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RESEARCH ARTICLE

Dynamic behavior of a time-dependent mass harmonic oscillator: Analytical, semi-analytical, and numerical investigation

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ABSTRACT

The harmonic oscillator with time-dependent mass represents an important model for describing dynamical systems in which the mass varies with time due to environmental interaction or internal processes. In this work, the dynamics of a time-dependent mass harmonic oscillator are investigated using analytical, semi-analytical, and numerical approaches. The governing equation of motion is derived using both Newtonian and Lagrangian formulations, with an exponentially decreasing mass function. An exact analytical solution is obtained by transforming the resulting differential equation into a Bessel-type equation. In addition, approximate solutions are computed using the Differential Transform Method (DTM) and its multistep extension (Ms-DTM). The numerical solution obtained with the classical fourth-order Runge–Kutta (RK4) method serves as a reference for evaluating the accuracy of the semi-analytical techniques. The results show excellent agreement between the analytical Bessel solution and the numerical RK4 solution, confirming the validity of the analytical formulation. The Ms-DTM method also provides accurate approximations for short time intervals and demonstrates rapid convergence. These findings indicate that the combined analytical–numerical framework offers a reliable approach for studying the dynamical behavior of harmonic oscillators with time-dependent mass and may be useful in the analysis of engineering systems where variable mass effects are significant.

1. Introduction

A harmonic oscillator is a standard model in classical and quantum physics, widely used to describe physical systems subjected to a restoring force that drives the system back to its equilibrium position. The harmonic oscillator is a fundamental model in many different areas of physics due to the relative ease of obtaining analytical solutions and its widespread use in describing physical systems. In classical physics, it is a primary tool in both Newtonian and Lagrangian formulations and provides a basis for studying more complicated systems by linearizing them.[1]

Research in science and engineering continues to focus on advanced models in which the physical properties of a system evolve with time. One area of advanced research includes the development of a harmonic oscillator with time-dependent mass.

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Time-dependent mass occurs when there are variations in the mass of a physical system caused by either interactions with the environment or changes in the internal structure of the system.[2]

Time-dependent mass harmonic oscillators are an important class of physical systems that continue to gain significant interest due to their ability to represent physical systems whose properties vary with time. Such systems include rocket propulsion systems, fluid ejection systems, and micro and nanoscale devices. Leach[3] established the mathematical foundation of the time-dependent mass harmonic oscillator, while Abdalla[4] further developed the theory for more general variable-mass systems. Building on these studies, Flores *et al.*[5] showed that several forms of the time-dependent mass oscillator can be transformed into Bessel-type differential equations, enabling exact analytical solutions under certain conditions. These analytical results are particularly useful because they provide insight into the behavior of variable-mass systems and serve as benchmarks for approximate methods. More recently, Rhoads *et al.*[6] highlighted the importance of time-dependent mass oscillators in engineering applications, particularly in the design of micro- and nano-resonators used in sensors and actuators. Taken together, these studies demonstrate the theoretical significance and practical relevance of time-dependent mass oscillator models. However, exact analytical solutions are generally limited to specific forms of mass variation, which motivates the use of semi-analytical techniques such as the Differential Transform Method (DTM) and the Multistep DTM (Ms-DTM).

The inclusion of time-dependent mass significantly complicates the mathematical description of the system. Unlike the constant mass harmonic oscillator, whose solution can easily be determined from characteristic equations, the variable mass harmonic oscillator generally leads to non-autonomous differential equations. Consequently, in numerous practical situations, exact analytical solutions are either difficult to obtain or exist only for special cases. Thus, semi-analytical and numerical methods are required to solve the governing differential equation.

One semi-analytical approach to solving nonlinear ordinary differential equations is the DTM. The DTM is a semi-analytical technique based on the Taylor series expansion that converts differential equations into recursive algebraic relations. Arikoglu and Ozkol[7] demonstrated the effectiveness of DTM for solving differential-difference equations, while Erturk *et al.*[8] successfully applied the method to analyze the kinematic behavior of coupled pendulum systems. In parallel, analytical approximation methods continue to play an important role in the study of nonlinear oscillatory systems, as illustrated by Lu.[9] These contributions highlight the growing use of analytical and semi-analytical techniques for investigating complex dynamical models.

However, the DTM may lose stability and accuracy when applied to systems with strong nonlinearities or time-dependent parameters. The Ms-DTM addresses this issue by applying the transformation iteratively over sub-intervals of the time domain. Several recent studies have shown the effectiveness of this method and demonstrated its utility in modeling complex dynamical systems.

The Ms-DTM is a powerful semi-analytic approach that has received considerable attention from researchers due to its ability to efficiently solve both linear and nonlinear differential equations. The Ms-DTM was introduced to overcome the limited convergence interval of the classical DTM. Odibat *et al.*[10] proposed the Ms-DTM framework by applying the differential transform sequentially over multiple subintervals and demonstrated its effectiveness for nonlinear dynamical systems, including the Lotka–Volterra, Chen, and Lorenz models. The results showed that Ms-DTM provides more accurate approximations over larger time intervals than the classical DTM while maintaining computational efficiency. Building on these developments, Xie *et al.*[11] proposed an improved DTM for solving nonlinear singular boundary value problems and demonstrated its ability to produce accurate numerical approximations for complex nonlinear equations. In addition to these methodological advances, Al-Ahmad and Abdullah[12] investigated the convergence properties of Ms-DTM and provided theoretical and numerical evidence supporting its stability and convergence. Together, these studies established both the practical effectiveness and the mathematical reliability of DTM-based approaches, motivating their application to the time-dependent mass harmonic oscillator considered in the present work. These contributions have enhanced the accuracy, efficiency, and applicability of the method to a wide range of nonlinear problems. As a result, Ms-DTM has gained increasing recognition as a reliable and effective analytical tool for investigating the dynamic behavior of nonlinear systems. Both the DTM and Ms-DTM have been effectively used to model physical systems whose masses evolve over time. Both methods have been successfully applied in both classical and quantum mechanical contexts as well as in engineering systems that exhibit oscillatory behavior. For example, Gitterman[13] used numerical and analytical techniques to analyze a harmonic oscillator whose mass varied with time.

