac{d}{d\omega}\mathcal{F}^\theta\{f\}(\omega)\right|^q u(\omega) d\omega\Big)^{\frac{1}{q}} \\ &\le 2C\Big(\int_{\mathbb{R}} \left|\omega\mathcal{F}^\theta\{f\}(\omega)\right|^{q'} u(\omega)^{-\frac{q'}{q}} d\omega\Big)^{\frac{1}{q'}} \\ &\quad \times \Big(\int_{\mathbb{R}} \left|\frac{d}{dx}\big(\mathcal{F}^\theta\{f\}\big)^\vee(x)\right|^p v(x)\Big|dx\Big)^{\frac{1}{p}} \\ &= 2C\Big(\int_{\mathbb{R}} \left|\omega\mathcal{F}^\theta\{f\}(\omega)\right|^{q'} u(\omega)^{-\frac{q'}{q}} d\omega\Big)^{\frac{1}{q'}} \\ &\quad \times \Big(\int_{\mathbb{R}} \Big|\big[(-x\cot\theta\big(\mathcal{F}^\theta\{f\}\big)^\vee(x) + \csc\theta) \\ &\quad \times \big(\omega\mathcal{F}^\theta\{f\}\big)^\vee(x)\big|^p\big] v(x)\Big|dx\Big)^{\frac{1}{p}}. \end{aligned} \quad (54)$$

Since $\frac{1}{q} + \frac{1}{q'} = 1$, we obtain

$$\begin{aligned} &\|\mathcal{F}^\theta\{f\}\|^2_{L^2(\mathbb{R})} \\ &= 2C\Big(\int_{\mathbb{R}} \big(\left|\omega\mathcal{F}^\theta\{f\}(\omega)\right|u(\omega)^{-\frac{1}{q}}\big)^{q'} d\omega\Big)^{\frac{1}{q'}} \\ &\quad \Big(\int_{\mathbb{R}} \left|(\csc\theta - \cot\theta)xf(x)\right|^p v(x)\Big|dx\Big)^{\frac{1}{p}} \\ &= 2C\Big(\int_{\mathbb{R}} \big(\left|\omega\mathcal{F}^\theta\{f\}(\omega)\right|u(\omega)^{\frac{1-q'}{q'}}\big)^{q'} d\omega\Big)^{\frac{1}{q'}} \\ &\quad \times \Big(\int_{\mathbb{R}} \left|(\csc\theta - \cot\theta)xf(x)\right|^p v(x)\Big|dx\Big)^{\frac{1}{p}}. \end{aligned} \quad (55)$$

Thus, the proof is complete. ☐

Some immediate consequences of Theorem 6 are summarized in the following remark.

Remark 5.

- *In the case $p = q = 2$ and $u = v = 1$, and $C = 1$, relation (51) becomes Parseval's formula for the FrFT, that is,*

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\mathcal{F}^\theta\{f\}(\omega)|^2 d\omega, \quad (56)$$

  which is the Parseval formula for the FrFT.
- *If we take $\theta = \frac{\pi}{2}$, the relation (52) reduces to*

$$\begin{aligned} &\|\mathcal{F}\{f\}\|^2 \\ &\le 2\sqrt{2\pi}C\|\omega\mathcal{F}\{f\}\|_{L^{u^{1-q'},q'}(\mathbb{R})}\|xf(x)\|_{L^{v,p}(\mathbb{R})}. \end{aligned} \quad (57)$$

  Equation (57) is usually known as the weighted uncertainty inequality for the FT (compare with[33]).

6. Application to probability modeling

The results for fractional Fourier transform (FrFT) can be applied to probabilistic modeling. In this section, we explore the utility of the FrFT in constructing characteristic functions and derive an inequality that describes the interaction between the probability density function and the fractional characteristic function. To illustrate the application of this inequality, we provide a representative example. To this end, we introduce the following definition.

Definition 5. *The characteristic function associated with the FrFT denoted by M^θ is defined by*

$$M^\theta(\omega)) = \int_{\mathbb{R}} e^{i(x^2+\omega^2)\frac{\cot\theta}{2} - ix\omega\csc\theta} P(x)dx, \quad (58)$$

where $P(x)$ is the probability density function satisfying

$$\int_{\mathbb{R}} P(x)dx = 1, \quad \text{and} \quad P(x) \geq 0, \forall x \in \mathbb{R}. \quad (59)$$

We then call M^θ the fractional characteristic function.

Next, the probability density function $P(x)$ can be recovered by the formula

$$P(x) = \frac{1}{2\sin\theta}\int_{\mathbb{R}} e^{-i(x^2+\omega^2)\frac{\cot\theta}{2} + ix\omega\csc\theta} M^\theta(\omega)d\omega. \quad (60)$$

provided that the integral exists. From Equation (58), we find that

$$|M^\theta(\omega)| \leq 1, \quad \forall\omega \in \mathbb{R}. \quad (61)$$

In particular,

$$M^{\frac{\pi}{2}}(0) = 1 \quad (62)$$

It is clear that relations (61) and (62) are consistent with the basic properties of the classical characteristic function (compare to[29]).

