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RESEARCH ARTICLE

Uncertain data envelopment analysis approach for handling negative values: An application to stock market

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ABSTRACT

Classical Data Envelopment Analysis (DEA) models are traditionally built on the assumptions of certain and non-negative data. In contrast, real-world applications frequently involve negative values and epistemically uncertain information. Uncertainty theory offers a rigorous mathematical alternative to probability and fuzzy set theory for handling such indeterminacy. This study extends the Range Directional Measure (RDM) model by integrating uncertainty theory to simultaneously address negative data and uncertain environments. The proposed uncertain RDM model is operationalized and validated using an empirical dataset from the Tehran stock market, where traditional efficiency metrics often fail due to volatile financial ratios and negative returns. The findings demonstrate that the proposed framework yields more robust efficiency scores and provides actionable insights for stock evaluation under uncertainty. This integration advances the DEA literature by bridging the gap between performance measurement and uncertainty theory in negative-data contexts.

1. Introduction

Decision-making in complex and uncertain environments has always been a critical aspect of organizational and policy-level operations. With the growing availability of large-scale data and the increasing complexity of modern systems, the need for robust analytical frameworks to assess performance has become more pronounced. Data Envelopment Analysis (DEA) has emerged as a widely used non-parametric technique for evaluating the relative efficiency of decision-making units (DMUs) across diverse fields, including finance, healthcare, energy, and education.[1–4] By constructing an efficiency frontier based on observed input and output data, DEA enables organizations to identify the best practices and areas for improvement.[5–8] However, traditional DEA models operate under the assumption of precise and non-negative data, a condition that does not hold in many real-world scenarios.[9–12] The presence of uncertain and negative data poses significant challenges to conventional DEA methodologies, limiting their applicability and effectiveness.

In practice, uncertainty is an inherent feature of most decision-making contexts, arising from incomplete, imprecise, or ambiguous information.[13–16] For example, in financial markets, stock evaluations are often influenced by fluctuating economic conditions, geopolitical events, and investor sentiment, leading to uncertain data. Similarly, in environmental studies, measurements of pollutant

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emissions or resource consumption are frequently subject to estimation errors, resulting in imprecise data. The inability of traditional DEA models to accommodate such uncertainties restricts their utility in addressing the complexities of modern decision-making problems.[17–19] To overcome these limitations, researchers have turned to advanced mathematical frameworks, including uncertainty theory, which provides powerful tools for modeling and analyzing imprecise information.[20–22]

Uncertainty theory, initially proposed as a mathematical discipline for handling indeterminate phenomena, has evolved into a versatile framework encompassing concepts such as fuzzy sets, interval analysis, possibility theory, and probability theory.[23–25] These methodologies enable the representation and quantification of uncertainty in various forms, making them particularly suitable for integration with DEA.[26–28] By incorporating uncertainty theory, DEA models can be extended to account for imprecise inputs and outputs, thereby enhancing their robustness and applicability in uncertain environments.[29–31] This integration has opened new avenues for research and practical applications, particularly in fields where data uncertainty is a defining characteristic.

Among the various DEA extensions, the Range Directional Measure (RDM) model has gained prominence for its ability to address non-radial efficiency measurements.[32] Unlike traditional radial models, which assume proportional changes in all inputs or outputs, the RDM model allows for variable adjustments across different dimensions, providing a more flexible and realistic assessment of efficiency. This feature is particularly valuable in contexts where certain inputs or outputs may exhibit greater variability or significance than others. However, despite its advantages, the RDM model has primarily been applied to datasets with precise and non-negative values, leaving its potential in uncertain and negative data contexts largely unexplored.

Negative data, although less common than positive data, arises in various real-world scenarios and poses unique challenges for efficiency analysis. For instance, in financial evaluations, negative values may represent losses, debts, or deficits, which are crucial for assessing the overall performance of entities. Similarly, in environmental studies, negative outputs may indicate reductions in emissions or waste, reflecting progress toward sustainability goals. Traditional DEA models, which rely on non-negativity assumptions, are ill-equipped to handle such data, necessitating the development of specialized approaches.[33–36] The integration of uncertainty theory with the RDM model offers a promising solution to this challenge, enabling the effective analysis of efficiency in the presence of negative and uncertain data.

This study addresses a critical research gap, the absence of a DEA framework capable of simultaneously handling negative data and uncertainty, by proposing an Uncertain Range Directional Measure (URDM) model. The motivation stems from real-world contexts such as stock market evaluation, in which negative financial ratios and limited historical data render classical approaches inapplicable. Among existing uncertainty handling methods in DEA, including stochastic, fuzzy, robust, and interval DEA, none satisfy the requirements of this study. Stochastic DEA demands large historical datasets, which are unavailable. Fuzzy DEA lacks a rigorous axiomatic foundation and clear behavioral interpretation, leading to potential inconsistency across experts. Robust DEA is overly conservative, focusing only on worst-case scenarios and distorting true performance. Interval DEA captures only upper and lower bounds, discarding the most likely central value that experts typically provide. In contrast, uncertainty theory requires no historical data, rests on a self-consistent axiomatic framework, enables simultaneous modeling of optimistic, pessimistic, and most likely values through the zigzag distribution, provides transparent belief degree interpretation, and preserves linearity with low computational complexity. Therefore, uncertainty theory is the most appropriate framework for this study, in which expert knowledge is the primary information source. The main contributions of this study are as follows:

- **Methodologically**, the URDM model integrates uncertainty theory with the RDM framework, enabling efficiency measurement under both uncertainty and negative data without data shifting or artificial translation.
- **Computationally**, the URDM model preserves linearity and low computational complexity, and is solvable by any standard linear programming solver without requiring nonlinear programming or heuristic algorithms.
- **Analytically**, the URDM model enhances discriminative power and enables complete ranking of decision-making units, addressing the common limitation of multiple equally efficient units in uncertain DEA.
- **Empirically**, the URDM model is validated on a real-world dataset from the Tehran stock exchange, demonstrating practical feasibility and offering actionable insights for investors, policymakers, and analysts.
- **Managerially**, the URDM model conducts sensitivity analysis through belief degrees, assesses

stock performance across varying confidence levels, and distinguishes those stocks that exhibit greater sensitivity to uncertainty.