Recent research into nonlinear and time-varying dynamic systems has significantly increased our knowledge of generalized oscillation models which incorporate position-dependent parameterization, harmonically potentialized oscillations, and fractional or non-standard kinetic terms. The use of analytic and semi-analytic methods, as described in articles from Physica Scripta and other associated journals,[14,15] describes the treatment of complex oscillator systems subject to a variety of physical constraints. Concurrently, recent publications in Applied Mathematical and Engineering Journals[16,17] emphasize numerical stability, perturbation methodology, and computational modeling of nonlinear motion. Additionally, other studies introduce new modifications to the mass framework for oscillating systems and provide improved dynamic formulations that better represent the dynamics of real-world physical systems than traditional harmonic approximations.[18–20] Lastly, recent work within the field of Mathematical Physics has provided enhanced solution methodologies for strongly nonlinear cases.[21] Overall, this body of work illustrates the growing desire to extend classical oscillator theories to more physically realistic, nonlinearly coupled, time-dependent systems applicable across the fields of physics and engineering.

Based on previous studies, this study aims to develop an efficient mathematical model of the time-dependent mass harmonic oscillator that uses both the DTM and Ms-DTM to obtain approximate solutions. The accuracy of the two methods will be evaluated by comparing them with the classical fourth-order Runge–Kutta (RK4) method. By combining analytical, semi-analytical, and numerical methods, this study provides a comprehensive analysis of the time-dependent mass harmonic oscillator and establishes a solid foundation for future practical applications of the model in engineering and other physical systems.

Numerical methods such as the classical fourth-order RK4 method will also be utilized as benchmarks to evaluate the accuracy of the semi-analytical results. The RK4 method has been extensively documented as being highly effective for solving stiff and complex differential equations.[22]

While numerous studies have investigated the harmonic oscillator with time-dependent mass using either analytical or numerical methods, exact solutions for such systems are limited. In addition to providing an independent derivation of a Bessel-based exact analytical solution, this study also integrates the exact solution with semi-analytical methods (DTM and Ms-DTM) and validates the results through the classical fourth-order RK4 method, thereby establishing a unified analytical-numerical framework that allows for high accuracy, rapid convergence, and reliable long-term prediction.

Together, the contributions of this study demonstrate the scientific significance of time-dependent mass oscillator models and emphasize the need for combining analytical, semi-analytical, and numerical techniques to better understand complex systems. The objective of this study is to establish an efficient analytical–numerical framework for the dynamics of a harmonic oscillator with time-dependent mass and to validate the results using numerical simulations based on the RK4 method. The subsequent sections of this document provide the necessary background, mathematical modeling, analytical and semi-analytical solution techniques, numerical simulation, and discussion to support these objectives.

2. The physical model

The physical model of the harmonic oscillator is formulated within the Lagrangian framework. We begin with the classical harmonic oscillator with constant mass m. Let $x(t)$ denote the displacement of the oscillator from its equilibrium position, where t represents time. The parameter k denotes the spring constant of the system, and $\dot{x}(t) = \frac{dx}{dt}$ represents the velocity of the oscillator. Using these variables, the kinetic and potential energies of the system can be defined, allowing the equation of motion to be derived using the Euler–Lagrange equation. The model is then generalized to include the case where the mass varies with time, which constitutes the main focus of this study (**Equation 1**).[23]

$$L = T - V \tag{1}$$

where T and V denote the kinetic and the potential energies, respectively. The kinetic energy is given by, $T = \frac{1}{2}m_o\dot{x}^2$, and potential energy is given by $V = \frac{1}{2}kx^2$. Substituting these into the Lagrangian yields **Equation 2**:

$$L = \frac{1}{2}m_o\,\dot{x}^2 - \frac{1}{2}kx^2 \tag{2}$$

To determine the equation of motion, the Lagrange equation (also known as the Euler–Lagrange equations)[1] is applied in one dimension (**Equation 3**):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \tag{3}$$

2.1. Newtonian formulation of the time-dependent mass harmonic oscillator

To establish the governing equation of motion rigorously, the model is first formulated using Newton's second law before being derived independently within the Lagrangian framework. The Newtonian approach provides a physically intuitive description based on momentum balance, whereas the variational formulation offers a systematic analytical framework for subsequent developments. The agreement between the two derivations confirms the consistency and validity of the proposed mathematical model for the time-dependent mass harmonic oscillator.

For a harmonic oscillator with time-dependent mass $m(t)$, the restoring force generated by the spring is: $F = -kx$, where k denotes the spring constant and $x(t)$ is the displacement from the equilibrium position.

According to Newton's second law, the net force acting on the system is equal to the time derivative of the linear momentum: $F = \frac{d}{dt}\left(m(t)\dot{x}(t)\right)$. Substituting the restoring force into Newton's law yields: $-kx = \frac{d}{dt}\left(m(t)\dot{x}(t)\right)$. Applying the product rule gives: $-kx = m(t)\ddot{x}+\dot{m}(t)\ \dot{x}(t)$. Dividing by $m(t)$ gives the general equation of motion for a harmonic oscillator with time-dependent mass (**Equation 4**):

$$\ddot{x}+\frac{\dot{m}(t)}{m(t)}\dot{x}+\frac{k}{m(t)}x=0 \tag{4}$$

Now consider the exponentially decreasing mass function $m(t) = m_o e^{-\lambda t}$, where m_0 is the initial mass and λ is the mass-decay parameter. Differentiating with respect to time gives **Equation 5**:

$$\dot{m}(t) = -\lambda m_o e^{-\lambda t} = -\lambda m(t) \tag{5}$$

The Newtonian formulation presented above provides a direct derivation of the governing equation of motion for the time-dependent mass harmonic oscillator. Starting from Newton's second law and the balance between the restoring spring force and the time-varying momentum, the equation of motion is obtained as **Equation 6**:

$$\ddot{x}-\lambda\dot{x}+\frac{k}{m_o}e^{\lambda t}x=0 \tag{6}$$

By introducing the parameter $\omega_o^2 = \frac{k}{m_o}$, the governing equation can be written in the compact form **Equation 7**:

$$\ddot{x}-\lambda\dot{x}+\omega_o^2 e^{\lambda t}x=0 \tag{7}$$

In the following subsection, the same physical system is reformulated within the Lagrangian framework, and the corresponding equation of motion is derived using the Euler–Lagrange equation. The recovery of the same governing differential equation through both the Newtonian and Lagrangian approaches confirms the consistency and validity of the mathematical model adopted in this study.

