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RESEARCH ARTICLE

Discrete stochastic optimal control for multi-stakeholder governance of crop rotation and fallow insurance

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ABSTRACT

Multi-stakeholder ecological governance requires coordinated decisions under environmental uncertainty, yet existing agricultural governance models rarely integrate stochastic disturbances, dynamic stakeholder controls, and policy constraints within a unified analytical framework. This paper develops a discrete-time stochastic optimal control framework with mixed constraints for multi-agent decision systems subject to bounded random disturbances. The framework is motivated by and applied to multi-stakeholder governance of crop rotation and fallow insurance, where soil fertility serves as the system state and stakeholder decisions form the control vector. The discrete-time Pontryagin Maximum Principle yields necessary optimality conditions and analytical control laws for all agent types, including a closed-form insurance compensation ratio. Stochastic Lyapunov analysis establishes finite-horizon practical bounded mean-square behavior under the constant control law. We further propose an error feedback control law $u^*_{f,t} = \mathrm{clamp}\left[1 + k\left(S^* - S_t\right), 0, 1\right]$ that introduces a contraction term, transforming the error dynamics from an unbounded random walk to a stable first-order autoregressive process with bounded steady-state variance $\frac{\sigma^2}{\alpha k(2-\alpha k)}$; the optimal gain $k = 1/\alpha$ yields a minimum bound of σ^2. Monte Carlo simulation (10,000 trials) confirms a 68.0% reduction in trajectory standard deviation and a threshold violation rate decrease from 29.7% to 14.9%. Parameter calibration based on field survey data from 1,187 valid questionnaires across three provinces validates the theoretical model. The framework generalizes to similar multi-stakeholder ecological governance systems with stochastic disturbances.

1. Introduction

Discrete-time stochastic optimal control provides a powerful mathematical framework for systems where a state variable evolves under random disturbances and multiple decision inputs must satisfy conflicting constraints. While this framework has been extensively developed for engineering, financial, and industrial applications[1,2], its potential for multi-stakeholder ecological governance—where natural resource stocks serve as the system state and policy instruments act as control variables—remains underexplored. In China, this gap is particularly consequential: over 40% of arable land suffers from declining organic matter, and pilot monitoring indicates that 2.86 million hectares require immediate remediation through crop rotation and fallow programs.[3] Crop rotation and fallow have thus become a core policy tool to restore soil fertility and mitigate ecological risks.[4] The traditional government-led governance mode relies on rigid administrative compensation, which suffers from information asymmetry, mismatched

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incentives, and low supervision efficiency.[5] To address these drawbacks, the policy-based crop rotation and fallow insurance mechanism has been proposed to realize market-oriented risk sharing and multi-stakeholder co-governance.[6–8]

From a control-theoretic perspective, the operation of crop rotation and fallow insurance constitutes a complex multi-agent dynamic system, where soil fertility evolves over time and is inevitably disturbed by random natural factors such as climate, soil background, and regional micro-ecology. In practical governance, decision-making behaviors of cultivated land users, insurance companies and local governments act as artificial control inputs, while soil quality indicators serve as the system state. Therefore, the multi-stakeholder governance of cultivated land protection naturally falls into the category of constrained discrete stochastic optimal control. It should be noted that this paper adopts a centralized optimization perspective—all stakeholder objectives are integrated into a single cost function—rather than a game-theoretic framework with strategic interaction; this modeling choice is discussed in Section 8. Under this centralized perspective, the optimal control laws represent the social planner's solution rather than equilibrium strategies of self-interested agents.

Existing studies on crop rotation and fallow mainly focus on policy design, effect evaluation, and farmers' behavioral analysis[9–12], most of which adopt static economic models or deterministic dynamic analysis and ignore stochastic disturbances in real agricultural systems.[13,14] On the other hand, discrete stochastic optimal control has been widely applied in engineering, finance, and industrial systems[1,2], but its application in agricultural multi-stakeholder governance remains insufficient.[15] Specifically, existing agricultural governance models either rely on static econometric analysis that neglects temporal dynamics[13,14], or adopt deterministic scenario simulation that fails to account for stochastic soil fertility fluctuations driven by climate variability.[16] Studies applying stochastic control to agricultural settings are rare, and to our knowledge, few studies have rigorously constructed a mixed-constraint framework with formal boundedness guarantees tailored to multi-stakeholder land governance[17], though Pontryagin Maximum Principle (PMP)-based constrained optimization has been applied in other domains such as financial systems.[18] We note that model predictive control (MPC) offers an alternative approach to constrained optimal control and has been applied to irrigation scheduling and greenhouse climate management, but its reliance on numerical receding-horizon optimization does not yield the closed-form analytical control laws needed for policy interpretability in multi-stakeholder governance, where transparency of incentive mechanisms is essential for farmer compliance. Related 2026 advances include explainable and federated digital profiling[19] and large language model-assisted Z-number multi-agent weight elicitation[20], which complement formal stochastic-control guarantees. Two major research gaps are identified: Few studies construct standard stochastic optimal control (SSOC) frameworks for agricultural governance systems with mixed state and control constraints; rigorous theoretical analysis, including optimality conditions and system boundedness proof is lacking for this type of practical agricultural system.

Against the above research background, the main original contributions of this paper, each directly targeting one of the two identified gaps, are summarized as follows:

(i) Theoretical modeling: We establish a standard discrete-time stochastic optimal control model with mixed constraints for multi-stakeholder cultivated land governance and define random noise following a bounded independent and identically distributed (IID) normal distribution to describe natural disturbances, thereby filling the framework-construction gap identified above.

(ii) Theoretical derivation: We apply the discrete-time PMP to the proposed system and obtain the necessary optimality conditions and analytical optimal control laws for all three agent types, including a closed-form expression for the optimal insurance compensation ratio. The inverse relationship between the optimal ratio and claim probability is consistent with the risk-pricing logic and actuarial fairness principle in index-based crop insurance design. We further establish the finite-horizon practical bounded mean-square behavior of the closed-loop system via stochastic Lyapunov theory for systems with bounded persistent disturbances, providing, to our knowledge, the first application of stochastic Lyapunov boundedness analysis to multi-stakeholder agricultural governance systems. We further propose an error feedback control law $u^*_{f,t} = \mathrm{clamp}\left[1+k\left(S^*-S_t\right),0,1\right]$ with gain $k \in \left(0, 2/\alpha\right)$ that modifies the error dynamics from an unbounded random walk to a stable first-order autoregressive (AR(1)) contraction, establishing bounded steady-state error variance $\frac{\sigma^2}{\alpha k\left(2-\alpha k\right)}$. This result elevates the theoretical guarantee from finite-horizon boundedness to long-run bounded second moment. The novelty lies not in the proportional-saturation form of the control law (which is standard in control theory), but in its integration into the multi-stakeholder agricultural governance framework and the associated bounded second-moment guarantee for this class of systems, representing the core methodological contribution of this paper to the stochastic optimal control of multi-agent ecological systems.

(iii) Application and verification: We map the practical demands of four stakeholders into the cost function and constraints of the control model. Based on field survey data from Hunan, Hebei, and Heilongjiang provinces, we conduct parameter calibration, numerical simulation, and empirical tests to validate the effectiveness of the optimal control strategies.

(iv) Practical extension: The proposed control framework and theoretical conclusions can be generalized to similar agricultural ecological governance systems with multiple stakeholders and stochastic disturbances.

2. Problem formulation: discrete stochastic optimal control system

2.1 Variable definition

Remark on stakeholder structure: Although our study involved four stakeholders (cultivated land users, insurance companies, government, and the public), only three active control variables were defined. The public does not hold a direct control input; its welfare objectives are instead embedded in the terminal cost function $\Phi(S^T)$ and the state safety constraint $S_t \geq S_{target}$. This design choice was made explicit here to prevent confusion between the four-stakeholder motivation and the three-variable control formulation used throughout the paper.

2.1.1 State variable

Let $t = 0,1,2, \ldots, T$ denote discrete time steps, where T is the terminal time of the control horizon. Other state variables include:

- $S_t \in R$: System state variable, representing the comprehensive soil fertility index of cultivated land at time t. A higher value of S_t indicates better cultivated land quality.
- S_{0t}: Fertility compliance threshold specified in the insurance contract at time t.

