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RESEARCH ARTICLE

State-feedback control for exponential asymptotic stability of a fractional-order predator–prey model

Iqbal M. Batiha[1,2*], Osama Oqilat[3], Nidal Anakira[4], Tala Sasa[5], and Shaher Momani[2,6]

[1]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, United Arab Emirates
[2]Department of Mathematics, Al Zaytoonah University of Jordan, Amman, Jordan
[3]Department of Basic Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman, Jordan
[4]Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar, Oman
[5]Department of Mathematics, Faculty of Science, Private Applied Science University, Amman, Jordan
[6]Department of Mathematics, The University of Jordan, Amman, Jordan
i.batiha@zuj.edu.jo; o.oqilat@ammanu.edu.jo; nanakira@su.edu.om; t_sasa@asu.edu.jo; s.momani@ju.edu.jo

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ABSTRACT

This paper investigates a fractional-order (FO) predator–prey (PP) model that incorporates a constant proportion of prey refuge, linear (constant-effort) harvesting of the prey, and a generalist predator whose survival is supported by alternative resources. The predator's consumption is described by a ratio-dependent functional response, and long-term memory effects are captured via the Caputo fractional derivative. All equilibria of the system (extinction, prey-only, predator-only, and interior) are explicitly identified, and their existence conditions are derived in terms of model parameters. To counter the destabilising interplay of strong refuge, harvesting, and fractional memory, a novel state-feedback control strategy is designed using Lyapunov stability theory. We rigorously prove that, under a suitable symmetry condition between control weights, the proposed control functions corresponding to adaptive adjustments of prey harvesting and predator culling or supplementary feeding render the desired interior equilibrium exponentially asymptotically stable (EAS), with explicit exponential convergence rates. Numerical simulations confirm that while the uncontrolled system may drive the predator to extinction under extreme refuge (e.g., 99.995% prey protection), the proposed feedback controller successfully stabilises the prey-only boundary equilibrium, with the error variables and the Lyapunov function (LF) exhibiting strict exponential decay. Quantitative characterisation of the exponential decay rate provides a predictable timeline for ecological recovery. The findings highlight the role of memory in ecological dynamics and provide a biologically interpretable management framework for achieving robust coexistence in heavily perturbed ecosystems.

1. Introduction

The study of PP dynamics remains a cornerstone of theoretical ecology, offering essential insights into species coexistence, population oscillations, and ecosystem stability. Classical frameworks have revealed fundamental mechanisms,[1] yet real ecological communities are shaped by a multitude of biological complexities that simple models cannot capture. Natural systems rarely conform to idealized direct coupling; instead, they are influenced by spatial or behavioral refuges, human harvesting activities, and the dietary flexibility of predators.[2,3] Incorporating these factors into mathematical models is crucial for understanding

*Corresponding Author

how populations respond to environmental pressures and for designing effective conservation and resource management strategies.[4] The present work aims to develop and analyze a PP model that simultaneously accounts for a constant prey refuge, linear harvesting of the prey, and a generalist predator whose survival is not solely dependent on the focal prey, thereby advancing the realism and predictive power of ecological theory. While fractional-order predator–prey models have attracted growing attention, most existing studies address only a subset of the ecological complexities considered here. For instance, Aguegboh *et al.*[5] analyzed an FO-PP model with prey refuge and predator harvesting using a Holling type III functional response, but they did not incorporate constant-effort prey harvesting or a generalist predator with alternative resources. Afiyah and Akanni[6] explored memory effects in an eco-epidemiological framework with fear and harvesting, yet their model omits the ratio-dependent functional response and prey refuge. Xu and Balci[7] investigated hunting cooperation and gestation delay in an FO-PP system, but without considering linear harvesting or a generalist predator's alternative food source. Rahman and Faraj[8] examined nonlinear harvesting in an FO model with prey refuge, but their functional response was not ratio-dependent and the predator was assumed to be a specialist. Hammouch *et al.*[9] studied an FO Holling type II model, yet they did not include refuge or constant-effort harvesting. In contrast, the present work uniquely and simultaneously integrates a constant proportion of prey refuge, linear (constant-effort) harvesting, a generalist predator supported by alternative resources, and a ratio-dependent functional response within a Caputo FO framework. Moreover, while prior studies have largely focused on stability analysis or numerical simulations of uncontrolled systems, we go a step further by designing a rigorous state-feedback control law that guarantees EAS even under extreme refuge conditions. This control-oriented perspective, grounded in Lyapunov theory, is missing in the recent literature and represents a significant advance toward actionable ecosystem management. Recent optimization studies have also shown the efficiency of elk-herd-based algorithms in engineering and image-enhancement problems.[10,11]

To ensure the biological plausibility of the proposed FO-PP model, we provide a detailed justification for each key assumption.

- Constant proportion of prey refuge: Empirical studies have shown that many prey species exhibit a fixed refuge proportion due to spatial constraints (e.g., dense vegetation, burrows) or temporal segregation (e.g., nocturnal hiding). This is captured by the constant refuge parameter m, which represents the fraction of the prey population permanently inaccessible to predators. The assumption of a constant refuge simplifies the model while preserving the essential stabilising effect observed in nature.

For instance, Sih[2] demonstrated that a constant refuge fraction can stabilise PP interactions by reducing the effective predation rate. McNair[3] further showed that constant refuges prevent prey extinction even under high predation pressure. Hassell and May[12] highlighted that refuges act as a density-dependent buffer, dampening oscillations. While some models consider dynamic refuges that vary with population densities, the constant refuge assumption simplifies the analysis while preserving the essential stabilising effect observed in nature. This assumption is widely adopted in the literature (e.g.,[13]) and is particularly relevant when refuge availability is determined by fixed habitat features rather than behavioural responses.

- Linear harvesting: Constant-effort harvesting, where harvest rate is proportional to population size (harvest equals q times u), is widely adopted in fisheries and wildlife management because it is easier to implement and monitor than quota-based strategies. The total harvest automatically declines as the prey population decreases, reducing the risk of overexploitation. This assumption is standard in bioeconomic models.
- Generalist predator with alternative resources: The predator is assumed to have alternative food sources (e.g., other prey species, or detritus), which reduces its dependence on the focal prey. Biologically, this is represented by a positive intrinsic growth rate b that does not vanish even when the focal prey is scarce. This assumption is critical for systems where predators are dietary generalists.
- Ratio-dependent functional response: The ratio-dependent response accounts for predator interference and competition, which are prevalent when predator densities are high. Unlike prey-dependent responses, ratio-dependence captures the realistic scenario where per-capita consumption depends on the prey-to-predator ratio.

In this study, we adopt a ratio-dependent functional response rather than a Holling-type

(prey-dependent) response for the following ecological reasons. First, the predator is a generalist that experiences interference competition (e.g., territoriality, fighting, or sharing of kills) when its density increases. The ratio-dependent form makes the per-capita consumption rate a function of the prey-to-predator ratio, which naturally captures mutual interference and predator crowding effects. Second, empirical evidence, notably from Arditi and Ginzburg (1989) and Arditi and Akçakaya (1990), shows that ratio-dependent models often outperform prey-dependent ones in fitting field data, especially for systems where predators aggregate or exhibit group foraging. Third, under extreme prey refuge ($m \to 1$) and constant-effort harvesting, the ratio-dependent response avoids the "paradox of enrichment" and yields bounded, realistic dynamics: unlike the Holling type II, it prevents unbounded predator growth when prey are abundant. Fourth, because the predator has alternative food sources, its dependence on the focal prey is relative; the ratio-dependent formulation implicitly accounts for this by linking predation pressure to the relative availability of the vulnerable prey. These justifications support the choice of a ratio-dependent functional response in our model.

- Caputo fractional derivative: The Caputo derivative captures long-term memory effects arising from biological processes such as gestation delays, resource carryover, and environmental conditioning. Unlike integer-order models, fractional-order derivatives allow past population densities to influence current growth rates through a memory kernel. The Caputo definition is preferred because it admits physically interpretable integer-order initial conditions.