2.2. Lagrangian formulation of the time-dependent mass harmonic oscillator

To investigate the dynamical behavior of a harmonic oscillator with time-dependent mass, the system is formulated within the Lagrangian framework. Using the Lagrangian defined in **Equation 1**, the Lagrangian function for a harmonic oscillator with time-dependent mass m(t) is given by **Equation 8**:

$$L=\frac{1}{2}m(t)\dot{x}^2-\frac{1}{2}kx^2 \tag{8}$$

and Euler–Lagrange equations from **Equation 3** lead to **Equation 9**:

$$\frac{d}{dt}\left(m(t)\dot{x}\right)+kx=0 \tag{9}$$

As a specific example, consider a mass that decreases exponentially with time according to $m(t) = m_o e^{-\lambda t}$. At $t = 0$, the mass is equal to m_o, whereas as $t \to \infty$, the mass approaches zero. Therefore, the derivative of the mass function with respect to time is given by **Equation 10**:

$$\frac{d}{dt}m(t) = -\lambda m_o e^{-\lambda t} \tag{10}$$

Substituting **Equation 6** into **Equation 5** yields **Equation 11**:

$$m_o e^{-\lambda t}\ddot{x}-\lambda m_o e^{-\lambda t}\dot{x}+kx=0 \tag{11}$$

dividing **Equation 7** by $m_o e^{-\lambda t}, gives\ \ddot{x}-\lambda\dot{x}+\frac{k}{m_0}e^{\lambda t}x = 0$. Since $\frac{k}{m_o}$ is a constant, it can be written as ω_o^2 . Therefore, **Equation 11** reduces to **Equation 12**:

$$\ddot{x}-\lambda\dot{x}+\omega_o^2 e^{\lambda t}x=0 \tag{12}$$

Equation 12 describes a time-dependent mass harmonic oscillator. It is a second-order linear homogeneous differential equation with variable coefficients, arising from the exponential variation of the system mass. Here, $\ddot{x}$, $\dot{x}$ denote the second and first derivatives of the displacement x with respect to time t, ω_o is the natural angular frequency, and λ represents the exponential rate of mass decrease.

3. Analytical and semi-analytical solution

In this section, the dynamical behavior of the harmonic oscillator with time-dependent mass is analyzed using both analytical and semi-analytical approaches. The analytical solution is obtained by transforming the governing differential equation into a standard form that can be solved in terms of Bessel functions. On the other hand, the semi-analytic methods, such as the differential transformation method (DTM) and the multistep differential transformation method (Ms-DTM), are utilized to obtain approximate solutions with relatively low computational effort. Therefore, by using both analytical and semi-analytical methods, it is possible to perform a comprehensive analysis of the system and compare the different approaches used to obtain exact and approximate solutions.

3.1. Analytical method via Bessel's equation for the time-dependent mass model

An analytical approach is applied to the harmonic oscillator with time-dependent mass (**Equation 12**). Using a suitable transformation, the equation is reduced to the standard form of Bessel's equation,[24,25] yielding exact solutions in terms of first and second kind Bessel functions.[26]

We consider the differential equation derived for the harmonic oscillator model with mass varying with time, given in **Equation 12**, with the initial conditions $x(0) = 1, \quad \dot{x}(0) = 0$.

Let $\tau(t) = \frac{2\omega_o}{\lambda} e^{\frac{\lambda t}{2}}$. We proceed by computing the derivatives of the transformed variable:[27]

The first derivative of x is given by **Equations 13 and 14**:

$$\frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} \tag{13}$$

$$\frac{dx}{dt} = \omega_o e^{\frac{\lambda t}{2}} \frac{dx}{d\tau} \tag{14}$$

The second derivative of x is obtained as follows (**Equation 15**):

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\omega_o e^{\frac{\lambda t}{2}} \frac{dx}{d\tau}\right) \tag{15}$$

$$\frac{d^2x}{dt^2} = \omega_o e^{\frac{\lambda t}{2}} \frac{d^2x}{d\tau^2}\frac{d\tau}{dt} + \frac{\lambda}{2}\omega_o e^{\frac{\lambda t}{2}} \frac{dx}{d\tau}$$

Let $\frac{d\tau}{dt} = \omega_o e^{\frac{\lambda t}{2}}$, then (**Equation 16**):

$$\frac{d^2x}{dt^2} = \omega_o e^{\frac{\lambda t}{2}} \left(\frac{\lambda}{2}\frac{dx}{d\tau} + \omega_o e^{\frac{\lambda t}{2}} \frac{d^2x}{d\tau^2}\right) \tag{16}$$

Substituting **Equation 16** into the original differential equation allows the equation of motion to be expressed in terms of the transformed variable τ. This yields **Equation 17**:

$$\omega_o^2 e^{\lambda t}\frac{d^2x}{d\tau^2} - \frac{\lambda}{2}\omega_o e^{\frac{\lambda t}{2}}\frac{dx}{d\tau} + \omega_o^2 e^{\lambda t} x = 0 \tag{17}$$

To simplify the expression and remove the common exponential factor, **Equation 17** is divided by $\omega_o e^{\frac{\lambda t}{2}}$, giving **Equation 18**:

$$\omega_o e^{\frac{\lambda t}{2}}\frac{d^2x}{d\tau^2} - \frac{\lambda}{2}\frac{dx}{d\tau} + \omega_o e^{\frac{\lambda t}{2}} x = 0 \tag{18}$$

Using the transformation relation $\tau(t) = \frac{2\omega_o}{\lambda} e^{\frac{\lambda t}{2}}$, the remaining time-dependent coefficient can be rewritten entirely in terms of τ. Substituting this relation into **Equation 18** gives **Equation 19**:

$$\frac{\lambda\tau}{2}\frac{d^2x}{d\tau^2} - \frac{\lambda}{2}\frac{dx}{d\tau} + \frac{\lambda\tau}{2} x = 0 \tag{19}$$

Dividing **Equation 19** by $\frac{\lambda}{2}$ leads to a more compact form (**Equation 20**):

$$\tau\frac{d^2x}{d\tau^2} - \frac{dx}{d\tau} + \tau x = 0 \tag{20}$$

and dividing by τ ($\tau \neq 0$) yields **Equation 21**:

$$\frac{d^2x}{d\tau^2} - \frac{1}{\tau}\frac{dx}{d\tau} + x = 0 \tag{21}$$

This is similar to a Bessel's equation, but it is not in the standard form. The standard Bessel equation of order v is:[24] $\tau^2\frac{d^2x}{d\tau^2} + \tau\frac{dx}{d\tau} + (\tau^2 - v^2)x = 0$.