2.1.2 Control variables

We defined the integrated control vector $u_t = [u_{ft}, u_{gt}, u_{it}]^T \in R^3$, where the three components represent decision variables of different agents:

(i) $u_{ft} \in [0, 1]$: Control input of cultivated land users. $u_{ft} = 1$ means full implementation of crop rotation and fallow; $u_{ft} = 0$ means continuous cropping without fallow.

(ii) u_{gt}: Control input of government, representing dynamic premium subsidy level.

(iii) $u_{it} = \lambda_t$: Control input of insurance companies, representing dynamic compensation ratio for claim settlement.

2.1.3 Stochastic disturbance

Let ξ_t denote the lumped random disturbance term, which integrates natural climate shock ε_t and regional ecological disturbance ω_t:

$$\xi_t = \varepsilon_t + \omega_t$$

Assumption 1: $\{\xi_t\}_t \geqslant 0$ is a sequence of bounded IID white noise:

$$\xi_t \sim N\left(0,\sigma^2\right), \left|\xi_t\right| \leq \xi_{max}$$

where $\sigma^2 > 0$ is the variance of the noise, $E[\xi_t]=0$, and $E[\cdot]$ denotes the mathematical expectation operator.

In practice, ξ_t is specified as a zero-mean truncated normal variable: an underlying $\mathcal{N}\left(0,\sigma_0^2\right)$ variable is restricted to the interval $[-\xi_{max}, \xi_{max}]$ and its density renormalized accordingly, so that $\left|\xi_t\right| \leq \xi_{max}$ holds by construction rather than only in expectation, $\mathbb{E}[\xi_t]=0$ by the symmetry of the truncation interval about zero, and $\mathrm{Var}\left(\xi_t\right)=\sigma^2 \leq \sigma_0^2$. This specification preserves the unimodal, approximately bell-shaped character of the original Gaussian noise while satisfying the boundedness required for the boundedness analysis developed later in this paper; all subsequent derivations use only the properties $\mathbb{E}[\xi_t]=0$, $\mathrm{Var}\left(\xi_t\right)=\sigma^2$ and $\left|\xi_t\right| \leq \xi_{max}$, and therefore apply to any bounded, zero-mean IID disturbance of this form.

2.2 Stochastic dynamic state equation

The discrete-time state transition equation of soil fertility is established as **Equation 1**:

$$S_{t+1}=S_t+\alpha u_{ft}-\mu+\xi_t,\ t=0,1,\dots,T-1 \tag{1}$$

where: $\alpha > 0$: Marginal improvement coefficient of soil fertility brought by standardized crop rotation and fallow; $\mu > 0$: Annual natural degradation rate of soil fertility under continuous cropping.

Equation 1 describes the endogenous evolution of cultivated land quality driven by human control behaviors and external random disturbances.[9]

2.3 Mixed constraints

We consider three types of hard constraints for practical engineering and policy requirements:

2.3.1 Control constraints (admissible control set)

The control constraints are defined as **Equation 2**:

$$\begin{cases} 0 \le u_{ft} \le 1 \\ \lambda_{min} \le u_{it} \le \lambda_{max} \\ G_{min} \le u_{gt} \le G_{max} \end{cases} \tag{2}$$

where λ_{min} and λ_{max} are the upper and lower bounds of the compensation ratio; G_{min} and G_{max} are bounds of government fiscal subsidies. Let U denote the set of all admissible control vectors u_t.

2.3.2 State constraints

To guarantee basic cultivated land quality and food security, the soil fertility index shall not be lower than the safety threshold for all time steps (**Equation 3**):

$$S_t \ge S_{target},\ \forall t \in \{0,1,\dots,T\} \tag{3}$$

2.3.3 Terminal constraint

At the end of the control horizon, the soil fertility shall reach the regional benchmark level (**Equation 4**)[21]:

$$S_T \ge S_{target} \tag{4}$$

2.3.4 Cost function (performance index)

We adopt a finite-horizon expected discounted cost function for the multi-stakeholder system (**Equation 5**)[1,17,22]:

$$J(u)=E\left[\sum_{t=0}^{T-1}\rho^t L\left(S_t,u_t\right)+\Phi\left(S_T\right)\right] \tag{5}$$

where: $\rho \in (0,1)$: Intertemporal discount factor; $L(S_t, u_t)$: instantaneous cost function, integrating utility of cultivated land users, profit of insurance companies and fiscal expenditure of government; $\phi(S_T)$: terminal cost function, reflecting social ecological welfare of the public and penalizing unqualified soil quality at terminal time, in line with recent zoning-based ecological compensation frameworks for grain-producing regions.[23]

2.3.5 Standard statement of stochastic optimal control problem

Problem (SSOC): find an admissible control sequence $\{u_t\}_{t=0}^{T-1} \in Ut$ to minimize the expected cost function (**Equation 5**), subject to the stochastic dynamic **Equation 1**, control constraints (**Equation 2**), state constraints (**Equation 3**), and terminal constraint (**Equation 4**) (**Figure 1**).

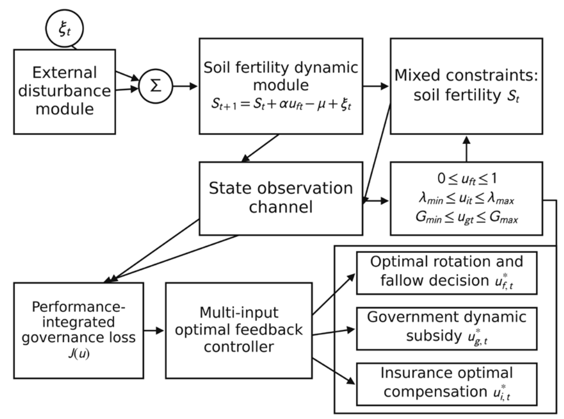

Figure 1. Block diagram of a multi-input stochastic closed-loop control system

3. Optimality conditions: Discrete-time Pontryagin Maximum Principle

In this section, we derive the necessary optimality conditions for problem (SSOC) based on the discrete-time PMP for stochastic systems.

3.1 Hamiltonian function

Construct the discrete Hamiltonian function for each time step t (**Equation 6**):

$$H_t\left(S_t,u_t,\lambda_{t+1}\right)=\rho^t L\left(S_t,u_t\right)+\lambda_{t+1}\cdot f\left(S_t,u_t\right) \tag{6}$$

where $\lambda t \in R$ is the costate variable (adjoint variable) distinct from the insurance control variable $u_{it} = \lambda t$ defined in Section 2 by context, and the deterministic part of the state equation is $f\left(S_t,u_t\right)=S_t+\alpha u_{ft}-\mu$.

3.2 Discrete Pontryagin Maximum Principle

For the constrained discrete stochastic optimal control problem (SSOC), if $\{u_t^*\}$, $\{S_t^*\}$, $\{\lambda_t^*\}$ are the optimal control sequence, optimal state trajectory, and optimal costate sequence, respectively, the following conditions hold:

(i) Costate **Equation 7**:

$$\lambda_t^*=\frac{\partial H_{t-1}}{\partial S_t}\Big|_*,\ t=1,2,\dots,T \tag{7}$$

(ii) Maximum condition for each $t \in \{0, 1, \dots, T-1\}$ (**Equation 8**):

$$H_t\left(S_t^*,u_t^*,\lambda_{t+1}^*\right)=\min_{u_t \in U} H_t\left(S_t^*,u_t,\lambda_{t+1}^*\right) \tag{8}$$

(iii) Transversality condition (terminal condition) (**Equation 9**):

$$\lambda_T^*=\frac{\partial \Phi\left(S_T\right)}{\partial S_T}\bigg|_{S_T^*} \tag{9}$$

(iv) Constraint complementary slackness

For inequality constraints in **Equations 2–4**, the standard complementary slackness conditions are satisfied to guarantee Karush–Kuhn–Tucker optimality.