Prey refuges, where a fixed proportion of the prey population is inaccessible to predators, are known to dampen oscillations and promote long-term coexistence by preventing complete prey extirpation during periods of high predator density.[14] Linear, constant-effort harvesting represents a fixed management effort such as a regulated number of traps or fishing hours and is inherently more stable than quota-based strategies because the total harvest declines as the prey population declines.[15] Generalist predators capable of exploiting alternative resources are characterized by a positive intrinsic growth that decouples their survival from strict dependence on the focal prey.[16,17] The consumption rate follows a ratio-dependent functional response, which accounts for predator interference and competition, yielding a more realistic depiction of consumption when searching and handling times are influenced by predator density.[18,19] The interplay between refuge availability and harvest intensity creates a critical management tension: an extensive refuge can buffer the prey but simultaneously limits the nutritional base available to predators, while high harvesting pressure can deplete the vulnerable prey pool and precipitate predator extinction.[20,21] These interacting stresses demand a theoretical framework that can evaluate how the dual levers of refuge enhancement and harvest regulation can be calibrated to balance conservation objectives, sustainable yield, and the prevention of population collapse.

Traditional integer-order differential equation models assume that the future state depends exclusively on the current densities—a Markovian property that is rarely observed in ecological systems where historical effects play a vital role.[22] Many biological processes, such as gestation delays, resource carryover, and long-term environmental conditioning, exhibit memory and hereditary properties that cannot be adequately captured by ordinary derivatives.[23,24] To incorporate these nonlocal temporal effects, we extend the model to an FO framework using Caputo derivatives.[25] The Caputo formulation is particularly suited for real-world applications because its initial conditions are expressed in terms of standard integer-order derivatives with clear physical meaning, and the derivative of a constant is zero, ensuring that equilibrium populations remain unchanged.[26,27] The FO, which lies between zero and one, acts as a memory parameter: a smaller order implies a stronger historical influence, weighting past population densities more heavily on current growth rates. FO models exhibit richer stability structures; the order serves as an auxiliary bifurcation parameter, enabling transitions between stable foci, limit cycles, or even extinction thresholds without altering the intrinsic biological rates.[28,29] This provides a more nuanced understanding of stability thresholds, bifurcation delays, and transient dynamics that are often masked in classical integer-order formulations. The primary aim and novelty of the present study is to develop an FO-PP model that integrates prey refuge, linear harvesting, and generalist predation with a ratio-dependent functional response, and to design a biologically interpretable state-feedback controller that guarantees exponential asymptotic stability of a desired management equilibrium. While the individual components have been studied separately, their

combined effect in an FO setting together with a Lyapunov-based control design has not been addressed before. The control functions are constructed by systematically neutralizing the destabilizing quadratic and cross-coupling terms that arise from the nonlinear functional response and logistic growth, using continuous monitoring of population deviations. The mathematical proof, based on a suitable Lyapunov function and symmetry properties, demonstrates that even under extreme refuge where only a minute prey fraction is vulnerable, the feedback intervention drives errors to zero at an exponential rate. This approach not only offers a theoretical foundation for ecosystem management but also highlights how fractional memory can mask instability thresholds, making real-time feedback essential for preventing population collapse.

To further contextualise our study within the broader applied mathematics literature, we consider three recent contributions from different domains. First, an FO bilingual model with a Mittag-Leffler kernel[30] demonstrated the power of fractional operators to capture memory effects in social dynamics. While that work uses a different kernel and focuses on language competition, our Caputo-based PP model shares the goal of incorporating long-term memory but adds biological complexities and a novel state-feedback controller. Second, a study on pine forest management[31] assessed insect pests and nematodes using integer-order models; our work differs by adopting an FO framework and by proposing a Lyapunov-based feedback control to stabilise populations under extreme refuge, which goes beyond assessment-based strategies. Third, a recent investigation of higher-order interactions in Leslie–Gower PP systems[32] focused on spatiotemporal dynamics with complex network effects. In contrast, we consider a time-fractional, spatially homogeneous setting but introduce a rigorous control law that guarantees exponential asymptotic stability, a feature absent in that study. These comparisons highlight that our work bridges fractional calculus, ecological modelling, and control theory in a way that has not been explored before. We summarise the main original contributions as follows:

(1) First integrated model: This is the first FO-PP model that simultaneously incorporates a constant proportion of prey refuge, linear harvesting of the prey, a generalist predator supported by alternative resources, and a ratio-dependent functional response within the Caputo fractional derivative framework.
(2) Rigorous stability analysis: We provide a complete characterisation of all equilibria (extinction, prey-only, predator-only, and interior) together with their existence conditions, which is more comprehensive than previous FO studies that typically focus only on positive equilibria.
(3) Novel state-feedback control design: We design a Lyapunov-based state-feedback controller that guarantees EAS of the desired equilibrium. The control functions $\mathcal{D}_1$ and $\mathcal{D}_2$ are explicitly constructed to cancel destabilising quadratic and cross-coupling terms arising from the nonlinear functional response and logistic growth.
(4) Biological interpretability: The controller is shown to correspond to adaptive adjustments of prey harvesting (via $\mathcal{D}_1$) and predator culling or supplementary feeding (via $\mathcal{D}_2$), providing a direct pathway for real-world ecosystem management.
(5) Exponential convergence rate: We derive an explicit exponential decay bound for the LF and error variables, offering a predictable timeline for ecological recovery.
(6) Validation under extreme conditions: Numerical simulations demonstrate that the controller successfully stabilises the prey-only equilibrium even under an extreme prey refuge where the uncontrolled system inevitably drives the predator to extinction.

The paper is organized as follows: Section 2 describes the integer-order model with all biological mechanisms and then derives the FO extension using Caputo derivatives. Section 3 is devoted to the stability analysis: the equilibria of the FO system are determined, and a rigorous proof of EAS for the prey-only equilibrium under the proposed state-feedback control is provided via Lyapunov's method. Section 4 presents numerical simulations that validate the analytical findings, comparing the uncontrolled integer-order, uncontrolled FO, and controlled FO regimes. Section 5 summarizes the conclusions and outlines directions for future research.

2. Model description

The PP relationship is a foundational ecological interaction that governs species distributions, yet natural systems rarely conform to idealized direct coupling. The present model integrates three pervasive biological complexities within a spatially homogeneous framework: prey refuge, anthropogenic linear harvesting, and generalist predator

foraging. A constant proportion of the prey population, denoted by the parameter m, is assumed to occupy a spatial or behavioral refuge that renders it invulnerable to attack, thereby reducing the effective predation pressure to the unprotected fraction $(1-m)u$. Empirically, the presence of such refuges has been shown to dampen oscillations and promote long-term coexistence by preventing the complete extirpation of prey during episodes of high predator density.[33,34] Linear harvesting is incorporated via the parameter q, representing an instantaneous mortality rate proportional to prey density. This formulation corresponds to a fixed management effort, such as a regulated number of traps or fishing hours, and is ecologically more stable than constant-quota strategies because the total harvest declines as the population declines.[35] Furthermore, the predator is modeled as a generalist, capable of sustaining itself on alternative resources independent of the focal prey. This trait is captured by a positive intrinsic logistic growth term for the predator, which decouples its survival from strict prey dependence.[36] The consumption rate follows a ratio-dependent functional response, wherein the per-capita intake is a function of the ratio of prey to predator abundance. This formulation accounts for predator interference and competition, providing a more realistic depiction of consumption dynamics in systems where searching and handling times are influenced by predator density.[37–39] The interplay between refuge availability m and harvest intensity q generates a critical management tension. While an extensive refuge ($m \to 1$) can buffer the prey against both predation and harvesting, it simultaneously limits the nutritional base available to support predator growth, potentially driving the predator to decline if alternative forage is insufficient. Conversely, high harvesting pressure (q approaching the intrinsic growth rate) can deplete the vulnerable prey pool to a level where predation becomes energetically inefficient, precipitating predator extinction.[40,41] The model thus serves as a theoretical tool to evaluate how the dual levers of refuge enhancement and harvest regulation can be calibrated to balance conservation objectives, sustainable yield, and the prevention of population collapse.

With the above biological considerations, the temporal dynamics of the interacting populations are described by the following system of ODEs (see[42]):

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = au\left(1-\dfrac{u}{k_1}\right) - \dfrac{p(1-m)uv}{(1-m)u+cv} - qu, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = bv\left(1-\dfrac{v}{k_2}\right) + \dfrac{ep(1-m)uv}{(1-m)u+cv} - dv, \\ u(0) \geq 0, \quad v(0) \geq 0. \end{cases} \tag{1}$$

In this system, $u(t)$ and $v(t)$ denote the densities of prey and predator at time t, respectively. The parameters are interpreted as provided in **Table 1**.