Our **Equation 21** has a negative sign on the first-derivative term, which means it does not directly match. To transform **Equation 21** into the standard Bessel form, we introduce the substitution $x(\tau) = \tau y(\tau)$. The first derivative with respect to t is given by **Equation 22**:

$$\frac{dx}{d\tau} = y + \tau\frac{dy}{d\tau} \tag{22}$$

Differentiating once more with respect to t, we obtain **Equations 23 and 24**:

$$\frac{d^2x}{d\tau^2} = \frac{dy}{d\tau} + \frac{dy}{d\tau} + \tau\frac{d^2y}{d\tau^2} \tag{23}$$

$$\frac{d^2x}{d\tau^2} = 2\frac{dy}{d\tau} + \tau\frac{d^2y}{d\tau^2} \tag{24}$$

Substituting **Equations 22 and 24** into **Equation 20**, becomes **Equation 25**:

$$\tau\left(2\frac{dy}{d\tau}+\ \tau\frac{d^2y}{d\tau^2}\right)-\left(y+\tau\frac{dy}{d\tau}\right)+\tau(\tau y)=0 \quad (25)$$

After substitution and simplification, the governing equation becomes **Equation 26**:

$$\tau^2\frac{d^2y}{d\tau^2}+\tau\frac{dy}{d\tau}+\left(\tau^2-1\right)y=0 \qquad (26)$$

Equation 26 is recognized as the standard Bessel equation of order one.[24] Consequently, its general solution can be expressed as a linear combination of the corresponding Bessel functions (**Equation 27**).[26]

$$y(\tau)=c_1J_1(\tau)+c_2Y_1(\tau) \qquad (27)$$

where $J_1(\tau)$ is the Bessel function of the first kind (order 1), $Y_1(\tau)$ is the Bessel function of the second kind (order 1), and c_1, c_2 are constant. Since $x(\tau)=\tau y(\tau)$, we have **Equation 28**:

$$x(\tau)=\tau\left(c_1J_1(\tau)+c_2Y_1(\tau)\right) \qquad (28)$$

The integration constants c_1 and c_2 are determined from the prescribed initial conditions: $x(0)=1$, and $\dot{x}(0)=0$. At $t=0$, the transformed variable takes the value $\tau(0)=\tau_0=\frac{2\omega_o}{\lambda}$. Substituting this condition into **Equation 28** gives **Equation 29**:

$$x\left(\tau_o\right)=\ \tau_o\left(c_1J_1\left(\tau_o\right)+c_2Y_1\left(\tau_o\right)\right)=1 \qquad (29)$$

which leads to **Equation 30:**

$$c_1J_1\left(\tau_o\right)+c_2Y_1\left(\tau_o\right)=\frac{1}{\tau_o} \qquad (30)$$

To obtain a second equation relating to the unknown coefficients, the derivative of the displacement is evaluated. Since $x(t)=x\left(\tau(t)\right)$, we use the chain rule to give: $\dot{x}(t)=\frac{dx}{d\tau}\cdot\frac{d\tau}{dt}$, where, $\frac{d\tau}{dt}=\omega_oe^{\frac{\lambda t}{2}}$. Differentiating **Equation 28** with respect to τ yields **Equations 31 and 32**:

$$\dot{x}(\tau)=\frac{d}{d\tau}\left(\tau\left(c_1J_1(\tau)+c_2Y_1(\tau)\right)\right) \qquad (31)$$

$$\dot{x}(\tau)=\left(c_1J_1(\tau)+c_2Y_1(\tau)\right)+\tau\frac{d}{d\tau}\left(c_1J_1(\tau)+c_2Y_1(\tau)\right) \qquad (32)$$

Using the Bessel function identities:[24] $\frac{d}{d\tau}\left(J_1(\tau)\right)=J_o(\tau)-\frac{J_1(\tau)}{\tau}$, and $\frac{d}{d\tau}\left(Y_1(\tau)\right)=Y_o(\tau)-\frac{Y_1(\tau)}{\tau}$, the velocity can be expressed as

$$\dot{x}(t)=\omega_oe^{\frac{\lambda t}{2}}\left(c_1\left(J_1(\tau)+\tau\left(J_o(\tau)-\frac{J_1(\tau)}{\tau}\right)\right)+c_2\left(Y_1(\tau)+\tau\left(Y_o(\tau)-\frac{Y_1(\tau)}{\tau}\right)\right)\right).$$

After simplification, this reduces to **Equation 33**:

$$\dot{x}(t)=\omega_oe^{\frac{\lambda t}{2}}\left(c_1\tau J_o(\tau)+c_2\tau Y_o(\tau)\right) \qquad (33)$$

Evaluating **Equation 33** at $t=0$, $\tau(0)=\tau_o=\frac{2\omega_o}{\lambda}$, gives **Equation 34**:

$$\dot{x}(0)=\omega_o\tau_o\left(c_1J_o\left(\tau_o\right)+c_2Y_o\left(\tau_o\right)\right) \qquad (34)$$

Since, $\omega_o\tau_o\neq0$, the initial condition $\dot{x}(0)=0$ implies that (**Equation 35**):

$$c_1J_o\left(\tau_o\right)+c_2Y_o\left(\tau_o\right)=0 \qquad (35)$$

Equations 30 and 35 form a system of two linear equations for the constants c_1 and c_2:

$c_1J_1\left(\tau_o\right)+c_2Y_1(\tau_o)=\frac{1}{\tau_o}$

$$c_1J_o\left(\tau_o\right)+c_2Y_o\left(\tau_o\right)=0$$

Solving this system using Cramer's rule,[27] the solution is:

$c_1=\frac{\frac{1}{\tau_0}Y_o(\tau_o)}{J_1(\tau_o)Y_o(\tau_o)-Y_1(\tau_o)J_o(\tau_o)}$, and $c_2=\frac{-\frac{1}{\tau_o}J_o(\tau_o)}{J_1(\tau_o)Y_o(\tau_o)-Y_1(\tau_o)J_o(\tau_o)}$.

The Wronskian identity for Bessel functions is used to simplify the expressions for the integration constants:[24]

$J_1(\tau)Y_o(\tau)-Y_1(\tau)J_o(\tau)=\frac{2}{\pi\tau}$,

Substituting this identity into the above expressions gives $c_1=\frac{\pi}{2}Y_o\left(\tau_o\right)$, and $c_2=-\frac{\pi}{2}J_o\left(\tau_o\right)$. Therefore, the solution of the transformed equation can be written as: $x(\tau)=\tau\left(c_1J_1(\tau)+c_2Y_1(\tau)\right)$. Substituting the evaluated constants into this expression leads to the analytical solution (**Equation 36**):

$$(x(t)=\tau(t)\left(\frac{\pi}{2}Y_o\left(\tau_o\right)J_1\left(\tau(t)\right)-\frac{\pi}{2}J_o\left(\tau_o\right)Y_1\left(\tau(t)\right)\right) \qquad (36)$$

where: $\tau(t)=\frac{2\omega_o}{\lambda}e^{\frac{\lambda t}{2}}$.