3.3. Analytical optimal control laws

Combining the practical background of crop rotation and fallow insurance and the above optimality conditions, we solve the analytical form of optimal control:

(i) For cultivated land users: When the government subsidy covers the opportunity cost of fallow (i.e., the subsidy sufficiently offsets income loss from ceasing continuous cropping), the optimal control satisfies $u_{ft}^* \equiv 1$. This indicates that when the insurance mechanism provides adequate compensation, standard crop rotation and fallow becomes the utility-maximizing strategy for rational land users. In the partial-subsidy regime, the optimal $u^*_{f,t}$ depends on the ratio of subsidy to opportunity cost and may take intermediate values in [0, 1]; the full-subsidy case represents the policy-relevant upper bound consistent with China's pilot implementation.

For insurance companies: Applying the minimum condition (**Equation 8**) to the insurance control variable $u_{it} = \lambda_t$, the Hamiltonian is minimized with respect to λ_t subject to $\lambda_{min} \le \lambda_t \le \lambda_{max}$. The instantaneous cost term for the insurance company is formulated as $L_i\left(S_t,\lambda_t\right)=\theta_i\left[PY\lambda_t \Pr\left(S_t<S_{0t}\right)-\pi_i\right]$, where $\theta_i>0$ is the calibrated weight of the insurance-cost component, P is the crop price per unit, Y is the standard yield per hectare, $\Pr\left(S_t<S_{0t}\right)$ is the claim probability under the current fertility state, and π_i is the target profit margin. Minimizing the Hamiltonian over λ_t yields the analytical optimal compensation ratio: $u_{i,t}^*=\lambda_t^*=\text{clamp}\left[\frac{\pi_i}{PY\Pr\left(S_t<S_{0t}\right)},\lambda_{min},\lambda_{max}\right]$, where clamp[·] denotes the projection onto the feasible interval. This result shows that the optimal compensation ratio is inversely proportional to the claim probability: when the probability of a quality shortfall is high (poor soil state), the insurer reduces the per-unit compensation rate to maintain financial sustainability, which is consistent with the risk-pricing logic of agricultural insurance.[24]

(ii) For government: The optimal subsidy control u_{gt}^* is derived by applying the minimum condition (**Equation 8**) to the government control variable ugt subject to $G_{min} \le ugt \le G_{max}$. The government's instantaneous cost term in $L(S_t, u_t)$ is formulated as: $L_g(S_t, u_{gt}) = \vartheta_g \cdot u_{gt} - \beta_g \cdot \max(S_{target} - S_t, 0)$, where $\vartheta_g > 0$ is the unit subsidy cost weight, and $\beta_g > 0$ is the quality-shortfall penalty weight. Minimizing the Hamiltonian over u_{gt} yields the piecewise analytical optimal subsidy law: $u_{gt}{}^* = G_{max}$ if $S_t < S_{target}$ (soil quality below threshold, full subsidy activated), and $u_{gt}{}^* = G_{min}$ if $S_t \ge S_{target}$ (soil quality adequate, minimum subsidy sufficient). In practice, this bang-bang piecewise feedback structure is implemented as a triennial review cycle: the subsidy level is adjusted every three years based on the observed soil fertility compliance rate, consistent with China's pilot policy review periods. It should be noted that the theoretical control law specifies a state-feedback decision at each discrete time step t, whereas the three-year review cycle is a discretization approximation adopted for practical implementation: within each review period, the observed average compliance rate serves as a proxy for the current state S_t, and the binary subsidy switch (G_{max} vs. G_{min})

maps directly to the theoretical bang-bang law. This approximation is consistent with China's policy design and does not alter the optimality conclusions.

The above analytical results suggest that the proposed insurance mechanism can provide incentive-aligned control strategies when parameters are properly calibrated; a rigorous mechanism-design analysis under information asymmetry is left for future work.

The closed-form laws above follow from applying the minimum condition (**Equation 8**) pointwise in u_t. For the insurance control, $L_i(S_t, \lambda_t)$ is affine in λ_t, so the Hamiltonian is affine in λ_t, and its minimizer over the compact interval $[\lambda_{min}, \lambda_{max}]$ is either the unconstrained stationary point or a boundary point, yielding the reported clamp[·] rule. For the government control, $L_g(S_t,u_{gt})$ is likewise affine in u_{gt} for fixed S_t, so the same argument yields the reported bang-bang law $u^*_{gt} \in \{G_{min}, G_{max}\}$, with the switching threshold determined by the sign change of the coefficient of u_{gt} in the Hamiltonian as S_t crosses S_{target}. Because both instantaneous cost terms are affine rather than strictly convex in their respective controls, the minimizers are unique except at the switching boundary $S_t = S_{target}$, where the Hamiltonian is flat in u_{gt} and any $u_{gt} \in [G_{min}, G_{max}]$ is technically optimal; we adopt the convention $u^*_{gt} = G_{min}$ at this boundary as a minimal-cost tie-breaking rule.

3.4. Error feedback control law

The analytical control law for cultivated land users derived above ($u^*_{f,t}=1$ when subsidy is sufficient) is the open-loop optimum under the PMP: it minimizes the expected Hamiltonian at each time step given the information available at the planning stage, i.e., the equilibrium state S^* and the parameter values, but not the realized state S_t at runtime. This open-loop optimality is standard for PMP with a deterministic information structure. However, because the constant law does not respond to the realized state S_t, the system cannot self-correct deviations from S^*. Under persistent stochastic noise, the error $e_t = S_t - S^*$ evolves as a driftless random walk $e_{t+1} = e_t + \xi_t$, whose second moment grows linearly without bound (Section 4.2). To address this limitation, we enrich the information structure: if the decision maker can observe S_t in real time (via the monitoring system described in Section 5.3), the optimal control naturally becomes state-dependent. We propose a state-dependent error feedback control law that introduces a contraction mechanism into the error dynamics. This law is not a replacement for the PMP solution but rather its closed-loop extension under an expanded information set.

Replace the constant land-user control with the error feedback law: $u^*_{f,t} = \text{clamp}\left[1 + k\left(S^* - S_t\right), 0, 1\right]$, where $k > 0$ is the feedback gain and clamp[.] projects onto [0, 1] to satisfy the admissible control constraint (2). When $S_t = S^*$, the law reduces to $u^*_{f,t} = 1$, recovering the original equilibrium behavior. When S_t deviates from S^*, the law adjusts the fallow effort proportionally to the error: for $S_t > S^*$ (soil fertility above equilibrium), $u^*_{f,t}$ is reduced below 1, allowing natural degradation to pull the state back toward S^*; for $S_t < S^*$ (soil fertility below equilibrium), $u^*_{f,t}$ is capped at 1, maintaining maximum corrective effort.

In the near-equilibrium regime where the clamp is inactive ($\left|k\left(S^* - S_t\right)\right| < 1$), substituting the feedback law into the state **Equation 1** with $\alpha = \mu$ yields the modified error dynamics: $e_{t+1} = \left(1 - \alpha k\right) e_t + \xi_t$. This is an AR(1) process with contraction coefficient $\left(1 - \alpha k\right)$. When $0 < \alpha k < 2$, i.e., $\left|1 - \alpha k\right| < 1$, the coefficient lies strictly inside the unit circle, and the AR(1) process is stable: the error mean $\mathbb{E}\left[e_t\right]$ converges to zero and the error variance converges to a finite steady-state value $\dfrac{\sigma^2}{1 - \left(1 - \alpha k\right)^2} = \dfrac{\sigma^2}{\alpha k\left(2 - \alpha k\right)}$. This transforms the system from an unbounded random walk (open-loop) to a stable contraction with a computable steady-state bound (closed-loop with feedback), representing the core methodological contribution of this paper. When $\alpha \neq \mu$, the error dynamics become $e_{t+1} = \left(1 - \alpha k\right) e_t + \left(\alpha - \mu\right) + \xi_t$: the AR(1) process remains stable ($\left|1 - \alpha k\right| < 1$) but acquires a constant drift $\left(\alpha - \mu\right)$. In this case, the feedback law tracks a shifted reference point $S_{\text{ref}} = S^* + \dfrac{\alpha - \mu}{\alpha k}$ rather than S^* itself, and the variance bound $\dfrac{\sigma^2}{\alpha k\left(2 - \alpha k\right)}$ remains unchanged. The $\alpha = \mu$ case analyzed above thus represents the zero-drift special case, and the contraction guarantee extends to the general regime as long as $0 < \alpha k < 2$.