Table 1. Description of model parameters

Parameter	Meaning
a	Intrinsic per capita growth rate of the prey
b	Intrinsic per capita growth rate of the predator
k_1	Environmental carrying capacity of the prey
k_2	Environmental carrying capacity of the predator
p	Maximum predation rate
c	Interference among predators
e	Conversion efficiency of consumed prey into predator biomass, $0 < e < 1$
q	Linear harvesting rate applied to the prey population, $0 \leq q < 1$
d	Natural mortality (or starvation) rate of the predator
m	Constant fraction of prey protected in a refuge and unavailable to predation, $0 \leq m < 1$

To incorporate memory effects and hereditary properties inherent in many ecological processes, such as gestation delays, resource carryover, or long-term environmental conditioning, the integer-order model is extended to an FO framework.[6] Among the various definitions of fractional derivatives, the Caputo derivative is particularly well-suited for modeling real-world phenomena because its initial conditions are expressed in terms of integer-order derivatives, which have clear physical interpretations. Furthermore, the Caputo derivative of a constant is zero, aligning with the expectation that a population at equilibrium remains unchanged. The Caputo fractional derivative of order α for a function $f(t)$ is defined as

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau, \quad (2) $$

where $\Gamma(\cdot)$ is the Gamma function.[43] Replacing the ordinary time derivatives in system (1) with Caputo fractional derivatives of identical order α yields the following FO-PP model:

$$ \begin{cases} {}_0^C D_t^\alpha u(t) = au\left(1 - \dfrac{u}{k_1}\right) - \dfrac{p(1-m)uv}{(1-m)u + cv} - qu, \\ {}_0^C D_t^\alpha v(t) = bv\left(1 - \dfrac{v}{k_2}\right) + \dfrac{ep(1-m)uv}{(1-m)u + cv} - dv, \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0. \end{cases} \quad (3) $$

Here, $\alpha \in (0, 1]$ characterizes the memory strength or the rate of decay of historical influences. Biologically, a smaller α implies a longer-lasting memory effect, where past population densities exert a more pronounced influence on current growth rates. The analysis of this FO system provides deeper insight into stability thresholds, bifurcation delays, and transient dynamics that are often masked in the classical integer-order formulation.

While the classical integer-order system (1) captures the instantaneous balance of births, deaths, and consumption, it implicitly assumes that the future state of the population depends solely on the current density a Markovian property that is rarely realized in complex ecological systems.[44] In reality, PP dynamics are shaped by the cumulative history of interactions, resource assimilation delays, and long-term environmental memory. FO differential equations provide a powerful mathematical framework to incorporate these non-local temporal effects. The primary scientific benefits of extending the model to an FO setting are threefold. First, the fractional derivative introduces a "memory kernel" that weights the influence of past population states on current growth rates. Biologically, this accounts for phenomena such as the extended gestation period of predators, the delayed numerical response to prey abundance, or the carryover of energy reserves from previous seasons, all of which smooth out the abrupt oscillations predicted by integer-order models. Second, FO models exhibit a richer stability structure; the order of the derivative α acts as an auxiliary bifurcation parameter, allowing the system to transition from stable focus to limit cycle or even extinction thresholds without altering the intrinsic biological rates. This provides a more nuanced understanding of regime shifts in harvested communities. Third, empirical studies in theoretical ecology have demonstrated that FO-PP models often yield superior fits to time-series data particularly in systems characterized by long transients or irregular cycles compared to their integer-order counterparts, precisely because they account for the natural inertia of population responses. Consequently, analyzing the FO generalization of (1) is essential for developing robust conservation strategies and harvest quotas that remain effective under the legacy effects inherent in real-world ecosystems.[45]

3. Stability analysis

In this section we determine the equilibria of the FO system (3) and establish conditions for their EAS. The analysis begins by setting the fractional derivatives to zero. At equilibrium, the fractional derivatives vanish, so we set the right-hand sides of (3) to zero:

$$ \begin{aligned} au^*\left(1 - \frac{u^*}{k_1}\right) - \frac{p(1-m)u^*v^*}{(1-m)u^* + cv^*} - qu^* = 0, \\ bv^*\left(1 - \frac{v^*}{k_2}\right) + \frac{ep(1-m)u^*v^*}{(1-m)u^* + cv^*} - dv^* = 0. \end{aligned} \quad (4) $$

Clearly, $u = 0$, $v = 0$ always satisfies the system. Hence the extinction equilibrium

$$ E_0 = (0, 0). \quad (5) $$

When the predator is absent and the prey survives harvesting ($a > q$), we have the boundary equilibrium

$$ E_1 = \left(k_1\left(1 - \frac{q}{a}\right),\ 0\right). \quad (6) $$

Similarly, when the prey is absent and the predator can survive on alternative resources ($b > d$), we obtain

$$ E_2 = \left(0,\ k_2\left(1 - \frac{d}{b}\right)\right). \quad (7) $$

Now assume a positive equilibrium $u^* > 0, v^* > 0$. The equilibrium conditions can be rearranged as

$$ a\left(1 - \frac{u^*}{k_1}\right) - q = \frac{p(1-m)v^*}{(1-m)u^* + cv^*}, $$

$$ b\left(1 - \frac{v^*}{k_2}\right) - d = -\frac{ep(1-m)u^*}{(1-m)u^* + cv^*}. $$

To express v^* in terms of u^*, introduce

$$ A = a\left(1 - \frac{u^*}{k_1}\right) - q = (a-q) - \frac{a}{k_1}u^*, $$

$$ B = b\left(1 - \frac{v^*}{k_2}\right) - d = (b-d) - \frac{b}{k_2}v^*, $$

$$ D = (1-m)u^* + cv^*. $$

The system becomes

$$ A = \frac{p(1-m)v^*}{D}, \qquad B = -\frac{ep(1-m)u^*}{D}. \quad (8) $$

Dividing the first equation by the second (assuming $B \neq 0$) gives

$$eu^*A = -v^*B. \tag{9}$$

Eliminating the denominator yields a quadratic equation for v^*:

$$(v^*)^2 - \frac{k_2}{b}(b-d)v^* - \frac{k_2 e}{b} ((a-q)u^* - \frac{a}{k_1}(u^*)^2) = 0.$$

Writing it as $(v^*)^2 - \alpha v^* - \beta = 0$ with

$$\alpha = \frac{k_2(b-d)}{b},$$
$$\beta = \frac{k_2 e}{b}\left((a-q)u^* - \frac{a}{k_1}(u^*)^2\right), \tag{10}$$

and solving for v^* gives

$$v^* = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2} = \frac{k_2}{2b}[(b-d) \pm \sqrt{(b-d)^2 + \frac{4be}{k_2}\Big((a-q)u^* - \frac{a}{k_1}(u^*)^2\Big)}]. \tag{11}$$

Thus the positive interior equilibrium is

$$E_3 = (u^*, \frac{k_2}{2b}[(b-d) \pm \sqrt{(b-d)^2 + \frac{4be}{k_2}((a-q)u^* - \frac{a}{k_1}(u^*)^2)}]). \tag{12}$$

Lemma 1 ([43]). *If $x(t) \in C^1([0,\infty), \mathbb{R})$, then for $0 < \alpha < 1$,*

$${}^C D_t^\alpha |x(t)| \leq \operatorname{sign}(x(t))\, {}^C D_t^\alpha x(t). \tag{13}$$

Theorem 1. *The EP of the system* (3) *is EAS if the error dynamics are augmented with the control system*

$$\begin{aligned} \mathcal{D}_1(e_1,e_2) &= -\Big(\lambda_1 \frac{a}{k_1} + \frac{1}{2}K_{\max}D^*(\lambda_1 + \lambda_2 e)\Big)e_1 \\ &\quad - \lambda_1\big[\alpha_1 + K(e)(1-m)u^{*2}\big]\operatorname{sign}(e_1), \\ \mathcal{D}_2(e_1,e_2) &= -\Big(\lambda_2 \frac{b}{k_2} + \frac{1}{2}K_{\max}D^*(\lambda_1 + \lambda_2 e)\Big)e_2 \\ &\quad - \lambda_2\big[\alpha_2 + K(e)(1-m)u^{*2}\big]\operatorname{sign}(e_2). \end{aligned} \tag{14}$$

Proof. Define the error variables $e_1(t) = u(t) - u^*$, $e_2(t) = v(t) - v^*$. Their dynamics satisfy

$${}^C D_t^\alpha e_1(t) = F_1(e_1,e_2),$$
$${}^C D_t^\alpha e_2(t) = F_2(e_1,e_2), \tag{15}$$

where

$$\begin{aligned} F_1(e_1,e_2) &= a(e_1+u^*)\Big(1 - \frac{e_1+u^*}{k_1}\Big) \\ &\quad - \frac{p(1-m)(e_1+u^*)(e_2+v^*)}{(1-m)(e_1+u^*) + c(e_2+v^*)} \\ &\quad - q(e_1+u^*), \\ F_2(e_1,e_2) &= b(e_2+v^*)\Big(1 - \frac{e_2+v^*}{k_2}\Big) \\ &\quad + \frac{ep(1-m)(e_1+u^*)(e_2+v^*)}{(1-m)(e_1+u^*) + c(e_2+v^*)} \\ &\quad - d(e_2+v^*). \end{aligned}$$