The analytical solution presented in **Equation 36** provides an exact representation of the displacement response of the harmonic oscillator with time-dependent mass under the specified initial conditions. Transforming the governing equation into the standard Bessel form allows the system dynamics to be expressed in terms of well-established special functions, yielding a closed-form solution. Although this analytical result offers valuable insight into the model's behavior, alternative solution techniques are also useful, particularly when exact solutions are difficult to obtain or when more complex forms of the governing equations are considered. For this reason, the DTM is introduced in the next section as a semi-analytical approach, and its results are subsequently compared with the exact Bessel-function solution obtained here.

3.2. Semi-analytical solution via the Differential Transform Method for the time-dependent mass model

A semi-analytical approach based on the DTM is employed to solve the harmonic oscillator with time-dependent mass (**Equation 12**). DTM combines analytical and numerical techniques by transforming differential equations into rapidly convergent series, providing efficient approximate solutions, especially when exact solutions are difficult to obtain or when the system exhibits complex time-dependent behavior.[28–31]

Definition 1. *The DTM is based on the Taylor series expansion of a function. The differential transform of a function $x(t)$ is defined as **Equation 37**:*[31]

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_o} \tag{37}$$

*where $f(t)$ is the original function and $F(k)$is the transformed function. The inverse differential transform of $F(k)$ is defined as **Equation 38**:*

$$f(t) = \sum\nolimits_{k=o}^{\infty} F(k)(t-t_o)^k \tag{38}$$

By substituting **Equation 37** into **Equation 38**, we obtain **Equation 39**:

$$f(t) = \sum\nolimits_{k=o}^{\infty} \left(\frac{(t-t_o)^k}{k!}\frac{d^k f(t)}{dt^k}\right)\Bigg|_{t=t_o} \tag{39}$$

which implies that the concept of the differential transform is derived from the Taylor series expansion. The transformation rules summarized in **Table 1** can be derived from the above definition.

The rules listed in **Table 1** are used to derive the recurrence relations required for the solution of the governing differential equation. To solve **Equation 12** using the DTM, the second-order differential equation is first converted into an equivalent system of first-order equations:

Letting $x_1 = x$, and $x_2 = \dot{x}$. Then the system becomes:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \lambda x_2 - \omega_o^2 e^{\lambda t} x_1.$$

The transformed functions $X_1(k)$and $X_2(k)$, satisfy the following recurrence relation (**Equation 40**):

$$X_1(k+1) = \frac{1}{k+1}\left(X_2(k)\right)$$

$$X_2(k+1) = \frac{1}{k+1}\left(\lambda X_2(k) - \omega_o^2 \sum\nolimits_{l=o}^{k} \frac{\lambda^l}{l!} X_1(k-l)\right) \tag{40}$$

To improve the accuracy of the solution and extend the range of approximation, the Ms-DTM is adopted.

Table 1. Basic transformed in the Differential Transform Method[31,32]

Original function $f(t)$	**Transformed function $F(k)$**
$f(t) = u(t) \pm v(t)$	$F(k) = U(k) \pm V(k)$
$f(t) = \alpha u(t)$	$F(k) = \alpha U(k)$ *where* $\propto$ *is constant*
$f(t) = u(t)v(t)$	$F(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$f(t) = \frac{du(t)}{dt}$	$F(k) = (k+1)U(k+1)$
$f(t) = t^m$	$F(k) = \delta(k-m)$
$f(t) = e^{\lambda t}$	$F(k) = \frac{\lambda^k}{k!}$

3.3. Semi-analytical solution via the Multistep-Differential Transform Method for the time-dependent mass model

This method extends the classical DTM by partitioning the time domain into multiple subintervals, thereby improving accuracy and enabling the solution to be obtained over longer time intervals.[33]

Although the DTM accurately captures the behavior of the problem, it generally provides reliable approximations only over relatively small time intervals. In contrast, the Ms-DTM generates series solutions over longer time intervals and improves the overall accuracy of the approximation.[34]

Definition 2. *Let* $[0,\ T]$ *be the interval for a nonlinear initial value problem (**Equation 41**):*[34]

$$f\left(t,\ u,\ u',\cdots,u^{(p)}\right)=0,\ u^{(k)}(0)=c_k \quad for \quad k=0,\ 1,\cdots,p-1 \tag{41}$$

*The solution can be approximated by the finite series (**Equation 42**):*

$$u(t)=\sum_{n=o}^{N} a_n t^n \tag{42}$$

Assume that the interval $[0,\ T]$ is divided into M subintervals: $[t_o,\ t_1],[t_1,\ t_2],\ldots,[t_{m-1},t_m]$, $m=1,\ 2,\cdots,M$. With a step size $h=\frac{T}{M}$, the nodes are defined as $t_m=mh$. For the first interval $[0,\ t_1]$, the initial conditions ${u_1}^{(k)}(0)=c_k$ are considered. Applying the DTM to **Equation 41**, and the solution is obtained as **Equation 43**:

$$u_1(t)=\sum_{n=o}^{K} a_{1n}t^n, \qquad t\in[0,\ t_1] \tag{43}$$

For $m\geq 2$, the initial conditions at each subinterval $[t_{m-1},\ t_m]$, are taken as: ${u_m}^{(k)}(t_{m-1})={u_{m-1}}^{(k)}(t_{m-1})$. The DTM is then applied to **Equation 41** over the interval $[t_{m-1},\ t_m]$, where the expansion point t_o in **Equation 42** is replaced by t_{m-1}. Repeating this procedure over all subintervals generates a sequence of approximate solutions $u_m(t),\ m=1,\ 2,\cdots,M$, given by **Equation 44**:[33,34]

$$u_m(t)=\sum_{n=o}^{K} a_{mn}\left(t-t_{m-1}\right)^n, \quad t\in[t_m,\ t_{m+1}] \tag{44}$$

where $N=K\cdot M$.