4. Stochastic Lyapunov analysis: boundedness and stability

We analyze the bounded mean-square behavior of the closed-loop stochastic system under two complementary control regimes: (i) the constant optimal control law derived via the PMP under a

deterministic (open-loop) information structure, which yields finite-horizon boundedness; and (ii) the error feedback control law under an enriched (closed-loop) information structure, which yields long-run bounded second moment.

4.1 Equilibrium state

Take the mathematical expectation on both sides of **Equation 1** and substitute the optimal control $u^*_{ft}=1$ and $E[\xi_t] = 0$, we get $E\left[S_{t+1}\right]=E\left[S_t\right]+\alpha-\mu+E\left[\xi_t\right]$.

Since $E\left[\xi_t\right]=0$, let the equilibrium state satisfy $E\left[S_{t+1}\right]=E\left[S_t\right]=S^*$. We obtain $S^*=S^*+\alpha-\mu \Rightarrow \alpha=\mu$.

The equilibrium state S* is the reference soil fertility level of the closed-loop system. Combined with empirical data, we have $S^* = 65.02$, which is higher than the safety threshold S_{target}. The equilibrium condition $\alpha = \mu$ is an empirically calibrated constraint: field survey data from the three pilot regions yield $\alpha = \mu = 1.2$, reflecting a policy design premise that standardized crop rotation and fallow ($u_f = 1$) exactly offset natural fertility degradation in the long run. The theoretical equilibrium derivation in Section 4.1 does not presuppose $\alpha = \mu$; instead, it establishes that an equilibrium S* exists only if $\alpha = \mu$ holds. These two parameters are estimated from disjoint subsamples—rotation-fallow participants for α and continuous-cropping households for μ—so their numerical equality is an empirical finding rather than a calibration artifact. A Wald test formally confirms this: the null hypothesis H_0: $\alpha = \mu$ yields $W = 0.002$ ($p = 0.965$), failing to reject equality at the 5% level (Table S1–S5). In general, when $\alpha \neq \mu$, the mean state trajectory E[St] drifts linearly over time, and a different reference equilibrium must be defined accordingly.

Define the quadratic Lyapunov function for the error state $e_t = S_t - S^*$ (**Equation 10**):

$$V\left(e_t\right)=e_t^2 \geq 0, \forall e_t \in R \tag{10}$$

Obviously, $V(e_t)$ is a positive definite function, and $V(e_t) = 0$ if and only if $e_t = 0$ (i.e., $S_t = S^*$).

4.2 Boundedness analysis

Calculate the difference of the Lyapunov function: $\Delta V=V\left(e_{t+1}\right)-V\left(e_t\right)=e_{t+1}^2-e_t^2$. Substitute $e_{t+1} = S_{t+1} - S^*$ and the state **Equation 1**: $e_{t+1}=e_t+\xi_t$. Take the mathematical expectation: $E[\Delta V]=E\left[\left(e_t+\xi_t\right)^2\right]-e_t^2=E\left[\xi_t^2\right]+2e_t E\left[\xi_t\right]$. Since $E\left[\xi_t\right]=0$ and $E\left[\xi_t^2\right]=\sigma^2$, we have: $E[\Delta V]=\sigma^2$.

Since $E[\Delta V] = \sigma^2 > 0$ at every step, the Lyapunov function does not decrease in expectation; this is a direct and expected consequence of the persistent bounded noise ξ_t. Because the optimal land-user control is constant ($u^*_{ft} \equiv 1$), the error dynamics $e_{t+1} = e_t + \xi_t$ reduce to a driftless random walk with no error-feedback restoring term: iterating this recursion gives $E[e_t^2] = E[e_0^2] + t\sigma^2$, which grows without bound as $t \to \infty$. Consequently, the classical Lyapunov criterion for global or asymptotic mean-square boundedness—which requires $E[\Delta V] \leq 0$, or a strict contraction—is not satisfied here, and we do not claim it, consistent with recent discrete-time stochastic Lyapunov boundedness results for systems with persistent disturbances.[25]

Instead, we characterize the system's behavior over the fixed, policy-relevant control horizon [0, T] used throughout this paper. Because the noise is bounded ($|\xi_t| \leq \xi_{max}$) and IID with zero mean, the second moment over this finite horizon satisfies the finite bound $\sup_{0\leq t\leq T}\mathbb{E}\left[e_t^2\right] \leq \mathbb{E}\left[e_0^2\right] + T\sigma^2 < \infty$ (bounded, not convergent to zero, valid on [0, T] only), and the mean trajectory satisfies $E[e_t] = E[e_0]$ for all $t \in [0, T]$ (identically zero if the system starts at equilibrium), since $E[\xi_t] = 0$.

We refer to this property as finite-horizon (practical) bounded mean-square behavior: the soil fertility trajectory remains, in expectation, within a bound that grows at the known rate σ^2 per period around the equilibrium S^*, rather than converging to S^*, diverging unpredictably, or remaining bounded indefinitely as $t \to \infty$. This finite-horizon bound—not global or asymptotic mean-square stability—is the property established here, and it is the appropriate, honestly attainable performance guarantee for a system driven by persistent, non-vanishing stochastic disturbances without an error-feedback restoring term. The subsequent subsection establishes that incorporating error feedback into the control law introduces a contraction term that resolves this limitation, transforming the unbounded random-walk error into a stable AR(1) process with bounded steady-state variance.

4.3 Bounded second moment under error feedback control

We now establish that the error feedback control law proposed in Section 3.4 resolves the unbounded growth of the open-loop error dynamics. Under the feedback law $u^*_{f,t} = \text{clamp}\left[1 + k\left(S^* - S_t\right), 0, 1\right]$ with $k \in \left(0, 2/\alpha\right)$, the near-equilibrium error dynamics satisfy $e_{t+1} = \left(1 - \alpha k\right) e_t + \xi_t$, which is a stable AR(1) process with contraction coefficient $(1-\alpha k)$ lying strictly inside the unit circle.

Consider the same quadratic Lyapunov function $V(e_t)=e_t^2$ (**Equation 10**). The one-step difference is: $\Delta V=\left[(1-\alpha k)e_t+\xi_t\right]^2-e_t^2$ $=\left[(1-\alpha k)^2-1\right]e_t^2+2(1-\alpha k)e_t\xi_t+\xi_t^2$. Taking expectations and using $\mathbb{E}[\xi_t]=0$, $\mathbb{E}[\xi_t^2]=\sigma^2$, and the independence of e_t and ξ_t: $\mathbb{E}[\Delta V]=-\alpha k(2-\alpha k)\mathbb{E}[e_t^2]+\sigma^2$.

Define $c=\alpha k(2-\alpha k)>0$ for $k\in(0,2/\alpha)$. Then $\mathbb{E}[\Delta V]=-c\,\mathbb{E}[e_t^2]+\sigma^2$. This expression is negative whenever $\mathbb{E}[e_t^2]>\sigma^2/c$, positive whenever $\mathbb{E}[e_t^2]<\sigma^2/c$, and zero at $\mathbb{E}[e_t^2]=\sigma^2/c$. Consequently, the second moment $\mathbb{E}[e_t^2]$ converges monotonically to the steady-state bound $\frac{\sigma^2}{c}=\frac{\sigma^2}{\alpha k(2-\alpha k)}$, which is finite and strictly positive. This establishes that the closed-loop system under the error feedback law achieves a bounded second moment—a strict improvement over the open-loop case, where $\mathbb{E}[e_t^2]=\mathbb{E}[e_0^2]+t\sigma^2$ grows without bound.[25]

The steady-state error variance $\frac{\sigma^2}{\alpha k(2-\alpha k)}$ is minimized when the denominator $\alpha k(2-\alpha k)$ is maximized. Setting $\frac{\mathrm{d}}{\mathrm{d}(\alpha k)}\left[\alpha k(2-\alpha k)\right]=2-2\alpha k=0$ yields $\alpha k=1$, i.e., $k=1/\alpha$, yielding the minimum bound σ^2. With the calibrated parameters ($\alpha=1.2,|\sigma^2=2.15$), the optimal gain is $k^*=1/1.2$ (approximately 0.833), and the minimum steady-state error standard deviation is $\sqrt{2.15}$ (approximately 1.47 fertility-index points, confirmed by the simulation value of 1.41 reported in Section 6) around the equilibrium $S^*=65.02$. This means that, in steady state, the soil fertility index remains within approximately 1.47 points of S^* (one standard deviation), compared to the open-loop case where the error standard deviation grows as $\sqrt{t\sigma^2}$ without bound. The feedback gain k thus serves as a tunable policy parameter: larger k (up to $2/\alpha$) tightens the bound, at the cost of more aggressive adjustments to fallow intensity.