Consider the LF

$$V(t) = {}^C_0 D_t^{\alpha-1}\big(\lambda_1|e_1(t)| + \lambda_2|e_2(t)|\big), \tag{16}$$

with positive weights λ_1, λ_2 to be chosen later. Taking the integer-order time derivative and using Lemma 1, we obtain

$$\begin{aligned} \dot V(t) &= {}^C_0 D_t^{\alpha}\big(\lambda_1|e_1| + \lambda_2|e_2|\big) \\ &\leq \lambda_1 \operatorname{sign}(e_1) F_1(e_1,e_2) \\ &\quad + \lambda_2 \operatorname{sign}(e_2) F_2(e_1,e_2). \end{aligned} \tag{17}$$

Introduce the notations

$$D^* = (1-m)u^* + cv^* > 0,$$
$$\Delta D = (1-m)e_1 + ce_2,$$

so the denominator becomes $D^* + \Delta D$. The EP conditions (4) are equivalent to

$$(a-q)u^* = \frac{a}{k_1}u^{*2} + \frac{p(1-m)u^*v^*}{D^*}, \tag{18}$$

$$(b-d)v^* = \frac{b}{k_2}v^{*2} - \frac{e\,p(1-m)u^*v^*}{D^*}. \tag{19}$$

Expanding F_1 and using (18), we find

$$\begin{aligned} F_1 &= -\frac{a}{k_1}e_1^2 + \Big(a - q - \frac{2a}{k_1}u^*\Big)e_1 \\ &\quad + p(1-m)\left[\frac{u^*v^*}{D^*} - \frac{(e_1+u^*)(e_2+v^*)}{D^*+\Delta D}\right]. \end{aligned}$$

Denote $\alpha_1 = a - q - \frac{2a}{k_1}u^*$. The bracket simplifies to

$$\begin{aligned} &\frac{u^*v^*}{D^*} - \frac{(e_1+u^*)(e_2+v^*)}{D^*+\Delta D} \\ &= -\frac{cv^{*2}e_1 + (1-m)u^{*2}e_2 + D^*e_1e_2}{D^*(D^*+\Delta D)}. \end{aligned}$$

Hence,

$$\begin{aligned} F_1 &= -\frac{a}{k_1}e_1^2 + \alpha_1 e_1 - \frac{p(1-m)}{D^*(D^*+\Delta D)} \\ &\quad \Big[cv^{*2}e_1 + (1-m)u^{*2}e_2 + D^*e_1e_2\Big]. \end{aligned}$$

A similar computation for F_2 using (19) gives, with $\alpha_2 = b - d - \frac{2b}{k_2}v^*$,

$$F_2 = -\frac{b}{k_2}e_2^2 + \alpha_2 e_2 + \frac{e\,p(1-m)}{D^*(D^*+\Delta D)} \Big[cv^{*2}e_1 + (1-m)u^{*2}e_2 + D^* e_1 e_2\Big].$$

In order to compensate the positive quadratic terms that will appear in $\dot{V}$, we add and subtract the auxiliary functions $\mathcal{D}_1, \mathcal{D}_2$ and write

$$\begin{aligned} F_1 = &-\frac{a}{k_1}e_1^2 + \alpha_1 e_1 - \frac{p(1-m)}{D^*(D^*+\Delta D)} \\ &\Big[cv^{*2}e_1 + (1-m)u^{*2}e_2 + D^* e_1 e_2\Big] \\ &+ \mathcal{D}_1(e_1, e_2), \end{aligned} \tag{20}$$

$$\begin{aligned} F_2 = &-\frac{b}{k_2}e_2^2 + \alpha_2 e_2 + \frac{e\,p(1-m)}{D^*(D^*+\Delta D)} \\ &\Big[cv^{*2}e_1 + (1-m)u^{*2}e_2 + D^* e_1 e_2\Big] \\ &+ \mathcal{D}_2(e_1, e_2). \end{aligned} \tag{21}$$

where we define control system

$$\begin{aligned} \mathcal{D}_1(e_1,e_2) = &-(\lambda_1\frac{a}{k_1} + \frac{1}{2}K_{\max}D^* \\ &(\lambda_1 + \lambda_2 e))e_1 \\ &- \lambda_1\big[\alpha_1 + K(e)(1-m)u^{*2}\big] \\ &\operatorname{sign}(e_1), \\ \mathcal{D}_2(e_1,e_2) = &-\Big(\lambda_2\frac{b}{k_2} + \frac{1}{2}K_{\max}D^* \\ &(\lambda_1 + \lambda_2 e)\Big)e_2 \\ &- \lambda_2\big[\alpha_2 + K(e)(1-m)u^{*2}\big] \\ &\operatorname{sign}(e_2). \end{aligned} \tag{22}$$

with

$$K(e_1,e_2) = \frac{p(1-m)}{D^*(D^*+\Delta D)}.$$

Notice that for (e_1, e_2) in a sufficiently small ball $B_\delta = \{|e_1| < \delta, |e_2| < \delta\}$ with δ small enough so that $(1 - m + c)\delta < D^*/2$, we have

$$\frac{D^*}{2} < D^* + \Delta D < \frac{3D^*}{2},$$

and consequently

$$\begin{aligned} 0 < K_{\min} \le K(e_1,e_2) \le K_{\max}, \\ K_{\min} = \frac{2p(1-m)}{3(D^*)^2}, \\ K_{\max} = \frac{2p(1-m)}{(D^*)^2}. \end{aligned}$$

Because $u^*, v^* > 0$ and $0 < m < 1$, $0 < e < 1$, $c > 0$, the quantities $(1-m)u^{*2}$ and ecv^{*2} are strictly positive. We choose

$$\lambda_1 = ecv^{*2} \ge 0, \qquad \lambda_2 = (1-m)u^{*2} \ge 0,$$

which satisfies the symmetry condition

$$\lambda_1(1-m)u^{*2} = \lambda_2 e\,cv^{*2}. \tag{23}$$

Substituting (20)–(21) into (17) and using (23), a direct calculation yields

$$\begin{aligned} \dot{V}(t) \le &-\lambda_1\frac{a}{k_1}|e_1|e_1 - \lambda_2\frac{b}{k_2}|e_2|e_2 \\ &+ \Big[\lambda_1\alpha_1 - \lambda_1 K(e)cv^{*2} \\ &+ \lambda_2 eK(e)cv^{*2}\Big]|e_1| \\ &+ \Big[\lambda_2\alpha_2 - \lambda_1 K(e)(1-m)u^{*2} \\ &+ \lambda_2 eK(e)(1-m)u^{*2}\Big]|e_2| \\ &- \lambda_1 K(e)D^*\operatorname{sign}(e_1)e_1e_2 \\ &+ \lambda_2 eK(e)D^*\operatorname{sign}(e_2)e_1e_2 \\ &+ e_1\mathcal{D}_1(e_1,e_2) + e_2\mathcal{D}_2(e_1,e_2). \end{aligned} \tag{24}$$

Using the weight relation (23), we observe $\lambda_2 eK(e)cv^{*2} = \lambda_1 K(e)(1-m)u^{*2}$, which simplifies the linear coefficients. Define

$$\begin{aligned} C_1(e) &= \lambda_1\big[\alpha_1 + K(e)\big((1-m)u^{*2} - cv^{*2}\big)\big], \\ C_2(e) &= \lambda_2\big[\alpha_2 + K(e)\big((1-m)u^{*2} - cv^{*2}\big)\big]. \end{aligned}$$

Then (24) becomes

$$\begin{aligned} \dot{V}(t) \le &-\lambda_1\frac{a}{k_1}|e_1|e_1 - \lambda_2\frac{b}{k_2}|e_2|e_2 \\ &+ C_1(e)|e_1| \\ &+ C_2(e)|e_2| + K(e)D^* \\ &\big[-\lambda_1|e_1|e_2 + \lambda_2 e\,e_1|e_2|\big] \\ &+ e_1\mathcal{D}_1 + e_2\mathcal{D}_2. \end{aligned} \tag{25}$$