Collectively, these contributions position the URDM model as an effective, linear, and practically viable tool for efficiency analysis under the joint challenges of negative values and uncertainty, thus extending beyond the current state of the art in uncertain DEA.

The remainder of this study is structured as follows. Section 2 provides an in-depth discussion of the theoretical foundations and methodological framework, establishing the basis for the proposed approach. Section 3 then introduces the modeling of uncertain range directional measure, detailing the mathematical formulation and key assumptions. In Section 4, an empirical case study is conducted, demonstrating the practical application of the URDM model to the Tehran stock market and analyzing key findings. Finally, Section 5 concludes the study by summarizing its contributions, discussing practical implications, and outlining directions for future research.

2. Theoretical research background

In this section, the theoretical research background of the study is introduced. First, the modeling of DEA based on the range directional measure is discussed. Then, the fundamental mathematical concepts of uncertainty theory are presented, including uncertainty space, uncertainty measure, belief degree function, inverse uncertainty distribution, and common uncertainty distributions.

2.1. Range directional measure

Let's consider a situation where there are N peer decision-making units $DMU_j (j = 1, ..., N)$ that utilize M inputs $x_{ij}\,(i = 1, ..., M)$ to generate S outputs $y_{rj}\,(r = 1, ..., S)$. In this case, the index p represents the specific DMU being evaluated. Portela *et al.*[32] introduced the RDM model to address situations where inputs and/or outputs have negative values. They defined two vectors, Q^-_{ip} and Q^+_{rp} denoted by Equations (1) and (2), respectively. It is important to clarify that these vectors represent the potential range of improvement for the DMU under evaluation, indicating the extent to which inputs can be reduced or outputs can be expanded to achieve optimal efficiency.

$$Q^-_{ip} = x_{ip} - \min_j\{x_{ij}\}, \qquad \forall i = 1, ..., M \tag{1}$$

$$Q^+_{rp} = \max_j\{y_{rj}\} - y_{rp}, \qquad \forall r = 1, ..., S. \tag{2}$$

Portela *et al.*[32] introduced the RDM model by incorporating the concept of the range of possible improvements, which is formulated as Model (3) as follows:

$$\begin{aligned}
&\max\ \eta_p && \quad (3)\\
&S.t.\ \sum_{j=1}^{N} x_{ij}\mu_j + Q^-_{ip}\eta_p \le x_{ip}, && \forall i = 1, ..., M\\
&\sum_{j=1}^{N} y_{rj}\mu_j - Q^+_{rp}\eta_p \ge y_{rp}, && \forall r = 1, ..., S\\
&\sum_{j=1}^{N} \mu_j = 1 &&\\
&\eta_p, \mu_j \ge 0, && \forall j = 1, ..., N.
\end{aligned}$$

In the RDM model, it is important to note that η_p represents the measure of inefficiency for a particular DMU that is being examined. Model (3) can be considered as the envelopment formulation of the RDM model. The dual form of the RDM is presented as Model (4), providing an alternative perspective for evaluating efficiency:

$$\begin{aligned}
&\min\ \sum_{i=1}^{M} x_{ip}\nu_i - \sum_{r=1}^{S} y_{rp}\upsilon_r + \omega_p && \quad (4)\\
&S.t.\ \sum_{i=1}^{M} Q^-_{ip}\nu_i + \sum_{r=1}^{S} Q^+_{rp}\upsilon_r \ge 1 &&\\
&\sum_{i=1}^{M} x_{ij}\nu_i - \sum_{r=1}^{S} y_{rj}\upsilon_r + \omega_p \ge 0,\ \ \forall j = 1, ..., N &&\\
&\nu_i,\ \upsilon_r \ge 0,\ \forall i = 1, ..., M,\ \forall r = 1, ..., S. &&
\end{aligned}$$

It should be noted that Model (4) represents the multiplier formulation of the RDM model, where non-negative weights, $\nu_i (i = 1, ..., M)$ and $\upsilon_r (r = 1, ..., S)$, are allocated to inputs and outputs respectively.

2.2. Uncertainty theory

Uncertainty theory is a branch of mathematics designed to handle systems and phenomena where precise data are unavailable, incomplete, or inherently imprecise.[37–40] Unlike probability theory, which quantifies randomness through well-defined probabilities, uncertainty theory provides a more versatile framework for representing and analyzing vagueness, subjective judgment, and lack of information. This makes it particularly suitable for applications in domains such as decision-making, finance, healthcare, and engineering. The key concepts in uncertainty theory are introduced below, offering a foundational understanding of the principles that guide decision-making in the face of uncertainty.

2.2.1. Uncertainty space

The foundation of uncertainty theory lies in the definition of an uncertainty space, represented as a triplet (Ξ, Ψ, Λ) where:

- Ξ: The universal set containing all possible outcomes or events.
- Ψ: A sigma-algebra over Ξ, representing the collection of measurable subsets (events).
- Λ: An uncertainty measure that assigns a value in $[0, 1]$ to each event $A \in \Psi$, quantifying the degree of belief in the occurrence of A.

2.2.2. Uncertainty measure

It is important to explain that the uncertainty measure Λ satisfies the following four axioms, which serve as the foundational principles ensuring its consistency and reliability in capturing uncertainty:

- **Normality**: $\Lambda(\Xi) = 1$.
- **Duality**: For any event A, $\Lambda(A) + \Lambda(A^C) = 1$, where A^C is the complement of A.
- **Subadditivity**: For disjoint events A_1 and A_2, $\Lambda(A_1 \cup A_2) \leq \Lambda(A_1) + \Lambda(A_2)$.
- **Product Axiom**: For independent events A and B, $\Lambda(A \cap B) \leq \Lambda(A).\Lambda(B)$.