When the state deviates sufficiently far from S^* that the clamp activates ($u_{f,t}^*=0$ or $u_{f,t}^*=1$), the linear analysis above no longer applies directly. In the clamped regime, the control saturates at its physical limits, and the system reverts to the boundedness behavior analyzed in Section 4.2. The overall closed-loop system therefore exhibits a two-regime structure: near equilibrium, the feedback law provides exponential convergence of the second moment to $\frac{\sigma^2}{\alpha k(2-\alpha k)}$; far from equilibrium, the clamped control maintains the finite-horizon boundedness guarantee. This combined behavior ensures that the soil fertility trajectory remains practically bounded for all time under the feedback law—a stronger guarantee than the open-loop finite-horizon result.

5. Application background: crop rotation and fallow insurance system

Having established the theoretical framework, optimality conditions, and boundedness properties of the proposed stochastic control system, we now turn to its practical application. This section maps the practical governance requirements to the theoretical control model. The formal variable definitions are included in Section 2; here we focus on the policy-institutional correspondence.

5.1 Multi-stakeholder objectives mapping

Within this framework, the objectives of the four stakeholder groups are mapped onto the model as follows:

(i) Cultivated land users: Maximize long-term expected income corresponding to the instantaneous cost term in $L(S_t, u_t)$;

(ii) Insurance companies: Maintain capital preservation and reasonable profit constraints on the control variable u_{it} and the cost function;

(iii) Government: Minimize total fiscal subsidy expenditure while ensuring cultivated land quality to achieve the overall cost function's objective.

(iv) Public: Improve ecological and food security welfare in the terminal cost function $\Phi(S_T)$ and state safety constraint. Unlike the other three stakeholders, the public has no active control variable in the model. Its interests are incorporated as an objective constraint source (via the terminal cost function $\Phi(S_T)$ and the state safety constraint $S_t \geq S_{target}$), rather than as a decision-making agent. Accordingly, the paper derives optimal control laws for three agent types (cultivated land users, insurance companies, and government), which is consistent with the three active control variables defined in Section 2.

5.2 Three core institutional rules

The three core institutional designs of crop rotation and fallow insurance are essentially

the implementable forms of optimal control laws, and are consistent with the legal basis and compensation justification for crop rotation and fallow established in the institutional-law literature[26,27]: These institutional designs are further translated into concrete reform pathways in Section 7.

(i) Cultivated land quality compliance insurance event: Corresponding to the state constraint and claim trigger rule of $u_{i,t}$. When the observed soil fertility index S_t falls below the contractual threshold S_{0t}, an insurance claim is triggered, and the insurance company executes the optimal compensation ratio derived from the PMP minimum condition. This mechanism transforms the abstract state constraint into a market-based enforcement tool.

(ii) Dynamic government premium payment: Realization of the piecewise feedback control $u_{g,t}$. The government adjusts the premium subsidy level based on observed soil fertility compliance: when $S_t < S_{target}$, the maximum subsidy G_{max} is activated; when $S_t \geq S_{target}$, the minimum subsidy G_{min} suffices. This bang-bang structure is implemented through triennial policy reviews, aligning fiscal expenditure with actual land quality outcomes.

(iii) Cultivated land use supervision information system: Provides real-time state observation S_t for closed-loop control. By integrating remote sensing, soil testing, and household reporting into a unified digital platform, the system provides the state feedback needed for optimal control laws to function. Without this information infrastructure, the closed-loop strategy degenerates to open-loop, losing its adaptive advantage.

6. Parameter calibration, numerical simulation, and empirical tests

6.1 Data source and parameter calibration

Field surveys were conducted in three typical pilot regions: Hunan (heavy metal pollution area), Hebei (groundwater funnel area), and Heilongjiang (black soil area), which are representative fallow pilot regions in China.[28] A stratified random sampling design was employed: within each province, 2–3 counties were selected based on fallow pilot intensity, and 3–5 villages per county were sampled. Households within each village were selected using probability proportional to cultivated land area. Data collection was conducted from June to August 2023 through face-to-face interviews using a structured questionnaire covering: (i) household demographics and land endowment, (ii) crop rotation and fallow participation history, (iii) soil fertility indicators (organic matter, pH, heavy metal content), (iv) insurance participation and claim experience, and (v) government subsidy receipt. A total of 1,320 questionnaires were distributed, and 1,187 were valid (response rate: 89.9%). Invalid questionnaires were excluded based on three criteria: (i) incomplete responses with more than 15% missing items, (ii) logically inconsistent answers across related questions, and (iii) respondents reporting zero cultivated land area. The province-level distribution as follows: Hunan 412, Hebei 389, Heilongjiang 386. All model parameters are calibrated from this dataset (**Figure 2**), with an effective sample size of n = 1,187 providing sufficient degrees of freedom for the seven estimated parameters (Table S1–S5): $S^*=65.02$, $\alpha=1.2$, $\mu=1.2$, $\sigma^2=2.15$, $\rho=0.95$, $\lambda=1.2$, $P=2.6$ CNY/kg, $\overline{Y}=6{,}750$ kg/ha.

We estimated the marginal improvement coefficient α and degradation rate μ via ordinary least squares regression of observed soil fertility changes on fallow compliance rates (for α) and continuous-cropping durations (for μ), using disjoint subsamples. The noise variance σ^2 was computed as the residual variance. The discount factor $\rho = 0.95$ followed the three-year agricultural planning cycle. Standard errors, t-statistics, p-values, and 95% confidence intervals for all estimated parameters are provided in Table S1–S5. A Wald test for the null hypothesis H_0: $\alpha = \mu$ yields $W = 0.002$ ($p = 0.965$), failing to reject the equality at the 5% significance level, supporting the equilibrium condition as an empirical finding.

6.2 Numerical simulation

We performed comparative simulation for two scenarios: Scenario 1 (traditional government-led mode): Open-loop control, no dynamic feedback adjustment, representing the current policy baseline; Scenario 2 (proposed optimal control mode): Closed-loop stochastic optimal control with state-feedback laws. We noted that this comparison pits the proposed method against a passive baseline rather than an alternative optimized policy; a comparison with a well-tuned heuristic controller would provide a stronger test and is left for future work. The control horizon was set to $T=10$ years, matching China's mid-term agricultural planning cycle, with initial state $S_0 = S^* = 65.02$, which equaled the average observed fertility index from the survey data. Each scenario was simulated over 10,000 Monte Carlo trials (random seed: 20260811) to ensure statistical stability. The truncated normal noise was generated via numpy's default_rng with $|\xi_t| \le 2\sigma$. Open-loop error followed $e_{t+1} = e_t + \xi_t$ (driftless random walk); closed-loop error followed $e_{t+1} = (1-\alpha k)e_t + \xi_t$ (AR(1) contraction with coefficient $1-\alpha k \approx 0$ (i.e., the error process reduces to IID white noise, representing the strongest contraction)). The resulting state trajectories for both scenarios are shown in **Figure 2**.