In general, the Ms-DTM solution given as **Equation 45**:

$$u(t)=\begin{cases} u_1(t), & t\in[0,\ t_1] \\ u_2(t), & t\in[t_1,\ t_2] \\ \quad\vdots & \\ u_M(t), & t\in[t_{M-1},\ t_M] \end{cases} \tag{45}$$

After converting the harmonic oscillator system into its DTM, the Ms-DTM is applied using a step size of $h=0.01$. The resulting approximate solutions are given by **Equation 46**:

$$x_1(t)=\begin{cases} \sum_{n=o}^{k} X_{11}(n)t^n, & t\in[0.00,\ 0.01] \\ \sum_{n=o}^{k} X_{12}(n)(t-0.01)^n, & t\in[0.01,\ 0.02] \\ \sum_{n=o}^{k} X_{13}(n)(t-0.02)^n, & t\in[0.02,\ 0.03] \\ \quad\vdots & \\ \sum_{n=o}^{k} X_{1M}(n)\left(t-t_{M-1}\right)^n, & t\in[t_{M-1},\ t_M] \end{cases}$$

$$x_2(t)=\begin{cases} \sum_{n=o}^{k} X_{21}(n)t^n, & t\in[0.00,\ 0.01] \\ \sum_{n=o}^{k} X_{22}(n)(t-0.01)^n, & t\in[0.01,\ 0.02] \\ \sum_{n=o}^{k} X_{23}(n)(t-0.02)^n, & t\in[0.02,\ 0.03] \\ \vdots & \\ \sum_{n=o}^{k} X_{2M}(n)\left(t-t_{M-1}\right)^n, & t\in[t_{M-1},\ t_M] \end{cases} \tag{46}$$

4. Simulation results and discussion

This section presents and analyzes the simulation results for the harmonic oscillator with time-dependent mass obtained using analytical, semi-analytical, and numerical approaches implemented in MATLAB. The main focus is on evaluating the accuracy and effectiveness of each method in capturing the system's dynamic behavior. Particular attention is given to the RK4 numerical method, which accounts for the additional complexity introduced by the variable mass. RK4 serves both as a reliable tool for approximating solutions and as a benchmark for validating results from analytical (Bessel's equation) and semi-analytical methods (Ms-DTM). The analysis also examines the effects of the natural frequency ω_0 and the time-dependent mass parameter λ on the system's response, highlighting how these parameters influence the accuracy and behavior of each solution method.

Figure 1 shows the variation of the mass $m(t)$ against time. It is clear that m decreases exponentially from 1 to 0, and the rapid exponential decrease occurs as λ increases.

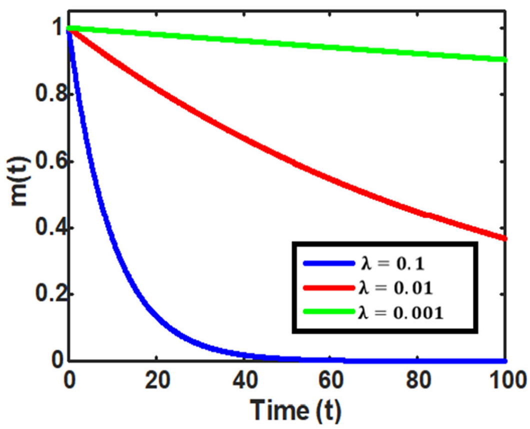

Figure 1. The behavior of $m(t)=m_oe^{-\lambda t}$ vs *time* with $m_o=1$.

Now, we discussed the variation of $x(t)$, and $\dot{x}(t)$ against time for the cases discussed below. Here, we simulate the plots using the Characteristic Equation.

For the time-dependent model, we discussed the variation of $x(t)$, and $\dot{x}(t)$ against time for the cases discussed below. Here we simulate the plots using the three methods discussed before (Bessel's analytical solution, Ms-DTM solution, and RK4).

Figures 2–4 display the dynamic responses of the time-dependent mass harmonic oscillator for a given value of natural frequency ($\omega_o=2$) and various values of the mass decay parameter λ. Comparisons of displacement, velocity, and phase trajectories were made using the analytical Bessel solution, the RK4 method, and the Mass-Dependent Time-Marching (Ms-DTM) approach. A striking result from **Figures 2–4** was the almost perfect coincidence between the analytical Bessel solution and the RK4 numerical solution throughout all simulations. This agreement verifies the correct transformation of the original equation into a Bessel-type equation and shows that the exact solution can represent the dynamics of a variable-mass oscillator. Consistency among these independent solution processes validated both the mathematical derivation and numerical implementations.

Physically, the value of λ determines how fast mass is lost. When moving from **Figure 2** to **Figure 4**, the mass changes less rapidly, so it exhibits properties similar to those of a classical constant-mass system. So, there will be smooth oscillatory motion and more clearly defined loops in the phase trajectory. With higher values of λ, the inertial terms will exhibit greater time dependence, leading to greater variation in displacement and velocity amplitudes. This happens because the effective inertia decreases continuously over time. Therefore, due to greater acceleration from stronger restoring forces, the oscillation pattern will change.