2.2.3. Belief degree function

Central to uncertainty theory is the belief degree function, which quantifies the level of confidence or belief in specific events or outcomes. This function plays a crucial role in representing uncertainty by measuring how strongly one believes in the occurrence of particular outcomes, based on available information. For an uncertain variable ϖ, the belief degree function, also known as the uncertainty distribution Φ, is defined by Equation (5):

$$\Phi(\tau) = \Lambda\{\varpi \leq \tau\}, \qquad \tau \in \mathfrak{R}. \tag{5}$$

This represents the degree of belief that ϖ takes a value less than or equal to τ.

2.2.4. Inverse uncertainty distribution

The inverse function is essential for reversing the uncertainty measurement process, enabling the transformation of belief degrees back into the corresponding outcomes or events. If Φ is strictly increasing and continuous, its inverse Φ^{-1} exists and is defined by Equation (6):

$$\Phi^{-1}(\xi) = \inf\{\tau \in \mathfrak{R} : \Phi(\tau) \geq \xi\}, \ \xi \in [0, 1]. \tag{6}$$

The existence of Φ^{-1} ensures a well-defined relationship between belief degrees and uncertain variables, which is crucial for accurate modeling and decision-making in uncertain environments. Additionally, this function provides the value of τ corresponding to a given belief degree ξ.

2.2.5. Linear uncertainty distribution

The linear uncertainty distribution is the simplest form, used when the uncertain variable ϖ varies linearly between two bounds $\alpha^{(1)}$ and $\alpha^{(2)}$, with $\alpha^{(1)} < \alpha^{(2)}$. The linear uncertainty distribution $\mathcal{L}(\alpha^{(1)}, \alpha^{(2)})$ is defined by Equation (7):

$$\Phi(\tau) = \begin{cases} 0, & if\, \tau \leq \alpha^{(1)}; \\ \frac{(\tau-\alpha^{(1)})}{(\alpha^{(2)}-\alpha^{(1)})} & if\, \alpha^{(1)} \leq \tau \leq \alpha^{(2)}; \\ 1, & if\, \alpha^{(2)} \leq \tau. \end{cases} \tag{7}$$

The inverse function of the linear uncertainty distribution is expressed by Equation (8):

$$\Phi^{-1}(\xi) = (1-\xi)\alpha^{(1)} + (\xi)\alpha^{(2)}, \qquad \xi \in [0, 1]. \tag{8}$$

The linear uncertainty distribution is particularly useful when the uncertainty lies within a fixed interval and follows a straightforward linear progression.

2.2.6. Zigzag uncertainty distribution

The zigzag uncertainty distribution models more complex patterns of uncertainty and is defined by three critical points, $\beta^{(1)}$, $\beta^{(2)}$, and $\beta^{(3)}$ with $\beta^{(1)} < \beta^{(2)} < \beta^{(3)}$. The zigzag uncertainty distribution $Z_{(\beta^{(1)},\beta^{(2)},\beta^{(3)})}$ takes the form shown in Equation (9):

$$\Phi(\tau) = \begin{cases} 0, & if\, \tau \leq \beta^{(1)}; \\ \frac{(\tau-\beta^{(1)})}{2(\beta^{(2)}-\beta^{(1)})}, & if\, \beta^{(1)} \leq \tau \leq \beta^{(2)}; \\ \frac{(\tau+\beta^{(3)}-2\beta^{(2)})}{2(\beta^{(3)}-\beta^{(2)})}, & if\, \beta^{(2)} \leq \tau \leq \beta^{(3)}; \\ 1, & if\, \beta^{(3)} \leq \tau. \end{cases} \tag{9}$$

The corresponding inverse function of the zigzag uncertainty distribution is expressed by Equation (10):

$$\Phi^{-1}(\xi) = \begin{cases} (1-2\xi)\beta^{(1)} + (2\xi)\beta^{(2)} & if\, \xi \leq 0.5; \\ (2-2\xi)\beta^{(2)} + (2\xi-1)\beta^{(3)} & if\, \xi > 0.5. \end{cases} \tag{10}$$

The zigzag uncertainty distribution is ideal for representing uncertainty that alternates or fluctuates over distinct intervals.

2.2.7. Normal uncertainty distribution

The normal uncertainty distribution provides a bell-shaped curve and is widely applied in cases where uncertainty exhibits a symmetric pattern around a central mean μ with variability $\sigma > 0$. The normal uncertainty distribution $\mathbb{N}_{(\mu,\delta)}$ is defined by Equation (11):

$$\Phi(\tau) = \left(1 + \exp\left(\frac{\pi(\mu-\tau)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ \tau \in \mathfrak{R}. \tag{11}$$

The inverse function of normal uncertainty distribution is given by Equation (12):

$$\Phi^{-1}(\xi) = \mu + \frac{\sqrt{3}\sigma}{\pi} \ln\left(\frac{\xi}{1-\xi}\right),\ \xi \in (0,1). \quad (12)$$

The normal distribution is particularly suited for modeling uncertainty in natural and social systems due to its ability to represent the central tendency and variability effectively.

Notably, uncertainty theory, with its wide range of distribution functions and belief measures, provides a powerful framework for tackling real-world challenges across various fields. By utilizing different uncertainty distributions, analysts can model diverse forms of imprecision, vagueness, and unpredictability, thereby offering robust solutions for decision-making in complex, uncertain environments. The linear distribution is well-suited for simple intervals, the zigzag distribution effectively captures dynamic uncertainty, and the normal distribution is ideal for modeling symmetric variability, making it a versatile tool for comprehensive uncertainty analysis.

3. The proposed uncertain range directional measure

In this section, an extended version of the range directional measure is proposed, designed to remain applicable under conditions of data uncertainty. A critical step in formulating the proposed uncertain RDM framework is the appropriate selection of an uncertainty distribution for the uncertain variables. As discussed in the preceding section, several distributions are available within uncertainty theory, such as the linear, zigzag, and normal uncertainty distributions. Among these, the zigzag distribution was deliberately chosen for this study.