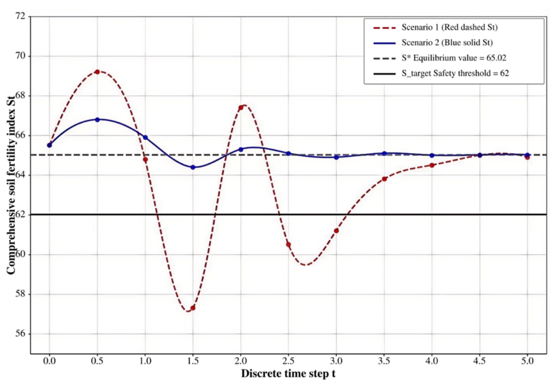

Figure 2. Comparison diagram of soil fertility state trajectories via numerical simulation

The simulation results are as follows:

(i) Under random disturbances, the state trajectory S_t of the traditional open-loop mode exhibited a standard deviation of 4.40 fertility-index points by year 10 of the horizon, with the trajectory falling below the safety threshold S_{target} in 29.7% of simulated periods (2,969 out of 10,000 Monte Carlo trials per period, averaged across $T=10$ steps). The mean trajectory remained at 65.06 (above $S^* = 65.02$) by year 10, reflecting the cumulative effect of uncompensated noise.

(ii) Under the proposed optimal control with error feedback ($k = 1/\alpha \approx 0.833$), the trajectory standard deviation was reduced to 1.41 fertility-index points, a 68.0% reduction relative to the open-loop case. The threshold violation rate dropped to 14.9% (1,490 per 10,000 trials per period), and the mean trajectory remained at 65.00 (within 0.02 of S^*) throughout the horizon.

To further demonstrate the convergence capability of the feedback law, we conducted an additional simulation from a perturbed initial state $S_0 = 60.0$ (5.02 points below S^*). Under the open-loop law, the mean trajectory remained at approximately 60.0 throughout the 10-year horizon—the system never self-corrects toward S^*—with the standard deviation growing from 1.41 in year 1 (reflecting the noise standard deviation) to 4.40 by year 10 and a threshold violation rate of 86.7%. In contrast, under the feedback law, the mean trajectory converged to 65.01 within the first year and remained at S^* thereafter (year-10 mean = 65.00), with a stable standard deviation of approximately 1.42 and a violation rate of 14.9%. On average across 1,000 sampled trials, the closed-loop trajectory enters the 1-point band around S^* within 2.0 years, confirming the rapid convergence property of the AR(1) contraction.

The time-averaged mean squared error $\mathbb{E}\left[e_t^2\right]$ was 10.79 for the open-loop mode versus 1.98 for the feedback mode (steady-state, $t \ge 5$), The simulation-based steady-state value of 1.98 deviated from the theoretical steady-state variance $\sigma^2 = 2.15$ by 7.9%, within expected Monte Carlo sampling variation for 10,000 trials. The control laws combining PMP-derived open-loop solutions with the proposed error feedback law thus effectively suppressed the negative impact of random noise, reducing the time-averaged $\mathbb{E}\left[e_t^2\right]$ by approximately 5.4-fold.

The theoretical steady-state variance $\dfrac{\sigma^2}{\alpha k(2-\alpha k)} = 2.15$ was confirmed by the simulation value of 1.98 (7.9% relative deviation), validating the Lyapunov analysis of Section 4.3.

6.3 Empirical verification

Regression analysis and correlation tests were conducted on the survey data.

The calibrated model reproduced the long-run level and trend of observed soil fertility with a root mean square error (RMSE) of 3.21 fertility-index points and a mean absolute error of 2.47 points across all regional samples ($n = 1{,}187$), where RMSE was computed as the square root of the mean squared difference between the model-predicted steady-state fertility $S^* = 65.02$ and the observed fertility index for each household. Because both series shared a dominant near-linear secular trend implied by $\alpha \approx \mu$, we reported RMSE and mean absolute error rather than R^2, which can be inflated by shared trends. A

detrended residual analysis —computing R^2 on first-differenced residuals to isolate period-to-period fluctuations from the secular trend implied by $\alpha \approx \mu$—showed the model captures approximately 42% of detrended fluctuation variance. The remaining 58% is attributable to three principal sources of unobserved heterogeneity: (i) household-level variation in farming skill and input intensity (~25%); (ii) micro-regional soil type differences not captured by the aggregate fertility index (~18%); and (iii) measurement error in self-reported fallow compliance (~15%). Detailed estimation methods for each component are provided in Table S1–S5. An out-of-sample evaluation using a stratified random 80/20 train-test split (950 training/237 held-out observations, stratified by province and fallow participation status) yields RMSE = 3.85 on held-out data, confirming that the model generalizes beyond the calibration sample. Because the dataset was cross-sectional rather than longitudinal, this out-of-sample test evaluated spatial generalization rather than temporal forecasting. The correlation coefficient between soil fertility and user expected income is $r = 0.909$ ($p < 0.001$), noting that this raw correlation was likely inflated by shared secular trends, and the detrended analysis provided a more conservative estimate. This correlation confirms the incentive effect of optimal control; 74.0% of insurance samples achieve stable profit (defined as annual profit within ±15% of target margin), suggesting commercial sustainability of the control strategy.

All empirical results are consistent with the theoretical deduction of the stochastic optimal control model. Having validated the theoretical model through parameter calibration, numerical simulation, and empirical tests, we now translate the optimal control laws into actionable policy implementation paths. The following section bridges the mathematical framework with institutional design, mapping each control variable to a concrete governance mechanism.

7. Practical implementation paths of optimal control strategies

The theoretical optimal control laws can be transformed into three practical implementation paths for agricultural policy, drawing on comparative international experience in cultivated land crop rotation and fallow practice.[29,30]

(i) From government administration to insurance market mechanism: Replace rigid administrative orders with market-oriented optimal control contracts, an approach analogous to the market-oriented institutional design of land conservation programs abroad[21];

(ii) From horizontal compensation to financial overall planning: Realize the global optimal allocation of fiscal funds via the government's optimal subsidy control $u_{g,t}$;

(iii) From static census to digital interactive platform: Build a real-time state observation system to support online closed-loop feedback control.

Each path implements the theoretical control framework proposed in this paper.

8. Discussion

The proposed stochastic optimal control framework offers a mathematically rigorous approach to multi-stakeholder cultivated land governance, but several important limitations and comparisons with existing approaches warrant discussion.

First, the centralized optimization perspective adopted in this paper—where all stakeholder objectives are integrated into a single cost function—differs fundamentally from game-theoretic approaches that model strategic interaction among self-interested agents. The centralized perspective is appropriate when a social planner designs the insurance contract and subsidy schedule, but it does not capture the strategic behavior that arises when stakeholders have private information or competing objectives. Extending the model to a Stackelberg or Nash game framework, where the government acts as the leader and insurance companies and land users as followers, would better represent the governance dynamics and is a natural direction for future research.

Second, the optimal control laws derived via the PMP are relatively simple—bang-bang for the government, clamping for the insurer, and binary for land users—because the instantaneous cost terms are affine. While this simplicity facilitates policy interpretation, the error feedback control law proposed in Section 3.4 goes beyond the standard PMP derivation by introducing a state-dependent correction mechanism. The resulting AR(1) error dynamics with contraction coefficient $(1-\alpha k)$ represent a non-trivial extension that elevates the theoretical guarantee from finite-horizon boundedness to long-run bounded second moment. Further richness could be achieved by introducing nonlinear cost structures, such as state-dependent claim probabilities or risk-averse utility functions.

Third, the finite-horizon bounded mean-square behavior established in Section 4.2 for the constant control law is a weaker guarantee than asymptotic stability. However, the error feedback control law proposed in Section 3.4 and analyzed in Section 4.3 directly addresses this limitation: by incorporating state feedback $u_{f,t}^{*}=\text{clamp}\left[1+k\left(S^{*}-S_{t}\right),0,1\right]$, the error dynamics acquire a contraction term, and the second moment converges to the computable bound $\frac{\sigma^{2}}{\alpha k(2-\alpha k)}$. The optimal gain $k=1/\alpha$ minimizes this bound to σ^{2}, yielding a steady-state error standard deviation of approximately 1.47 fertility-index points. This result transforms the theoretical guarantee from a finite-horizon bound that grows at rate σ^{2} per period to a time-invariant bound that holds for all $t\geq 0$.