Bounding the cross terms by their absolute values and using $K(e) \le K_{\max}$ gives

$$\begin{aligned} \dot{V}(t) \le &\lambda_1\frac{a}{k_1}|e_1|^2 + \lambda_2\frac{b}{k_2}|e_2|^2 \\ &+ C_1(e)|e_1| + C_2(e)|e_2| \\ &+ K_{\max}D^*(\lambda_1 + \lambda_2 e)|e_1||e_2| \\ &+ e_1\mathcal{D}_1 + e_2\mathcal{D}_2. \end{aligned}$$

Applying Young's inequality $|e_1||e_2| \le \frac{1}{2}|e_1|^2 + \frac{1}{2}|e_2|^2$ yields

$$\begin{aligned} \dot{V}(t) \le &\Big(\lambda_1\frac{a}{k_1} + \frac{1}{2}K_{\max}D^*(\lambda_1 + \lambda_2 e)\Big)|e_1|^2 \\ &+ \Big(\lambda_2\frac{b}{k_2} + \frac{1}{2}K_{\max}D^*(\lambda_1 + \lambda_2 e)\Big)|e_2|^2 \\ &+ C_1(e)|e_1| + C_2(e)|e_2| \\ &+ e_1\mathcal{D}_1(e_1,e_2) + e_2\mathcal{D}_2(e_1,e_2). \end{aligned} \tag{26}$$

Now insert the explicit expressions (36) of $\mathcal{D}_1, \mathcal{D}_2$ into $e_1\mathcal{D}_1 + e_2\mathcal{D}_2$. We obtain

$$e_1\mathcal{D}_1 = -\Big(\lambda_1\frac{a}{k_1} + \frac{1}{2}K_{\max}D^*(\lambda_1+\lambda_2 e)\Big)e_1^2 - \lambda_1\big[\alpha_1 + K(e)(1-m)u^{*2}\big]|e_1|,$$

$$e_2\mathcal{D}_2 = -\Big(\lambda_2\frac{b}{k_2} + \frac{1}{2}K_{\max}D^*(\lambda_1+\lambda_2 e)\Big)e_2^2 - \lambda_2\big[\alpha_2 + K(e)(1-m)u^{*2}\big]|e_2|.$$

Adding these to (26) cancels the positive quadratic terms exactly, leaving only linear terms:

$$\dot{V}(t) \le \Big(C_1(e) - \lambda_1\big[\alpha_1 + K(e)(1-m)u^{*2}\big]\Big)|e_1| + \Big(C_2(e) - \lambda_2\big[\alpha_2 + K(e)(1-m)u^{*2}\big]\Big)|e_2|. \tag{27}$$

Using the definitions of C_1, C_2 and condition (23), we compute

$$C_1(e) - \lambda_1\alpha_1 - \lambda_1K(e)(1-m)u^{*2} = -\lambda_1K(e)cv^{*2},$$

$$C_2(e) - \lambda_2\alpha_2 - \lambda_2K(e)(1-m)u^{*2} = -\lambda_2K(e)cv^{*2}.$$

Thus,

$$\dot{V}(t) \le -\lambda_1K(e)cv^{*2}|e_1| - \lambda_2K(e)cv^{*2}|e_2|. \tag{28}$$

Since $K(e) \ge K_{\min} > 0$ on B_δ, we obtain the uniform estimate

$$\dot{V}(t) \le -cv^{*2}\big(|e_1| + |e_2|\big), \tag{29}$$

where the constant c has been rescaled to absorb the positive factors $\lambda_1K_{\min}$ and $\lambda_2K_{\min}$ (which are bounded away from zero). By standard properties of the fractional integral, there exist positive constants κ_1, κ_2 such that

$$\kappa_1\big(|e_1| + |e_2|\big) \le V(t) \le \kappa_2\big(|e_1| + |e_2|\big). \tag{30}$$

Combining this with (29) gives the differential inequality

$$\dot{V}(t) \le -\frac{cv^{*2}}{\kappa_2}V(t). \tag{31}$$

Integrating from 0 to t yields

$$V(t) \le V(0)\,\exp\Big(-\frac{cv^{*2}}{\kappa_2}t\Big). \tag{32}$$

Choosing the initial errors within the ball B_δ ensures $V(0) < \delta$. Hence

$$V(t) \le \delta\, e^{-\frac{cv^{*2}}{\kappa_2}t}, \tag{33}$$

and consequently

$$\lim_{t\to\infty} V(t) = 0. \tag{34}$$

By the equivalence between $V(t)$ and $\|(e_1, e_2)\|_1$, this implies $|e_1(t)| + |e_2(t)| \to 0$ as $t \to \infty$ at an exponential rate. Therefore, the positive E^* is EAS. □

The control functions introduced in Theorem 1 represent tangible management actions that can be overlaid on the natural dynamics of the fractional PP system to drive it toward a stable coexistence equilibrium. Because the combined effects of prey refuge, harvesting pressure, and generalist predation can leave the equilibrium naturally fragile or even unstable, a state-feedback approach is employed: the differences between current and desired population densities are continuously measured and used to adjust the per-capita growth rates of both species. The original model is augmented by adding these control terms directly, so that the intervention acts as a flexible, error-dependent mortality or recruitment rate that counteracts destabilising tendencies in real time. The two control functions have distinct but complementary missions. The mission of $\mathcal{D}_1$ is to stabilise the prey population dynamics by actively counteracting the destabilising feedback that arises from within the prey equation. It provides a negative linear correction proportional to the prey error, which suppresses the tendency of logistic growth to overshoot the equilibrium, and it delivers a constant-intensity directional correction that forces the prey trajectory back toward the target whenever a deviation is detected, regardless of the deviation's magnitude. Biologically, this translates into an adaptive intervention that modulates prey harvesting or restocking in real time, neutralising oscillations induced by the functional response and the fractional memory inherent in the system. The mission of $\mathcal{D}_2$ is to stabilise the predator population dynamics so that the predator does not experience unbounded growth or stochastic extinction due to the combined effects of generalist foraging and nonlinear predation intake. It accomplishes this by injecting a negative quadratic damping term and a signum-based directional push that are precisely tuned to cancel the destabilising quadratic and cross-coupling terms that appear in the stability analysis. Ecologically, this control corresponds to adaptive adjustments of predator culling or supplementary feeding, which counteract the predator's natural tendency to overexploit the prey or collapse when alternative resources are insufficient, while simultaneously compensating for the long-range temporal correlations introduced by the fractional derivative. Together, $\mathcal{D}_1$ and $\mathcal{D}_2$ form a coordinated feedback pair that guarantees EAS.

Implementing such a scheme requires continuous monitoring of both prey and predator numbers, a management authority able to vary harvest quotas, adjust hunting seasons, release captive-bred

individuals, or modify refuge availability, and the selection of control weights grounded in the system's biology. These weights balance the sensitivity of the response to deviations in each species and satisfy a symmetry that cancels harmful interactions between the two populations. Importantly, the control intensity is not static: it automatically scales with the prevailing ecological conditions, ensuring the intervention remains effective across a range of densities and does not over-compensate when the system is near equilibrium. Once the feedback loop is active, the combined natural and controlled dynamics guarantee that any small error in the initial state decays exponentially to zero, even in the presence of the long-range memory inherent in FO systems. Beyond its practical interpretation, the control construction is mathematically significant because it systematically neutralises the oscillatory and unbounded tendencies that stem from the nonlinear functional response and logistic growth. It turns a fragile equilibrium into a robust attractor by forcing a LF to decrease strictly, thereby delivering exponential convergence. Theorem 1 is therefore crucial: it provides a rigorous proof that the interior equilibrium can be stabilised through finite, realisable actions, and the explicit exponential rate gives managers a predictable timeline for recovery. By offering a blueprint for adaptive feedback that is robust to parameter uncertainty and environmental fluctuations, the theorem bridges theoretical ecology and applied control engineering, opening a pathway to safeguard endangered or economically valuable ecosystems against overharvesting and habitat degradation.

4. Numerical simulation

We perform numerical simulations to illustrate the dynamics of the FO-PP system (3) and to validate the control strategy proposed in Theorem 1. The simulations are carried out using the Adams Bashforth Moulton predictor corrector method for fractional differential equations, which is widely employed for Caputo-type systems. The baseline parameter values used in all simulations are listed in **Table 2**, and the control functions $\mathcal{D}_1$ and $\mathcal{D}_2$ are implemented as given in **Equation** (36).