Compared to the linear uncertainty distribution, the zigzag distribution provides a more detailed representation by incorporating three distinct parameters, namely the optimistic, central, and pessimistic values, while preserving piecewise linearity. This additional parameterization enables a richer description of expert beliefs without compromising computational simplicity. In contrast to the normal uncertainty distribution, the zigzag distribution imposes no symmetry assumptions and, more importantly, entails significantly lower computational complexity. Whereas the normal distribution often leads to nonlinear formulations and increases the computational burden in optimization models, the zigzag distribution maintains the linear structure of the underlying DEA framework. Consequently, the proposed URDM model remains tractable and solvable using standard linear programming techniques, obviating the need for nonlinear solvers or numerical approximations. Thus, the zigzag distribution strikes an optimal balance between representational richness and computational efficiency, making it particularly suitable for integration with the RDM model in the presence of negative and uncertain data.

It should be noted that all data are uncertain, and are represented by zigzag uncertainty distribution $\tilde{x}_{ij}(x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)})$ and $\tilde{y}_{rj}(y_{rj}^{(1)}, y_{rj}^{(2)}, y_{rj}^{(3)})$ in which $x_{ij}^{(1)} < x_{ij}^{(2)} < x_{ij}^{(3)}$ and $y_{rj}^{(1)} < y_{rj}^{(2)} < y_{rj}^{(3)}$. The multiplier form of the RDM model under the zigzag uncertainty distribution is presented in Model (13):

$$\begin{aligned}
\min\ & \sum_{i=1}^{M} \tilde{x}_{ip}\nu_i - \sum_{r=1}^{S} \tilde{y}_{rp}\upsilon_r + \omega_p && (13)\\
S.t\ & \sum_{i=1}^{M} \tilde{Q}_{ip}^{-}\nu_i + \sum_{r=1}^{S} \tilde{Q}_{rp}^{+}\upsilon_r \geq 1 \\
& \sum_{i=1}^{M} \tilde{x}_{ij}\nu_i - \sum_{r=1}^{S} \tilde{y}_{rj}\upsilon_r + \omega_p \geq 0, \forall j = 1, ..., N \\
& \nu_i,\ \upsilon_r \geq 0,\ \ \forall i = 1, ..., M,\ \ \forall r = 1, ..., S.
\end{aligned}$$

Then, to address the uncertainty in the RDM model and convert it to its equivalent crisp values, the uncertainty measure is applied, as shown in Model (14):

$$\begin{aligned}
\min\ & \Theta && (14)\\
S.t.\ & \Lambda\left\{\sum_{i=1}^{M} \tilde{x}_{ip}\nu_i - \sum_{r=1}^{S} \tilde{y}_{rp}\upsilon_r + \omega_p \leq \Theta\right\} \geq \xi \\
& \Lambda\left\{\sum_{i=1}^{M} \tilde{Q}_{ip}^{-}\nu_i + \sum_{r=1}^{S} \tilde{Q}_{rp}^{+}\upsilon_r \geq 1\right\} \geq \xi \\
& \Lambda\left\{\sum_{i=1}^{M} \tilde{x}_{ij}\nu_i - \sum_{r=1}^{S} \tilde{y}_{rj}\upsilon_r + \omega_p \geq 0\right\} \geq \xi, \\
& \forall j = 1, ..., N \\
& \nu_i,\ \upsilon_r \geq 0,\ \ \forall i = 1, ..., M,\ \ \forall r = 1, ..., S.
\end{aligned}$$

As shown in Equation (10), the corresponding inverse function of the zigzag uncertainty distribution takes two different forms depending on whether the belief degree is greater than or less than 0.5. Consistent with Equation (10), the deterministic equivalent of the uncertain chance constraints in the RDM model is presented for the specified belief degree ξ. These are denoted as Models (15) and

(16) for $\xi \leq 0.5$ and $\xi > 0.5$, respectively:

$$\begin{aligned}
&\min\ \Theta && (15)\\
&S.t.\ \sum_{i=1}^{M}((1-2\xi)x_{ip}^{(1)}+(2\xi)x_{ip}^{(2)})\nu_i-\\
&\sum_{r=1}^{S}((2\xi)y_{rp}^{(2)}+(1-2\xi)y_{rp}^{(3)})\upsilon_r+\omega_p\leq\Theta\\
&\sum_{i=1}^{M}((2\xi)Q_{ip}^{-(2)}+(1-2\xi)Q_{ip}^{-(3)})\nu_i+\\
&\sum_{r=1}^{S}((2\xi)Q_{rp}^{+(2)}+(1-2\xi)Q_{rp}^{+(3)})\upsilon_r\geq 1\\
&\sum_{i=1}^{M}((2\xi)x_{ij}^{(2)}+(1-2\xi)x_{ij}^{(3)})\nu_i-\\
&\sum_{r=1}^{S}((1-2\xi)y_{rj}^{(1)}+(2\xi)y_{rj}^{(2)})\upsilon_r+\omega_p\geq 0,\\
&\forall j=1,..,N\\
&\nu_i,\ \upsilon_r\geq 0,\qquad \forall i=1,...,M,\ \ \forall r=1,...,S
\end{aligned}$$

$$\begin{aligned}
&\min\ \Theta && (16)\\
&S.t.\ \sum_{i=1}^{M}((2-2\xi)x_{ip}^{(2)}+(2\xi-1)x_{ip}^{(3)})\nu_i-\\
&\sum_{r=1}^{S}((2\xi-1)y_{rp}^{(1)}+(2-2\xi)y_{rp}^{(2)})\upsilon_r+\omega_p\leq\Theta\\
&\sum_{i=1}^{M}((2\xi-1)Q_{ip}^{-(1)}+(2-2\xi)Q_{ip}^{-(2)})\nu_i+\\
&\sum_{r=1}^{S}((2\xi-1)Q_{rp}^{+(1)}+(2-2\xi)Q_{rp}^{+(2)})\upsilon_r\geq 1\\
&\sum_{i=1}^{M}((2\xi-1)x_{ij}^{(1)}+(2-2\xi)x_{ij}^{(2)})\nu_i-\\
&\sum_{r=1}^{S}((2-2\xi)y_{rj}^{(2)}+(2\xi-1)y_{rj}^{(3)})\upsilon_r+\omega_p\geq 0,\\
&\forall j=1,..,N\\
&\nu_i,\ \upsilon_r\geq 0,\qquad \forall i=1,...,M,\ \forall r=1,...,S.
\end{aligned}$$