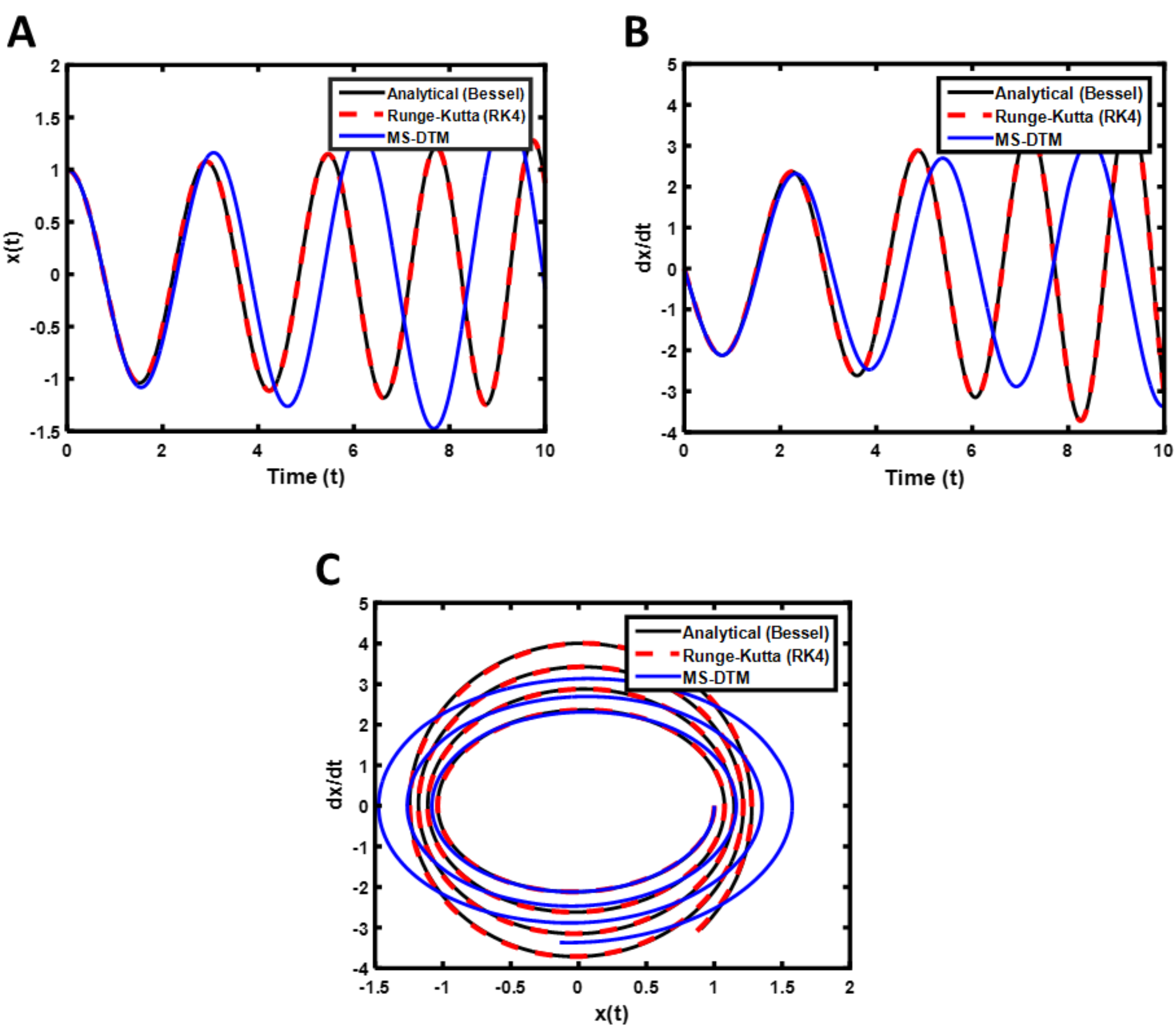

Figure 2. Graphical representation for the time-dependent mass of (A) $x(t)$ vs t , (B) $\frac{dx}{dt}$ vs t, and (C) $\frac{dx}{dt}$ vs $x(t)$ for case one.

At first, the Ms-DTM solution agrees well with both the analytical and RK4 solutions, but as time progresses, they appear to diverge. This is expected because the Ms-DTM uses piecewise Taylor series approximations whose truncation errors accumulate over successive subintervals. While the multistep strategy extends the convergence interval much further than standard DTM, its accuracy is still limited by the time it can integrate, especially when the governing equation contains strong time-dependent terms. Thus, this discrepancy is simply a function of local truncation error accumulation instead of a lack of quality in either the mathematical model or the algorithms used.

Phase portraits are presented in **Figures 2C–4C**, giving further insight into the system's dynamics. Both analytical and RK4-generated phase space structures have almost identical characteristic features of system dynamics. On the other hand, phase space structures generated by Ms-DTM deviated from the reference orbits over time and showed effects from increasing approximation errors, which affected predictions of the energy distribution and phase evolution.

Case one: $\lambda = 0.1$ and $\omega_o = 2$.

Case two: $\lambda = 0.01$ and $\omega_o = 2$.

Case three: $\lambda = 0.001$ and $\omega_o = 2$.

In addition to varying values of λ, we studied the influences of natural frequency ω_o on dynamics. We did this while keeping the mass-decay parameter λ constant. Increased ω_o produces a more intense restoring force, leading to faster oscillation frequency of both displacement and velocity. Results show that as ω_o increases, the system undergoes more oscillation cycles in a shorter amount of time. These behaviors agree with classical oscillating theories, where oscillation periods are inversely proportional to the natural frequency ω_o.

Even though there is an increase in oscillational complexity, the analytical Bessel solution remained extremely close to the RK4 results. The analytical formulation had maintained its reliability even in high-frequency cases. The Ms-DTM solution has a larger difference at higher frequencies. For example, rapid oscillations require higher-order series expansion to get an accurate solution variation locally; thus, as ω_o increases, approximation errors accumulate more rapidly, which is the reason why Ms-DTM has larger differences than it does in **Figure 5**.

Case four: $\lambda = 0.01$ and $\omega_o = 4$.

Case five: $\lambda = 0.01$ and $\omega_o = 8$.