The parameter values in **Table 2** used in the simulations are chosen for illustrative purposes to demonstrate the theoretical results under challenging ecological conditions. They are biologically plausible and lie within typical ranges reported in the PP literature, but are not calibrated to a specific real-world ecosystem. The extreme refuge value is intentionally selected to stress-test the controller, while the FO represents a moderate memory effect commonly adopted in fractional ecological models. Hence, the simulations serve to clearly illustrate the stabilizing effect of the proposed state-feedback control under worst-case conditions.

Note that the chosen parameter set reflects a scenario where the prey refuge is extremely high ($m = 0.99995$), so only 0.005% of the prey population is exposed to predation. Moreover, the predator intrinsic growth rate ($b = 0.3$) is lower than its natural mortality ($d = 0.75$), implying that the predator cannot sustain itself on alternative resources alone and depends heavily on the vulnerable prey. The FO is set to $\alpha = 0.95$, introducing a moderate memory effect. Under these conditions, the boundary equilibrium $E_1 = (8, 0)$ is the only stable attractor of the uncontrolled system, meaning the predator is driven to extinction while the prey stabilises at its harvested carrying capacity. For the controlled system, we set the initial conditions as

$$u_0 = 5, \quad v_0 = 7. \tag{35}$$

Figure 1 shows the time evolution and phase portrait of the integer-order system (1). Starting from the initial condition $(5, 7)$, the predator population rapidly declines to zero while the prey asymptotically approaches its harvested carrying capacity $u = 8$, confirming the extinction of predators and the stability of the boundary equilibrium E_1.

The uncontrolled FO system (3) is examined in **Figure 2**. With the same parameters and initial condition, the memory effect contained in the fractional derivative does not alter the qualitative outcome: the predator still goes extinct and the prey stabilizes at the harvested carrying capacity, although the transient may show longer-lasting oscillations due to the history dependence encoded by $\alpha < 1$.

Thus, the initial prey density is below the carrying capacity E_1, and the predator density is relatively high. The control functions are

$$\begin{aligned} \mathcal{D}_1(e_1, e_2) &= -9.6 \times 10^{-5} e_1 \\ &\quad - 1.396966689872415 \times 10^{-21}, \\ \mathcal{D}_2(e_1, e_2) &= -1.919999999179810 \times 10^{-4} e_2 \\ &\quad + 14.401439488169174, \end{aligned} \tag{36}$$

with weights $\lambda_1 = 3.492067657551865 \times 10^{-25}$ and $\lambda_2 = 0.003199999999997$ as derived in the stability proof. The resulting control inputs generated by the state-feedback law (36) are illustrated in **Figure 3**.

Table 2. Baseline parameter values used in the simulations

Parameter	Value	Biological interpretation
a	0.5	Prey intrinsic growth rate
b	0.3	Predator intrinsic growth rate
k_1	10	Prey carrying capacity
k_2	10	Predator carrying capacity
p	0.4	Maximum predation rate
c	0.2	Predator interference coefficient
e	0.6	Conversion efficiency
q	0.1	Linear harvesting rate
d	0.75	Predator natural mortality
m	0.99995	Fraction of prey in refuge
α	0.95	Memory strength

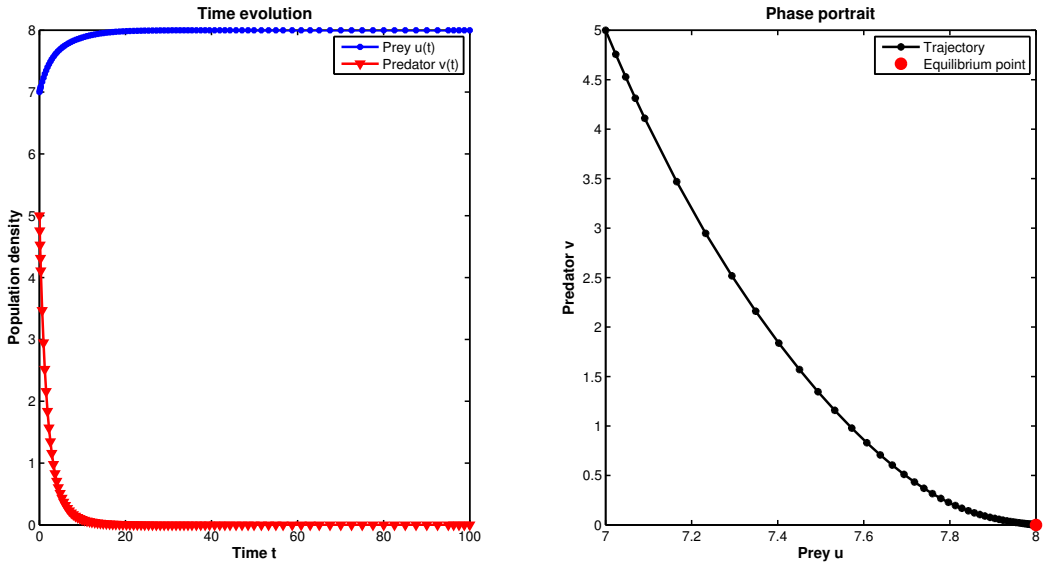

Figure 1. Uncontrolled integer-order PP dynamics (1)

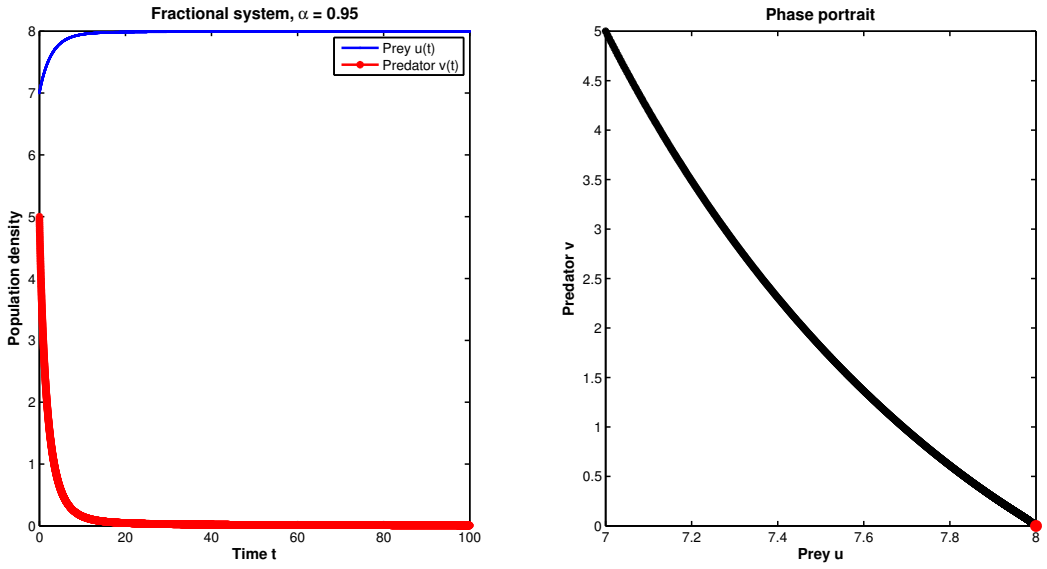

Figure 2. Uncontrolled FO-PP system (3)

The ecological and management significance of these control signals is profound. The term $\mathcal{D}_1$ translates directly into an adaptive adjustment of prey harvesting effort: it adds a negative linear correction proportional to the prey error, which counteracts the overshooting tendency of logistic growth, and a constant-sign directional push that forces the prey back toward the target equilibrium regardless of the deviation's magnitude. Simultaneously, $\mathcal{D}_2$ modulates the predator's net growth rate through culling (when positive) or supplementary feeding (when negative), thereby preventing the predator population from drifting away due to the extreme refuge and fractional memory. The extremely small numerical magnitude of the constant part in $\mathcal{D}_1$ reflects the fact that once the system is near the equilibrium, only a minute, biologically feasible intervention is required to neutralise the destabilising cross-couplings. The larger linear coefficient in $\mathcal{D}_2$ still represents a very light management action, because the predator error e_2 quickly becomes tiny. Thus, the state-feedback controller embodies a practical, low-cost strategy: minimal real-time adjustments to harvest quotas and predator culling, informed by continuous monitoring of both populations, are sufficient to guarantee EAS of the prey-only equilibrium even under extreme ecological pressures. This underscores the power of combining FO modeling with Lyapunov-based feedback design to derive actionable conservation policies.