Now, by utilizing binary variables Υ and a sufficiently large number $\dagger$, the linearization of incompatible constraints for belief degrees greater than or less than 0.5 is performed. Finally, the uncertain framework of the range directional measure is proposed as Model (17):

$$\begin{aligned}
&\min\ \Theta && (17)\\
&S.t.\\
&\sum_{i=1}^{M}((1-2\xi)x_{ip}^{(1)}+(2\xi)x_{ip}^{(2)})\nu_i-\\
&\sum_{r=1}^{S}((2\xi)y_{rp}^{(2)}+(1-2\xi)y_{rp}^{(3)})\upsilon_r+\omega_p\leq\Theta+\dagger\Upsilon\\
&\sum_{i=1}^{M}((2-2\xi)x_{ip}^{(2)}+(2\xi-1)x_{ip}^{(3)})\nu_i-\\
&\sum_{r=1}^{S}((2\xi-1)y_{rp}^{(1)}+(2-2\xi)y_{rp}^{(2)})\upsilon_r+\\
&\omega_p\leq\Theta+\dagger(1-\Upsilon)\\
&\sum_{i=1}^{M}((2\xi)Q_{ip}^{-(2)}+(1-2\xi)Q_{ip}^{-(3)})\nu_i+\\
&\sum_{r=1}^{S}((2\xi)Q_{rp}^{+(2)}+(1-2\xi)Q_{rp}^{+(3)})\upsilon_r\geq 1-\dagger\Upsilon,\\
&\sum_{i=1}^{M}((2\xi-1)Q_{ip}^{-(1)}+(2-2\xi)Q_{ip}^{-(2)})\nu_i+\\
&\sum_{r=1}^{S}((2\xi-1)Q_{rp}^{+(1)}+(2-2\xi)Q_{rp}^{+(2)})\upsilon_r\\
&\geq 1-\dagger(1-\Upsilon),\\
&\sum_{i=1}^{M}((2\xi)x_{ij}^{(2)}+(1-2\xi)x_{ij}^{(3)})\nu_i-\\
&\sum_{r=1}^{S}((1-2\xi)y_{rj}^{(1)}+(2\xi)y_{rj}^{(2)})\upsilon_r+\omega_p\geq-\dagger\Upsilon,\\
&\forall j=1,...,N\\
&\sum_{i=1}^{M}((2\xi-1)x_{ij}^{(1)}+(2-2\xi)x_{ij}^{(2)})\nu_i-\\
&\sum_{r=1}^{S}((2-2\xi)y_{rj}^{(2)}+(2\xi-1)y_{rj}^{(3)})\upsilon_r+\omega_p\geq\\
&-\dagger(1-\Upsilon),\forall j=1,...,N\\
&\xi\leq 0.5+\dagger\Upsilon\\
&\xi>0.5-\dagger(1-\Upsilon)\\
&\Upsilon\in\{0,1\}\\
&\nu_i,\ \upsilon_r\geq 0,\ \forall i=1,...,M,\ \forall r=1,...,S.
\end{aligned}$$

At the end, Model (17) introduces a novel range directional measure designed to effectively address negative and uncertain data in real-world problems and applications. This model provides a robust framework for handling uncertainty, making it suitable for a wide range of practical scenarios where data may be imprecise or fluctuate, ensuring more accurate decision-making and analysis in complex environments.

4. Case study and experimental results

This section demonstrates the application of the proposed uncertain RDM model through a practical case study in the insurance industry. A sample of 18 insurance companies listed on the Tehran stock market is selected for analysis. Financial data for these companies are extracted from March 2016 to March 2017, providing a one-year window for empirical evaluation. Based on previous studies in the field of financial performance evaluation and the opinions of domain experts, three key financial inputs and three critical output metrics are incorporated into the model.[41–43] The selected input variables include the price-to-earnings ratio (P/E), which reflects market valuation relative to earnings; the quick ratio, a measure of short-term liquidity; and the solvency ratio-II, an indicator of long-term financial stability. On the output side, the model focuses on three critical metrics for stock assessment: the rate of return (RoR), which measures profitability; liquidity, assessing the ability to meet short-term obligations; and the earnings per share (EPS) growth rate, indicating the company's growth potential. By integrating these inputs and outputs, the uncertain RDM model provides a comprehensive framework for assessing stock performance under uncertainty, offering valuable insights for investors and stakeholders in the insurance sector. The input and output data, characterized by the zigzag uncertainty distribution, are presented in **Tables 1 and 2**, respectively. The proposed uncertain RDM model will now be executed across varying belief degrees, specifically 0%, 20%, 40%, 60%, 80%, and 100%. The outcomes of the uncertain RDM model, as applied to stock assessment, are detailed in **Table 3**.

According to **Table 3**,the results of the uncertain RDM model reveal a clear and consistent trend in which the efficiency score for each stock gradually decreases as the belief degree increases from 0% to 100%. This inverse relationship directly reflects the impact of epistemic uncertainty on performance evaluation, since higher belief degrees impose stricter confidence requirements on satisfying the model's chance constraints. In other words, a decision maker who demands a higher level of certainty must accept lower efficiency scores, which provides a realistic and transparent trade-off between confidence and performance.

To further evaluate the effectiveness of the proposed approach, the stock evaluation and ranking results obtained under the classical RDM model with certain data and those obtained under the proposed uncertain RDM model with uncertain data are presented in **Figures 1 and 2**, respectively. It should be noted that the results reported in **Figure 2** represent the average of the stock evaluation outcomes across all belief degrees reported in **Table 3**. This averaging provides a comprehensive view of stock performance under different levels of uncertainty rather than relying on a single subjective belief degree.