According to (36) the Heisenberg's inequality related to the fractional characteristic function takes the following form

$$\int_{\mathbb{R}} x^2|P(x)|^2dx \int_{\mathbb{R}} \omega^2|M^\theta(\omega)|^2d\omega \geq \frac{\pi|\sin\theta|^3}{2}\left(\int_{\mathbb{R}}|P(x)|^2dx\right)^2. \quad (63)$$

Let us consider a Gaussian signal (function) of the following form

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}. \quad (64)$$

We first observe that $P(x)$ is a probability density function. With the help of (34) its fractional characteristic function is

$$\begin{aligned} M^\theta(\omega) &= \frac{1}{\sigma\sqrt{2\alpha - i\cot\theta}} e^{-\frac{\omega^2}{2}\left(\frac{2\alpha}{\cos^2\theta+4\alpha^2\sin^2\theta} - i\frac{\sin\theta\cos\theta(4\alpha^2-1)}{\cos^2\theta+4\alpha^2\sin^\theta}\right)} \\ &= \frac{1}{\sigma}\sqrt{\frac{\sin\theta}{2\alpha\sin\theta - i\cos\theta}} \\ &\times e^{-\frac{\omega^2}{2}\left(\frac{2\alpha}{\cos^2\theta+4\alpha^2\sin^2\theta} - i\frac{\sin\theta\cos\theta(4\alpha^2-1)}{\cos^2\theta+4\alpha^2\sin\theta}\right)}, \end{aligned} \quad (65)$$

where $\alpha = \frac{1}{2\sigma^2}$. Next, a direct computation gives

$$\int_{\mathbb{R}} |P(x)|^2dx = \int_{\mathbb{R}} \frac{1}{2\pi\sigma^2}e^{-\frac{x^2}{\sigma^2}}dx = \frac{1}{2\sigma\sqrt{\pi}}, \quad (66)$$

and

$$\int_{\mathbb{R}} x^2|P(x)|^2dx = \int_{\mathbb{R}} x^2\frac{1}{2\pi\sigma^2}e^{-\frac{x^2}{\sigma^2}}dx = \frac{\sigma}{4\sqrt{\pi}}. \quad (67)$$

In fact, we have

$$\begin{aligned} |M^\theta(\omega)|^2 &= \frac{1}{\sigma^2}\left|\frac{\sin\theta}{2\alpha\sin\theta - i\cos\theta}\right| e^{-\omega^2\left(\frac{2\alpha}{\cos^2\theta+4\alpha^2\sin^2\theta}\right)} \\ &= \frac{|\sin\theta|}{\sigma^2}\left(\cos^2\theta + 4\alpha^2\sin^2\theta\right)^{-\frac{1}{2}} \\ &\times e^{-\omega^2\left(\frac{2\alpha}{\cos^2\theta+4\alpha^2\sin^2\theta}\right)}. \end{aligned} \quad (68)$$

This equation gives

$$\begin{aligned} \int_{\mathbb{R}} \omega^2|M^\theta(\omega)|^2d &= \frac{|\sin\theta|}{\sigma^2}\left(\cos^2\theta + 4\alpha^2\sin^2\theta\right)^{-\frac{1}{2}} \\ &\times \int_{\mathbb{R}} \omega^2 e^{-\omega^2\left(\frac{2\alpha}{\cos^2\theta+4\alpha^2\sin^2\theta}\right)}d\omega \\ &= \frac{|\sin\theta|}{\sigma^2}\left(\cos^2\theta + 4\alpha^2\sin^2\theta\right)^{-\frac{1}{2}} \\ &\times \frac{\sqrt{\pi}}{4\sqrt{2}\alpha^{\frac{3}{2}}}(\cos^2\theta + 4\alpha^2\sin^2\theta)^{\frac{3}{2}} \\ &= \frac{|\sin\theta|}{\sigma^2}\frac{\sqrt{\pi}}{4\sqrt{2}\alpha^{\frac{3}{2}}}(\cos^2\theta + 4\alpha^2\sin^2\theta). \end{aligned} \quad (69)$$

For $\alpha = \frac{1}{2\sigma^2}$, we obtain

$$\int_{\mathbb{R}} \omega^2|M^\theta(\omega)|^2d\omega = \frac{\sqrt{\pi}\sigma|\sin\theta|(\sigma^2\cos^2\theta + \sin^2\theta)}{2\sigma^2}. \quad (70)$$

Considering equations (70), (67) together with equation (66) we obtain

$$\frac{|\sin\theta|(\sigma^2\cos^2\theta + \sin^2\theta)}{8} \geq \frac{|\sin\theta|^3}{8\sigma^2}, \quad \forall\sigma^2 \geq 1. \quad (71)$$

7. Conclusion

In this study, we introduced the FrFT and summarized its essential properties. Using two different approaches in deriving Heisenberg-type uncertainty principles related to the FrFT, we obtained two different forms of the uncertainty principle related to this transformation. We also proposed the weighted uncertainty inequality related to the FrFT. Additionally, the use of the FrFT in constructing the characteristic function in the context of probability modeling was discussed. Finally, we obtained the inequality that describes the relationship between the probability density function and its fractional characteristic function.

Builing on the references,[21–28] future work will investigate the usefulness of the FrFT in solving several fractional mathematical models.

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Mawardi Bahri *earned a Doctoral degree from the University of Fukui, Japan in 2007. He is a professor at Hasanuddin University, Indonesia. His research interests include fractional Fourier transform, quaternion Fourier and wavelet transforms in Clifford algebra.*

https://orcid.org/0000-0003-1628-9163

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