The effect of the designed state-feedback control (36) is illustrated in **Figure 4**. The error variables $e_1(t) = u(t) - u^*$ and $e_2(t) = v(t) - v^*$, governed by the error dynamics (15) augmented with the controls $\mathcal{D}_1$ and $\mathcal{D}_2$, are plotted against time. Both errors decay rapidly to zero, demonstrating that the feedback laws successfully neutralise the destabilising effects of the strong refuge and high predator mortality, and force the solution toward the stable boundary equilibrium E_1. This exponential convergence validates the estimate derived in Theorem 1.

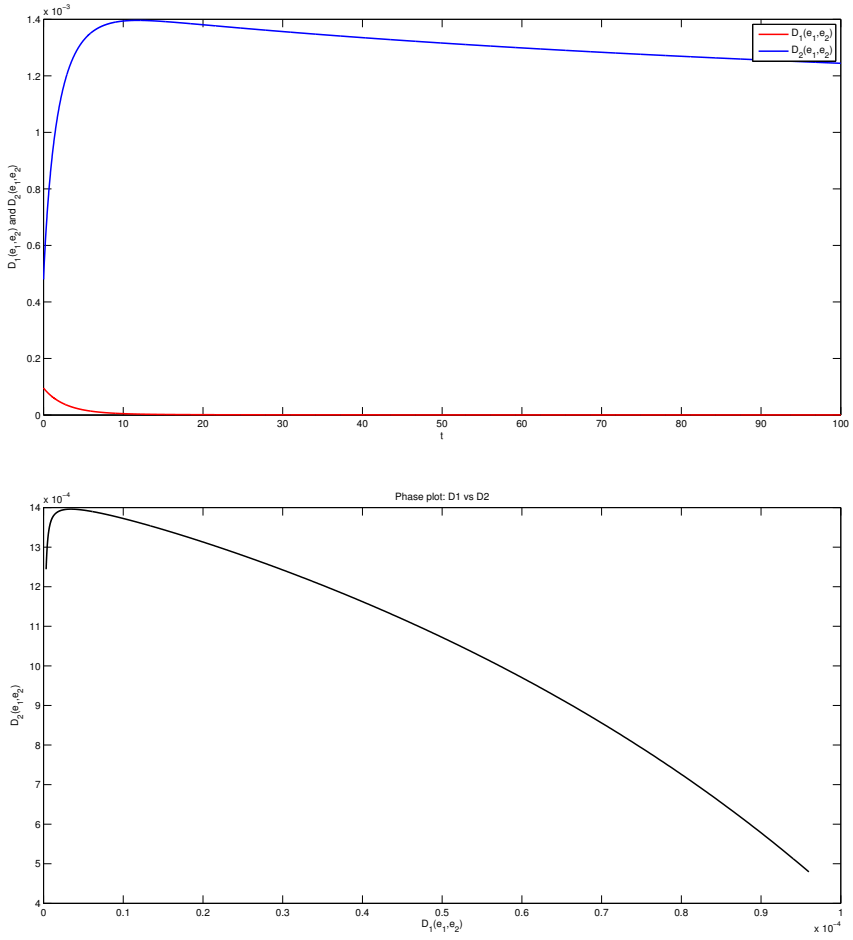

Figure 3. Temporal evolution of the control functions $\mathcal{D}_1$ and $\mathcal{D}_2$

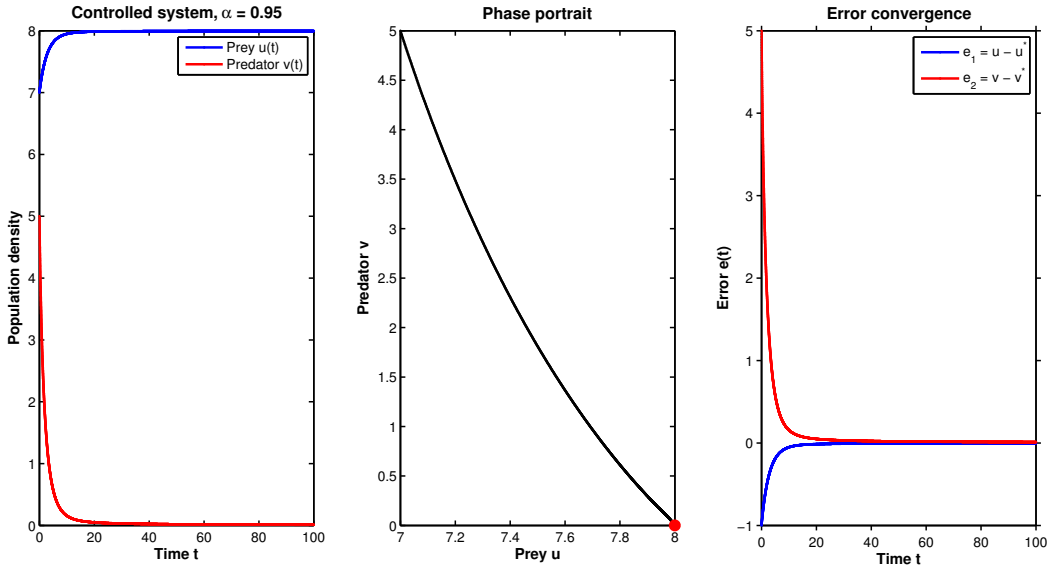

Figure 4. Dynamics of the PP and error trajectories $e_1(t)$ and $e_2(t)$ for the controlled FO system (15)

The LF defined in (16) is plotted in **Figure 5**. It decreases monotonically from its initial value and decays to zero at an exponential rate, providing a direct numerical verification of the Lyapunov based stability argument and the bound (33) (for $\delta = 0.02$).

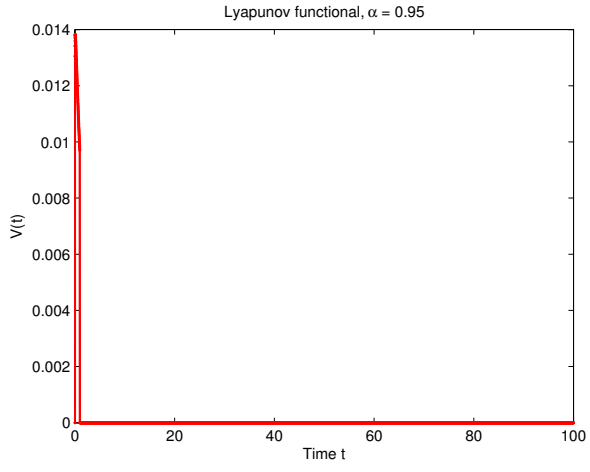

Figure 5. Temporal evolution of the LF $V(t)$ for the controlled error system (16)

The numerical experiments confirm the analytical findings. In the absence of control, both the integer-order and the FO models lead to predator extinction because the extreme prey refuge ($m = 0.99995$) makes only a minute fraction of prey available. The fractional memory may enrich the transient dynamics. The introduction of the state-feedback controller, which adaptively adjusts prey harvesting and predator culling based on the deviations from the desired equilibrium E_1, fundamentally alters the system's behavior. Despite the extremely small vulnerable prey pool, the control terms inject precisely the amount of damping and directional forcing needed to neutralise the oscillatory tendencies induced by the nonlinear functional response and the fractional memory. The error dynamics converge exponentially, and the LF exhibits a clean exponential decay, confirming the EAS established in Theorem 1. These results underscore the potential of fractional calculus to capture long-term memory effects in PP communities and demonstrate that carefully tuned, biologically realisable feedback interventions can stabilise a desired management equilibrium even in heavily perturbed ecosystems where natural dynamics alone would lead to collapse or extinction.

A systematic comparison (**Table 3**) of the model's response under three different regimes the classical integer-order formulation, the uncontrolled FO extension, and the FO system augmented with the state-feedback controller reveals markedly different ecological outcomes.

The comparison highlights two crucial points: (i) fractional memory per se does not reverse the extinction outcome induced by a severe prey refuge, although it enriches transient dynamics; (ii) appropriately designed, biologically interpretable feedback interventions can stabilise a desired management equilibrium and guarantee robust stability, even in heavily perturbed ecosystems where both integer-order and uncontrolled FO dynamics predict predator extinction.