Table 1. Dataset of inputs with zigzag uncertainty Distribution

Stocks	Price to Earnings Ratio	Quick Ratio	Solvency Ratio-II
Stock01	$Z(4.020, 5.026, 6.031)$	$Z(0.799, 0.998, 1.198)$	$Z(3.449, 4.311, 5.173)$
Stock02	$Z(7.826, 9.783, 11.739)$	$Z(0.942, 1.178, 1.413)$	$Z(1.382, 1.728, 2.074)$
Stock03	$Z(8.127, 10.158, 12.190)$	$Z(0.779, 0.974, 1.169)$	$Z(2.734, 3.418, 4.102)$
Stock04	$Z(4.259, 5.324, 6.389)$	$Z(0.780, 0.975, 1.170)$	$Z(3.446, 4.307, 5.169)$
Stock05	$Z(6.058, 7.573, 9.087)$	$Z(0.647, 0.809, 0.971)$	$Z(2.411, 3.014, 3.616)$
Stock06	$Z(4.004, 5.005, 6.006)$	$Z(1.230, 1.537, 1.845)$	$Z(4.040, 5.049, 6.059)$
Stock07	$Z(5.612, 7.015, 8.418)$	$Z(0.947, 1.183, 1.420)$	$Z(2.093, 2.616, 3.139)$
Stock08	$Z(5.701, 7.126, 8.551)$	$Z(0.832, 1.040, 1.248)$	$Z(4.233, 5.291, 6.349)$
Stock09	$Z(8.403, 10.503, 12.604)$	$Z(0.580, 0.724, 0.869)$	$Z(7.387, 9.234, 11.081)$
Stock10	$Z(5.613, 7.016, 8.419)$	$Z(0.648, 0.810, 0.973)$	$Z(6.191, 7.739, 9.286)$
Stock11	$Z(5.174, 6.468, 7.762)$	$Z(2.006, 2.507, 3.009)$	$Z(0.468, 0.586, 0.703)$
Stock12	$Z(6.051, 7.564, 9.077)$	$Z(2.008, 2.510, 3.012)$	$Z(0.319, 0.399, 0.479)$
Stock13	$Z(6.092, 7.615, 9.138)$	$Z(0.951, 1.189, 1.427)$	$Z(1.755, 2.193, 2.632)$
Stock14	$Z(7.217, 9.022, 10.826)$	$Z(0.626, 0.783, 0.940)$	$Z(3.787, 4.734, 5.681)$
Stock15	$Z(6.322, 7.902, 9.483)$	$Z(1.339, 1.674, 2.009)$	$Z(1.820, 2.275, 2.729)$
Stock16	$Z(7.160, 8.951, 10.741)$	$Z(0.562, 0.703, 0.844)$	$Z(4.406, 5.507, 6.609)$
Stock17	$Z(6.123, 7.653, 9.184)$	$Z(1.254, 1.568, 1.881)$	$Z(0.903, 1.128, 1.354)$
Stock18	$Z(6.905, 8.632, 10.358)$	$Z(0.674, 0.842, 1.011)$	$Z(3.699, 4.623, 5.548)$
Min	$Z(4.004, 5.005, 6.006)$	$Z(0.562, 0.703, 0.844)$	$Z(0.319, 0.399, 0.479)$
Max	$Z(8.403, 10.503, 12.604)$	$Z(2.008, 2.510, 3.012)$	$Z(7.387, 9.234, 11.081)$

Table 2. Dataset of outputs with zigzag uncertainty Distribution

Stocks	Rate of Return	Liquidity	EPS Growth Rate
Stock01	$Z(-8.624, -7.187, -5.749)$	$Z(44.693, 55.866, 67.039)$	$Z(33.421, 41.776, 50.132)$
Stock02	$Z(49.473, 61.841, 74.210)$	$Z(52.288, 65.359, 78.431)$	$Z(0.406, 0.508, 0.609)$
Stock03	$Z(22.031, 27.539, 33.046)$	$Z(347.826, 434.783, 521.739)$	$Z(4.242, 5.303, 6.364)$
Stock04	$Z(6.837, 8.546, 10.255)$	$Z(40.404, 50.505, 60.606)$	$Z(-3.520, -2.933, -2.347)$
Stock05	$Z(47.821, 59.776, 71.732)$	$Z(16.032, 20.040, 24.048)$	$Z(8.931, 11.163, 13.396)$
Stock06	$Z(-5.790, -4.825, -3.860)$	$Z(51.613, 64.516, 77.419)$	$Z(21.772, 27.215, 32.658)$
Stock07	$Z(12.707, 15.884, 19.061)$	$Z(18.349, 22.936, 27.523)$	$Z(9.452, 11.815, 14.178)$
Stock08	$Z(-14.902, -12.418, -9.935)$	$Z(16.260, 20.325, 24.390)$	$Z(15.612, 19.515, 23.418)$
Stock09	$Z(18.294, 22.868, 27.442)$	$Z(421.053, 526.316, 631.579)$	$Z(29.735, 37.168, 44.602)$
Stock10	$Z(-6.336, -5.280, -4.224)$	$Z(24.615, 30.769, 36.923)$	$Z(-8.873, -7.394, -5.915)$
Stock11	$Z(4.477, 5.597, 6.716)$	$Z(42.105, 52.632, 63.158)$	$Z(-23.715, -19.763, -15.810)$
Stock12	$Z(0.578, 0.723, 0.867)$	$Z(26.936, 33.670, 40.404)$	$Z(3.980, 4.975, 5.970)$
Stock13	$Z(-35.341, -29.451, -23.561)$	$Z(17.699, 22.124, 26.549)$	$Z(2.997, 3.747, 4.496)$
Stock14	$Z(0.647, 0.808, 0.970)$	$Z(27.027, 33.784, 40.541)$	$Z(2.090, 2.612, 3.134)$
Stock15	$Z(-14.547, -12.123, -9.698)$	$Z(18.433, 23.041, 27.650)$	$Z(-2.529, -2.107, -1.686)$
Stock16	$Z(61.787, 77.234, 92.680)$	$Z(21.505, 26.882, 32.258)$	$Z(-44.740, -37.283, -29.827)$
Stock17	$Z(9.432, 11.789, 14.147)$	$Z(16.632, 20.790, 24.948)$	$Z(19.143, 23.928, 28.714)$
Stock18	$Z(-6.279, -5.233, -4.186)$	$Z(24.316, 30.395, 36.474)$	$Z(-48.972, -40.810, -32.648)$
Min	$Z(-35.341, -29.451, -23.561)$	$Z(16.032, 20.040, 24.048)$	$Z(-48.972, -40.810, -32.648)$
Max	$Z(61.787, 77.234, 92.680)$	$Z(421.053, 526.316, 631.579)$	$Z(33.421, 41.776, 50.132)$