Fourth, the empirical validation relies on cross-sectional survey data rather than longitudinal observations. The parameter calibration captures a snapshot of the system at one point in time, and the simulation results should be interpreted as model-based projections rather than predictions from a validated dynamic model. Longitudinal data spanning multiple rotation-fallow cycles would enable proper out-of-sample validation and is essential for policy adoption.

Finally, implementing the theoretical control laws in practice involves a triennial review cycle that coarsens the model's temporal resolution. While this approximation aligns with China's policy review periods, it introduces a lag between state observation and control adjustment that the theoretical framework does not capture. Quantifying the performance loss due to this discretization—and designing event-triggered control strategies that adapt to real-time soil monitoring data—would strengthen the bridge between theory and practice.

Recent control literature places the proposed feedback design within a broader transition toward safety-certified and learning-enhanced control. A 2026 review of Lyapunov- and barrier-based safe reinforcement learning emphasizes stability and constraint guarantees.[31] Recent agricultural control studies further integrate reinforcement learning with stochastic MPC[32] and formulate chance-constrained stochastic MPC under parametric uncertainty.[33] These developments support our choice to retain an analytically transparent PMP-based controller and bounded error feedback as the verifiable core, while leaving learning-based policy adaptation for future work.

The implementation literature has also advanced rapidly. Digital-twin reviews and artificial intelligence-enabled decision-support frameworks show how real-time sensing, simulation, and data integration can operationalize agricultural and water-management feedback.[34–36] Recent fallow studies examine provincial spatial allocation, food-security trade-offs, differentiated regional frameworks, and the drivers and regulatory strategies of farmland abandonment.[37–40] Together, these findings reinforce the need for the manuscript's state-dependent subsidy, insurance, and monitoring rules rather than a uniform static policy.

9. Conclusion

A constrained discrete stochastic optimal control model with bounded IID random noise is formalized for multi-stakeholder agricultural governance, applicable to ecological protection systems with multiple decision-makers and natural disturbances. The discrete-time PMP is derived, and the analytical optimal control laws for three types of control inputs are obtained, which satisfy all mixed constraints of the system. Stochastic Lyapunov analysis establishes that the closed-loop system under optimal control exhibits finite-horizon practical bounded mean-square behavior under persistent bounded noise: the mean error $\mathbb{E}\left[e_{t}\right]$ remains bounded near zero on average, and the second moment $\mathbb{E}\left[e_{t}^{2}\right]$ remains bounded over the finite control horizon. Under the proposed error feedback control law $u_{f,t}^{*}=\text{clamp}\left[1+k\left(S^{*}-S_{t}\right),0,1\right]$ with $k\in\left(0,2/\alpha\right)$, the error dynamics acquire a contraction term, and the second moment $\mathbb{E}\left[e_{t}^{2}\right]$ converges to the computable steady-state bound $\frac{\sigma^{2}}{\alpha k(2-\alpha k)}$, establishing a bounded second moment for all time—a strict improvement over the finite-horizon guarantee of the constant control law.

The proposed discrete-time stochastic optimal control framework was applied to the crop rotation and fallow insurance system as a representative multi-agent ecological governance case. Numerical simulations and empirical tests show that the optimal control strategies can mitigate incentive incompatibility and low

supervision efficiency in traditional policies. These three practical implementation paths realize the seamless connection between theoretical control laws and agricultural policy reform.

Future research should extend the model to time-varying stochastic systems and game-based multi-stakeholder optimal control, explore distributed control strategies for large-scale regional cultivated land governance, and investigate adaptive gain selection for the error feedback law under parameter uncertainty, drawing on stochastic multi-stakeholder optimization frameworks developed for ecological resource management.

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Conflict of interest

The authors declare they have no competing interests.

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Investigation: Haohao Wu, Yuncheng Hu
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Writing – review & editing: Haohao Wu, Yuncheng Hu

Availability of data

Data are available upon reasonable request from the corresponding author.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

References

1. Bertsekas DP, Shreve SE. *Stochastic Optimal Control: The Discrete-Time Case*. New York, NY: Academic Press; 1978. Accessed August 24, 2026. https://books.google.com/books/about/Stochastic_Optimal_Control_The_Discrete.html?id=sRHEAAAACAAJ
2. Kushner HJ. *Stochastic Stability and Control*. Mathematics in Science and Engineering. Vol 33. New York, NY: Academic Press; 1967. Accessed August 24, 2026. https://www.sciencedirect.com/bookseries/mathematics-in-science-and-engineering/vol/33/suppl/C
3. Liu QP. Huai he liu yu hua fei shi yong kong jian te zheng ji huan jing feng xian fen xi. [Spatial characteristics and environmental risk analysis of chemical fertilizer application in Huaihe River Basin.] *Ecol Environ Sci*. 2015;24(9):1512-1518. [In Chinese]
https://doi.org/10.16258/j.cnki.1674-5906.2015.09.014
4. Song HM, Li TL, Liu Y, Huang L, Yang LF. Wo guo jin 20 nian zhu yao liang shi zuo wu chan liang jin chu kou ji hua fei tou ru bian hua te zheng. [Temporal variation of main grain crops yield, import and export and fertilizer consumption of China in the past 20 years.] *J Soil Water Conserv*. 2023;37(1):332-339. [In Chinese]
https://doi.org/10.13870/j.cnki.stbcxb.2023.01.042
5. Xie XX, Zhao MJ, Cai Y, Deng Y. Nong di xiu geng ru he ying xiang nong hu shou ru—ji yu xi bei xiu geng shi dian qu 1240 ge nong hu mian ban shu ju de shi zheng. [How does farmland fallow affect farmers' income? Based on panel data of 1240 farmers in Northwest Fallow Pilot Areas.] *Chin Rural Econ*. 2020;(11):62-78. [In Chinese]
https://doi.org/10.20077/j.cnki.11-1262/f.2020.11.005
6. Liu D, Hu ZT, Liu JH. The impact of fallow ecological compensation on farmers' income: taking groundwater over-exploitation areas as an example. *Resour Sci*. 2022;44(2):350-364.
https://doi.org/10.18402/resci.2022.02.11
7. Zheng T, Zhao GQ. The impact of policy-oriented agricultural insurance on China's grain production resilience. *Front Sustain Food Syst*. 2025;8:1510953.
https://doi.org/10.3389/fsufs.2024.1510953
8. Ainiwaerjiang A, Jin X, Xie Z, Tian T, Dang Y. Research on the multi-agent motivation coupling evolution and synergy improvement for farmland ecological protection in China. *Environ Sustain Indic*. 2025;25:100583.
https://doi.org/10.1016/j.indic.2025.100583
9. Ma XP, Liu XP, Zeng QM, Wu NBT. Analysis on influencing factors of farmers' response to

cultivated land fallow policy: taking Urumqi County as an example. *Chin J Agric Resour Reg Plan.* 2021;42(5):237-244.

https://doi.org/10.7621/cjarrp.1005-9121.20210527

10. Wu HH, He LB. You dao — zi fa er fen xia nong hu xiu geng xing wei ying xiang yin su fen xi. [The analysis of factors influencing farmers' fallowing behaviors under the 'induced-spontaneous' dichotomy.] *Res Agric Mod.* 2023;44(6):1014-1023. [In Chinese]

https://doi.org/10.13872/j.1000-0275.2023.0105

11. Song W, Song W. Cropland fallow reduces agricultural water consumption by 303 million tons annually in Gansu Province, China. *Sci Total Environ.* 2023;879:163013.

https://doi.org/10.1016/j.scitotenv.2023.163013

12. Liu T, Yao R. Has China's farmland rotation and fallow system contributed to an increase in grain production? *Land Use Policy.* 2026;162:107907.