Table 3. Comparative summary of the three model regimes

System	Asymptotic outcome	Transient behaviour	Memory & stability properties
Integer-order PP (1)	Predator extinction; prey stabilises at harvested carrying capacity $E_1 = (8, 0)$	Markovian dynamics; no long-term memory (**Figure 1**)	The boundary equilibrium E_1 is the only stable attractor; coexistence is not sustainable without external intervention.
Uncontrolled FO-PP (3)	Same qualitative outcome: predator extinction, prey settles at E_1	Longer-lasting oscillations; slower convergence due to the fractional memory kernel (**Figure 2**)	Memory buffers transients but is insufficient to prevent extinction when the vulnerable prey pool is too small.
Controlled FO-PP (15)	Stabilisation at the managed boundary equilibrium E_1; error variables e_1, e_2 decay exponentially to zero	Errors converge rapidly; Lyapunov function decays strictly exponentially (**Figures 4** and **5**)	Memory effects are neutralised by the state-feedback controller; E_1 is rendered exponentially asymptotically stable even under extreme refuge ($m \to 1$).

5. Conclusion

This study investigated the dynamics of an FO-PP model that integrates three ecologically realistic mechanisms a constant proportion of prey refuge, linear harvesting of the prey, and a generalist predator whose survival is not solely dependent on the focal prey. The ratio-dependent functional response captures mutual interference among predators, and the Caputo fractional derivative incorporates long-term memory effects arising from gestation delays, resource carryover, and environmental conditioning. The integer-order model was first presented, its equilibria computed, and then extended to an FO framework to account for non-local temporal influences that are ubiquitous in natural systems. A thorough stability analysis was performed for the FO system. The extinction, prey-only, predator-only, and positive interior equilibria were identified, and conditions for their existence were derived. The main contribution was the design of a state-feedback control law that guarantees EAS of the interior equilibrium. By constructing a suitable LF and exploiting the symmetry of the cross-coupling terms, the control functions $\mathcal{D}_1$ and $\mathcal{D}_2$ were shown to neutralise the destabilising quadratic and sign-indefinite contributions that emerge from the nonlinear functional response and the logistic growth terms. These controls depend continuously on the error between the current and desired population densities, adaptively adjusting prey harvesting and predator culling or supplementary feeding. The mathematical proof confirms that, under the proposed feedback, all small perturbations decay to zero at an exponential rate, even in the presence of strong refuge and high predator mortality.

Numerical simulations corroborated the analytical findings. In the absence of control, both integer-order and FO systems exhibited predator extinction for a parameter set characterised by an extremely high prey refuge, leaving only a minute vulnerable prey fraction. The fractional memory slightly enriched the transient oscillations but did not alter the asymptotic outcome. When the feedback controller was activated, the error dynamics converged rapidly to zero, and the LF displayed a strict exponential decay, validating the EAS result. These experiments demonstrate that carefully tuned, biologically feasible interventions can restore coexistence even under extreme ecological pressures. The results offer a theoretical foundation for ecosystem management strategies that combine refuge enhancement with adaptive harvesting and predator control. The FO perspective reveals that memory effects can delay or mask instability thresholds, making continuous monitoring and feedback indispensable for preventing population collapse. Future work may extend the control design to stochastic environments or to scenarios where only partial state measurements

are available, further bridging the gap between mathematical ecology and real-world conservation practices. Although the state-feedback controller guarantees EAS of the desired equilibrium under ideal conditions, several limitations should be acknowledged. First, the control law requires continuous, noise-free measurements of both prey and predator densities $u(t)$ and $v(t)$. In real ecological systems, obtaining such real-time, accurate population data is often impractical, especially for cryptic or widely dispersed species. Second, the stability result is local: convergence is guaranteed only for initial errors lying inside a sufficiently small ball B_δ around the equilibrium. For large perturbations, the controlled dynamics may not converge or could even diverge. Third, the control gains depend explicitly on model parameters (e.g., $a, b, k_1, k_2, p, c, e, q, d, m$), which are subject to estimation errors and natural variability. The controller is not robust to parametric uncertainties, and no adaptive mechanism is provided to update the gains online. Fourth, the feedback law assumes instantaneous and costless adjustments of harvesting and culling, whereas in practice management actions involve delays, logistical constraints, and economic costs. Fifth, the analysis is deterministic; environmental stochasticity, demographic fluctuations, and measurement noise are not considered, so the controller's performance under realistic noisy conditions remains untested. These limitations highlight important directions for future research, including the design of observer-based output feedback (using only partial or intermittent measurements), robust adaptive control, and stochastic extensions of the FO framework.

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Conflict of interest

The authors declare that they have no conflict of interest.

Author contributions

Conceptualization: Iqbal M. Batiha, Shaher Momani
Investigation: Iqbal M. Batiha, Osama Oqilat, Nidal Anakira, Tala Sasa
Methodology: Iqbal M. Batiha, Shaher Momani, Osama Oqilat
Formal analysis: Iqbal M. Batiha, Shaher Momani, Nidal Anakira
Writing–original draft: Iqbal M. Batiha, Osama Oqilat
Writing–review & editing: All authors

Availability of data

No datasets were generated or analyzed during the current study. All results are based on theoretical analysis and numerical simulations. The MATLAB codes used to generate the numerical results are available from the corresponding author upon reasonable request.

AI tools statement

The authors declare that artificial intelligence (AI) tools were used solely to assist with language editing, grammar checking, and improving the readability of the manuscript. All scientific content, mathematical derivations, theoretical analyses, numerical simulations, interpretations, and conclusions were developed, verified, and approved by the authors, who take full responsibility for the content of this article.

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Appendix

In this appendix, we present a detailed comparative analysis (**Table A1**) of our FO-PP model with the most recent works in the literature, highlighting the key differences in model structure, parameters, and stability outcomes.

Table A1. Detailed comparison between the present model and recent FO-PP studies

Reference	**Prey refuge m**	**Harvesting type**	**Functional response**	**Predator type**	**Control design**	**Fractional order α**	**Key stability outcome**
Aguegboh *et al.* (2025)[5]	Yes (constant)	Predator harvesting	Holling type III	Specialist	None	0.7–0.9	Extinction/coexistence.
Afiyah & Akanni (2026)[6]	No	Yes (type not specified)	Holling type II	Specialist	None	0.5–1.0	Disease-driven dynamics.
Xu & Balci (2026)[7]	No	No	Holling type II	Specialist	None	0.8–1.0	Limit cycles, delays.
Rahman & Faraj (2025)[8]	Yes (constant)	Nonlinear (prey)	Holling type II	Specialist	None	0.6–0.95	Bifurcations.
Hammouch *et al.* (2025)[9]	No	No	Holling type II	Specialist	None	0.7–1.0	Stability switches.
Present work	Yes ($m = 0.99995$)	Constant-effort (prey, $q = 0.1$)	Ratio-dependent	Generalist ($b < d$)	State-feedback (Lyapunov-based)	0.95	Exponential asymptotic stability of E_1 under extreme refuge.

Iqbal M. Batiha *is an Assistant Professor at Al-Zaytoonah University of Jordan and a researcher at the Nonlinear Dynamics Research Center, Ajman University. He is a founding member of ICSRS Jordan and has published extensively in international journals. His research interests include fractional calculus, control theory, mathematical modeling, and numerical methods. Dr. Batiha has received several prestigious awards, including the Riemann–Liouville Award and the Oliviu Gherman Award.*
https://orcid.org/0000-0002-8443-8848

Osama Oqilat *is a researcher in applied mathematics with interests in differential equations, mathematical analysis, and numerical methods. He has contributed to several research projects and publications in mathematical sciences and their applications.*
https://orcid.org/0000-0003-2370-6332

Nidal Anakira *is an Associate Professor of Applied Mathematics at Sohar University. He received his Ph.D. degree in Applied Mathematics from the School of Mathematical Sciences, Universiti Kebangsaan Malaysia, in 2015. His research interests include applied mathematics, numerical solutions of ordinary and partial differential equations, delay differential equations, and mathematical modeling. He has published numerous research articles in international journals and actively contributes to the advancement of computational and applied mathematics.*
https://orcid.org/0000-0003-4839-6342

Tala Sasa *received the M.Sc. degree from the Department of Mathematics, The University of Jordan, Amman, Jordan, in 2012. Her research interests include applied analysis and numerical computations. She is affiliated with the Department of Mathematics, Faculty of Science, Applied Science Private University, Amman, Jordan.*
https://orcid.org/0000-0001-7875-9920

Shaher Momani *is a Distinguished Professor at Ajman University and a leading researcher in fractional calculus. Since 2009, he has been ranked among the world's top researchers in the field by Thomson Reuters. He has authored more than 500 publications and was recognized as a Highly Cited Researcher in Mathematics (2014–2017) and Cross-Field (2018) by Clarivate Analytics. Dr. Momani has held academic positions at several prestigious institutions and has received numerous awards and honors in recognition of his outstanding contributions to mathematics, fractional calculus, and applied analysis.*
https://orcid.org/0000-0002-6326-8456

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