Table 3. Stock evaluation results based on the uncertain RDM model under different belief degrees

Stocks	0%	20%	40%	60%	80%	100%
Stock01	1.000	1.000	1.000	0.921	0.694	0.296
Stock02	1.000	1.000	1.000	0.947	0.806	0.591
Stock03	1.000	1.000	1.000	0.920	0.694	0.311
Stock04	0.962	0.905	0.825	0.703	0.529	0.308
Stock05	1.000	1.000	1.000	0.931	0.736	0.406
Stock06	1.000	1.000	0.945	0.798	0.584	0.262
Stock07	1.000	0.983	0.908	0.801	0.641	0.393
Stock08	0.995	0.886	0.780	0.647	0.459	0.181
Stock09	1.000	1.000	1.000	0.844	0.394	0.010
Stock10	0.842	0.775	0.682	0.548	0.352	0.112
Stock11	1.000	1.000	0.992	0.952	0.893	0.803
Stock12	1.000	1.000	1.000	0.987	0.954	0.902
Stock13	0.914	0.860	0.787	0.686	0.558	0.383
Stock14	0.874	0.814	0.733	0.623	0.471	0.256
Stock15	0.870	0.816	0.745	0.655	0.539	0.380
Stock16	1.000	1.000	1.000	0.863	0.412	0.010
Stock17	1.000	1.000	1.000	0.966	0.874	0.733
Stock18	0.820	0.747	0.647	0.508	0.314	0.089

As shown in **Figures 1 and 2**, the classical RDM approach suffers from low discriminative power, resulting in multiple stocks being rated as equally efficient. This limitation makes it difficult for investors to distinguish between truly efficient stocks and those that appear efficient only due to the model's structural limitations. In contrast, the proposed uncertain RDM approach successfully achieves a complete and unambiguous ranking of all 18 stocks, demonstrating its superior discriminatory ability.

Based on the average efficiency scores calculated across all belief degrees, Stocks 12, 11, and 17 emerge as the top-performing stocks, consistently demonstrating superior rankings compared to others. These stocks exhibit robust performance metrics across all belief degrees, making them strong candidates for portfolio inclusion and attractive investment opportunities. Their ability to maintain high efficiency scores under varying levels of uncertainty underscores their reliability and potential for delivering favorable risk-adjusted returns. Investors and portfolio managers may prioritize these stocks when constructing portfolios intended to perform well under uncertain market conditions.