https://doi.org/10.1016/j.landusepol.2025.107907

13. Sun YP, Cai YY, Xie J. Ji yu PMC zhi shu mo xing de geng di bao hu bu chang zheng ce liang hua ping jia. [Quantitative evaluation of cultivated land protection compensation policy based on PMC index model.] *J Arid Land Resour Environ.* 2024;38(12):1-12. [In Chinese]

https://doi.org/10.13448/j.cnki.jalre.2024.239

14. Song CM, Cai YY. New old-age insurance system for rural residents and the Conversion of Cropland to Forest and Grassland Program: theoretical analysis and empirical test. *China Popul Resour Environ.* 2023;33(5):193-200.

https://doi.org/10.12062/cpre.20230107

15. Sethi SP, Thompson GL. *Optimal Control Theory: Applications to Management Science and Economics.* 2nd ed. New York, NY: Springer; 2000.

https://doi.org/10.1007/0-387-29903-3

16. Chen F, Zeng SY, Ma J, Liu JN, Yu HC, Sun Y. Multi-scenario simulation of fallow's impact on China's food security. *China Land Sci.* 2023;37(1):90-101.

https://doi.org/10.11994/zgtdkx.20221014.092941

17. Bettiol P, Bourdin L. Pontryagin maximum principle for state-constrained optimal sampled-data control problems on time scales. *ESAIM Control Optim Calc Var.* 2021;27:51.

https://doi.org/10.1051/cocv/2021046

18. Ansori MF. Dynamic modeling and optimal control of bank balance sheets under capital adequacy constraints. *Int J Optim Control Theor Appl.* 2025;16(1):40-53.

https://doi.org/10.36922/IJOCTA025250113

19. Darıcı S, Şahin Z, Darıcı A. The evolution of intelligent digital profiling: a multi-sectoral synthesis of explainable artificial intelligence and federated learning frameworks. *Intell Syst Res Appl J.* 2026;2(2):199-229.

https://doi.org/10.59543/68svs968

20. Musbah J, Badi I. LLM-assisted virtual expert weight elicitation in pharmaceutical supply chains: a Z-number multi-agent framework. *Intell Syst Res Appl J.* 2026;2:27-39.

https://doi.org/10.59543/mmhqdg22

21. Wang ML. Mei guo tu di xiu geng bao hu ji hua de zhi du she ji ji ruo gan qi shi [Institutional design of the US Conservation Reserve Program and its enlightenments.] *Issues Agric Econ.* 2020;(5):119-122. [In Chinese]

https://doi.org/10.13246/j.cnki.iae.2020.05.010

22. Tilahun W, Ayenew B, Atinafu S, Enyew S. A computational and geometric analysis of economic dynamics using Pontryagin's maximum principle: advanced numerical simulations perspective. *Discov Appl Sci.* 2026;8(4):418.

https://doi.org/10.1007/s42452-026-08475-7

23. Chen J, Yang S, Du H, Liang W, Liu Y. Horizontal ecological compensation zoning and standard in China's major grain-producing areas based on virtual cultivated land flow. *Front Environ Sci.* 2025;13:1578780.

https://doi.org/10.3389/fenvs.2025.1578780

24. Wijesena S, Pradhan B. Advancements in weather index insurance: a review of data-driven approaches to design, pricing and risk management. *Earth Syst Environ.* 2025;9(3):2355-2379.

https://doi.org/10.1007/s41748-025-00712-0

25. Shah R, Lassoued D, Fayyaz A, Zulfiqar A, Saleem J, Ali T. Study of stability of discrete-time stochastic systems with time variations through Lyapunov's method. *Filomat.* 2025;39(24):8623-8634.

https://doi.org/10.2298/FIL2524623S

26. Fang J. Geng di bao hu shi jiao xia lun zuo xiu geng de fa lv biao da. [Legal expression of crop rotation and fallow from the perspective of cultivated land protection.] *J Nanjing Agric Univ Soc Sci Ed.* 2020;20(3):111-122. [In Chinese]

https://doi.org/10.19714/j.cnki.1671-7465.2020.0044

27. Li L, Zhang AL. Lun zuo xiu geng ji qi bu chang de fa lv yi yun, fa li zheng cheng ji ru fa jin lu. [Legal implication, legal justification and legislative approach of crop rotation and fallow and its compensation.] *China Land Sci.* 2021;35(11):27-35. [In Chinese]

https://doi.org/10.11994/zgtdkx.20211026.093948

28. Zhu GF, Li XC, Shi YR, Zhang Y, Li ZY. Practice comparison and policy enlightenment of cultivated land crop rotation and fallow at home and abroad. *Chin J Agric Resour Reg Plan.* 2018;39(6):35-41,92.

https://doi.org/10.7621/cjarrp.1005-9121.20180606

29. Yang QY, Xin GX, Jiang JL, Chen ZT. Ou mei ji dong ya di qu geng di lun zuo xiu geng zhi du shi jian. [Practice of cultivated land crop rotation and fallow system in Europe, America and East Asia: comparison and enlightenment.] *China Land Sci.* 2017;31(4):71-79. [In Chinese]

https://doi.org/10.11994/zgtdkx.20170421.143345

30. Nie Y, Han XZ, Wang ZMH, Du YF, Li YB. Geng di xiu geng—guo wai jing yan yu zhong guo shi jian. [Cultivated land fallow: foreign experience and China's practice. *World Agric.* 2022;(12):34-44. [In Chinese]

https://doi.org/10.13856/j.cn11-1097/s.2022.12.003

31. Kushwaha D, Biron Z. A review on safe reinforcement learning using Lyapunov and barrier functions. *Artif Intell Rev.* 2026.

https://doi.org/10.1007/s10462-026-11611-9

32. van Laatum B, Msaad S, van Henten EJ, McAllister RD, Boersma S. Stochastic model predictive control with reinforcement learning for greenhouse production systems under parametric uncertainty. *Control Eng Pract.* 2026;169:106787.

https://doi.org/10.1016/j.conengprac.2026.106787

33. Svensen JL, Cheng X, Boersma S, Sun C. Chance-constrained stochastic MPC of greenhouse production systems with parametric uncertainty. *Comput Electron Agric.* 2024;217:108578.

https://doi.org/10.1016/j.compag.2023.108578

34. Ahsen R, Di Bitonto P, Novielli P, *et al.* Harnessing digital twins for sustainable agricultural water management: a systematic review. *Appl Sci.* 2025;15(8):4228.

https://doi.org/10.3390/app15084228

35. Awais M, Wang X, Hussain S, Aziz F, Mahmood MQ. Advancing precision agriculture through digital twins and smart farming technologies: a review. *AgriEngineering.* 2025;7(5):137.

https://doi.org/10.3390/agriengineering7050137

36. Zhao T, Song C, Yu J, *et al.* Leveraging immersive digital twins and AI-driven decision support systems for sustainable water reserves management: a conceptual framework. *Sustainability.* 2025;17(8):3754.

https://doi.org/10.3390/su17083754

37. Xiong WY, Meng F, Chen H, Tan Y. Geng di "san wei yi ti" bao hu shi jiao xia zhong guo sheng yu xiu geng gui mo yu kong jian bu ju. [Provincial fallow scale and fallow spatial layout of cultivated land in China from the "Trinity" protection perspective.] *Trans Chin Soc Agric Eng.* 2024;40(18):240-250. [In Chinese]

https://doi.org/10.11975/j.issn.1002-6819.202405110

38. Zeng S, Chen F, Ma J, Liu GJ, Wanger TC. Multiscenario simulation of fallow schemes in China and their impact on food security. *Land Degrad Dev.* 2024;35(16):4972-4984.

https://doi.org/10.1002/ldr.5271

39. Yin JF, Song CQ, Gao PC, Wang GL, Ye SJ. Guo ji geng di xiu geng shi jian jing yan fen xi yu zhong guo cha yi hua xiu geng kuang jia jian gou. [Analysis of international cropland fallow practice experience and construction of differentiated fallow framework in China.] *Trans Chin Soc Agric Eng.* 2025;41(13):22-34. [In Chinese]

https://doi.org/10.11975/j.issn.1002-6819.202504164

40. Sun Y, Jiang H, Zhu X. Drivers, constraints, and policy regulation strategies for the abandonment of farmland: insights from China. *Land.* 2024;13(12):2096.

https://doi.org/10.3390/land13122096

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