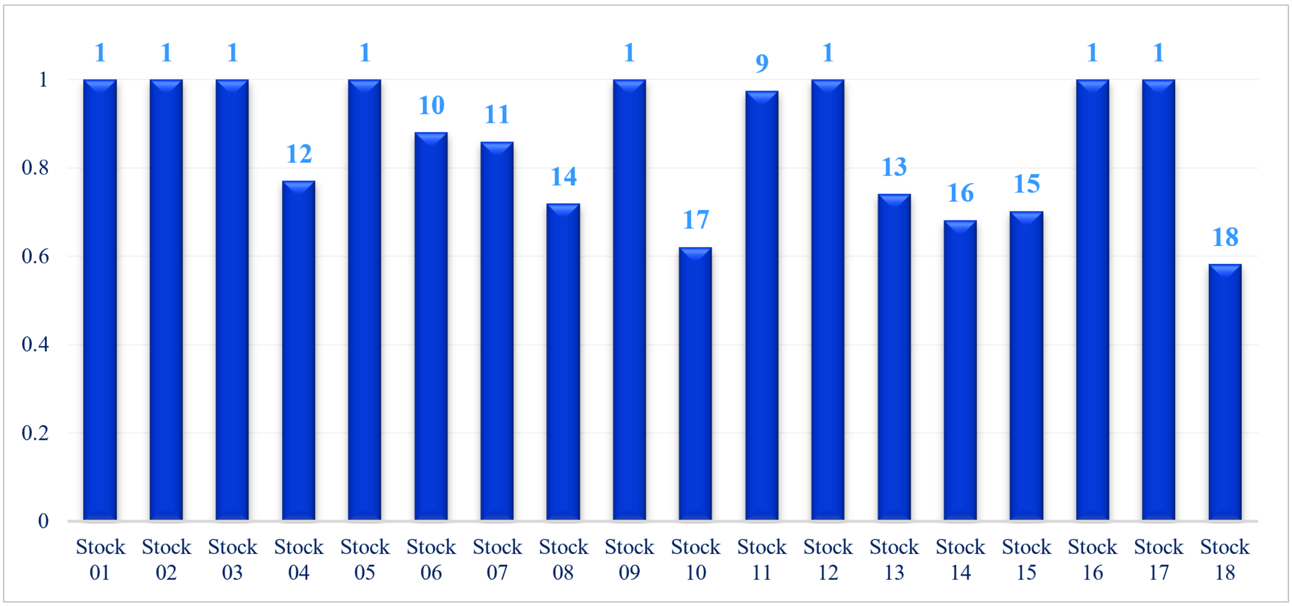

Figure 1. Stock evaluation and ranking results using the classical RDM model

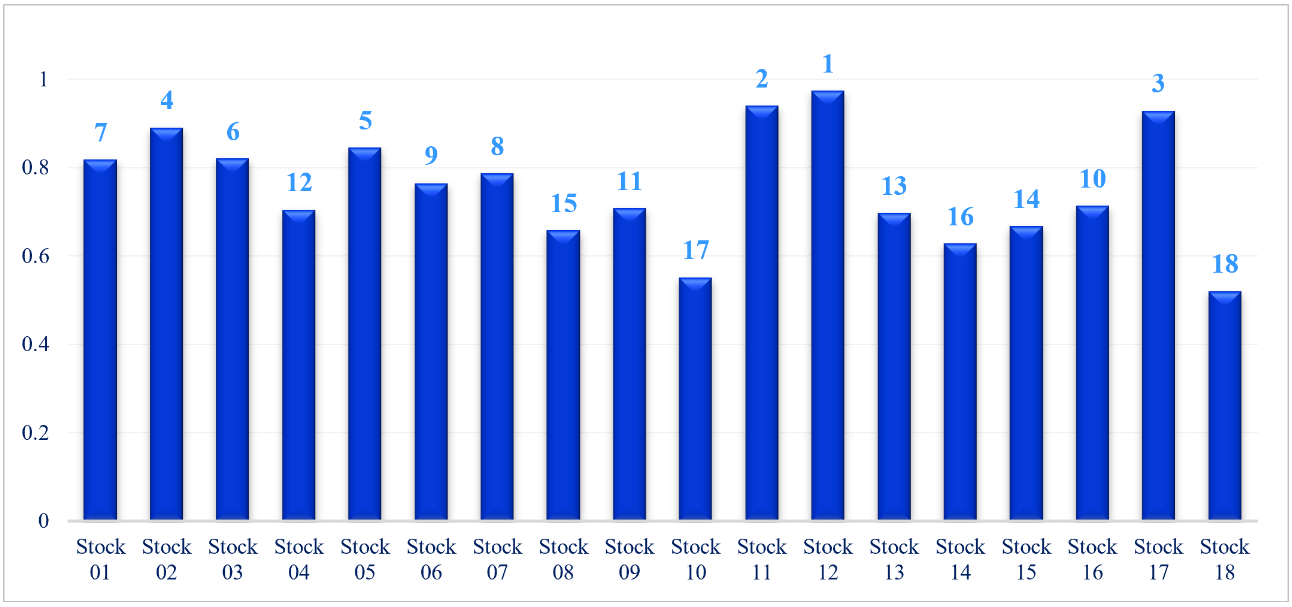

Figure 2. Stock evaluation and ranking results using the uncertain RDM model

Furthermore, it is clearly evident that Stocks 18, 10, 14, 8, 15, and 13 exhibit the worst performance under both certainty and uncertainty conditions. The consistency of this result across both classical and uncertain frameworks confirms the robustness of this finding. Based on these observations, these stocks can be confidently excluded from the investment portfolio, while preference can be given to stocks with more favorable performance profiles.

The proposed uncertain RDM model serves as a practical and effective tool for investment portfolio selection. Its first function is to filter out undesirable stocks by systematically identifying those that consistently underperform under uncertain environments. After the set of desirable stocks has been identified based on their efficiency rankings, the next logical step is to determine the optimal capital allocation among them. To this end, portfolio optimization models including mean-variance, mean-VaR, mean-CVaR, and other modern portfolio theory approaches can be employed to determine the precise investment weight of each favorable stock.[44–46] By integrating the filtering capability of the URDM model with the allocation power of optimization models, investors can construct portfolios that are both performance efficient and uncertainty resilient. This integrated framework thus provides a practical and theoretically grounded decision support system for investors, portfolio managers, and financial analysts operating in volatile and uncertain markets.

5. Conclusions and future research directions

The integration of uncertainty theory with DEA methodologies constitutes a significant advancement in the field of efficiency analysis. By addressing the inherent limitations of traditional DEA models, the proposed URDM framework contributes to the development of more robust and flexible analytical tools. These tools are relevant not only to academic research but also to practical applications across various industries, including finance, manufacturing, healthcare, and environmental management. The findings of this study underscore the potential of combining uncertainty theory with the RDM model to deliver more accurate and reliable efficiency assessments, thereby supporting informed decision-making under uncertain conditions. Furthermore, the adoption of uncertainty-based DEA models aligns with the broader trend of recognizing uncertainty as an integral aspect of decision-making. In an era defined by rapid technological change and global interconnectedness, uncertainty is no longer an anomaly but rather a fundamental characteristic of modern systems. By developing methodologies that explicitly account for uncertainty, researchers and practitioners can enhance the resilience and adaptability of decision-making processes. The proposed URDM model exemplifies this approach, offering a versatile framework for addressing the intertwined challenges of uncertainty and negative data in efficiency analysis.

For future research, it is suggested to extend the proposed approach by integrating it with the Best-Worst Method (BWM) to improve the weighting process and reduce pairwise comparison inconsistencies. This integration would allow decision-makers to incorporate subjective preferences more systematically while preserving the objectivity of the uncertain DEA framework. By combining BWM with the proposed uncertain RDM model, the overall robustness and discriminative power of efficiency evaluations under negative and uncertain data can be significantly enhanced. Such a hybrid methodology could offer a more balanced trade-off between stakeholder judgments and data-driven efficiency measurement.[47–51]

Another promising avenue is to incorporate time-varying data and dynamic performance measurement techniques. In this context, several established methodologies can be considered, including Dynamic DEA, Window DEA, and the Malmquist Productivity Index (MPI). Dynamic DEA explicitly accounts for intertemporal dependencies by linking consecutive periods through carry-over variables, while Window DEA evaluates efficiency by treating each time period as a separate window and averaging scores across overlapping intervals. The MPI, on the other hand, enables the measurement of productivity changes over successive time periods and decomposes efficiency shifts into technical change and technological change components. By adopting any of these dynamic perspectives, the proposed framework could better reflect the evolving nature of real-world systems, where decision-making units continuously adapt to changing conditions and uncertain environments.[52–57]

A further direction would be to enhance the model's capability to operate within a network structure. This enhancement would enable the approach to address more complex and interconnected systems, such as bank branches, insurance companies, mutual funds, supply chains and electricity distribution companies, where interactions between entities play a critical role. By incorporating network-based analysis, the model could account for dependencies, resource flows, and collaborative dynamics, thereby improving its applicability to real-world scenarios. Additionally, integrating network-specific metrics and constraints could provide deeper insights into system efficiency, resilience, and optimization under uncertainty.[58–63]

Such an advancement would not only broaden the scope of the proposed approach but also offer valuable tools for decision-making in multi-dimensional and interdependent environments.[64–66]

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Availability of data

All data generated or analyzed during this study are included in this manuscript.

AI tools statement

During the preparation of this work, the authors used Generative AI and AI-assisted technologies to improve the readability and language of the manuscript. After using these tools/services, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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