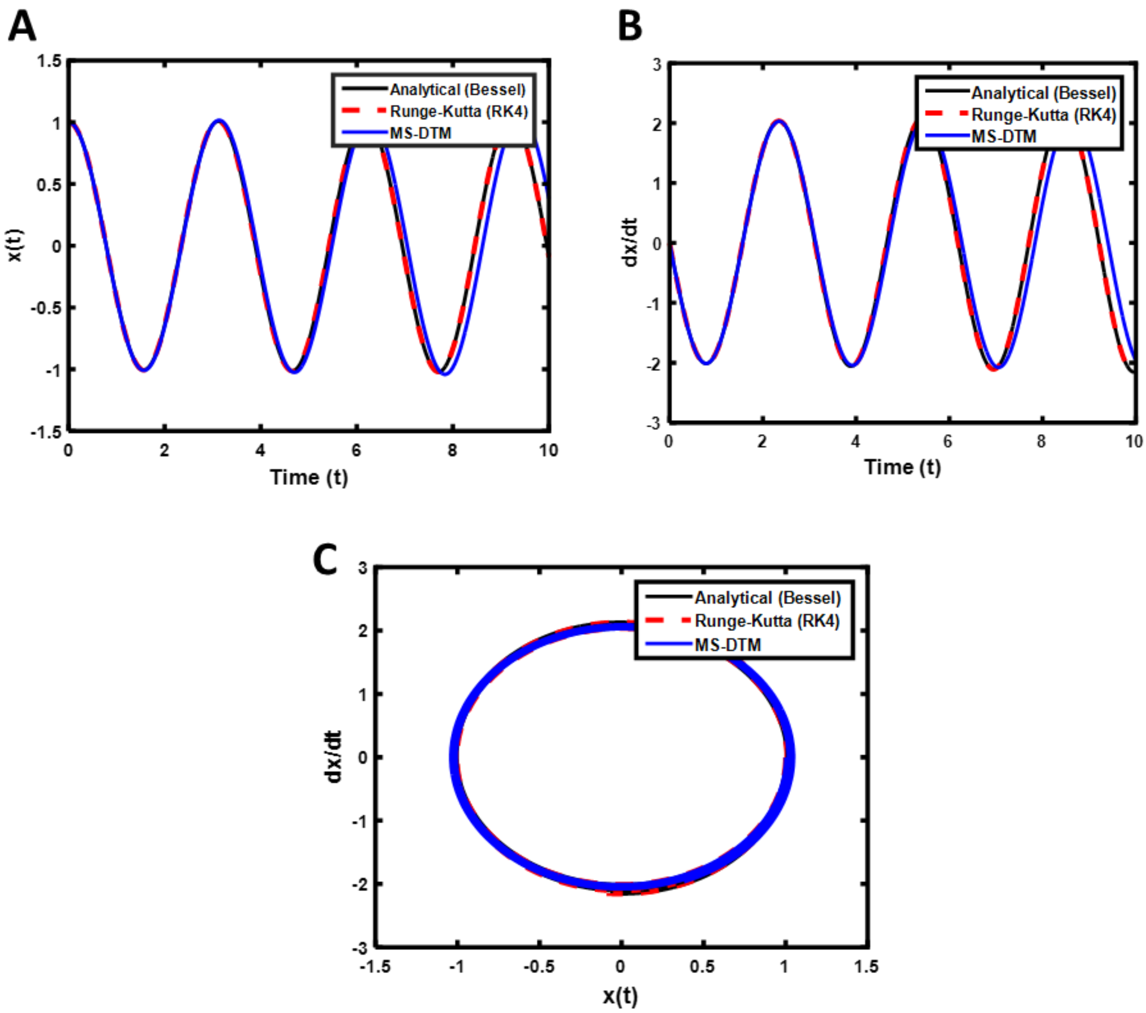

Figure 3. Graphical representation for the time-dependent mass of (A) $x(t)$ vs t , (B) $\frac{dx}{dt}$ vs t, and (C) $\frac{dx}{dt}$ vs $x(t)$ for case two.

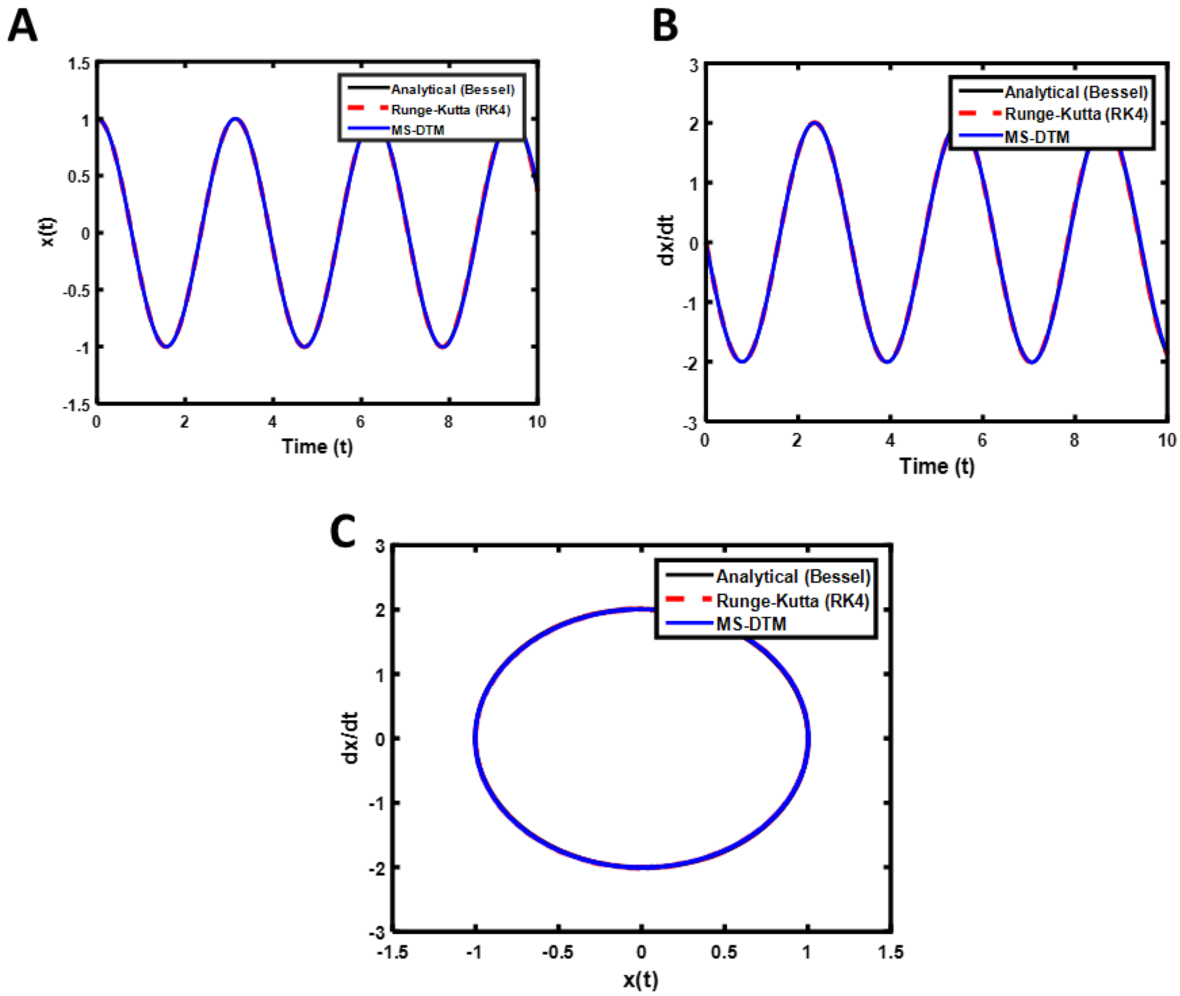

Figure 4. Graphical representation for the time-dependent mass of (A) $x(t)$ vs t , (B) $\frac{dx}{dt}$ vs t, and (C) $\frac{dx}{dt}$ vs $x(t)$ for case three.

Overall, **Figures 2–6** demonstrate that the analytical Bessel solution provided an exact and excellent representation of the oscillator dynamics, while Ms-DTM gave an efficient semi-analytical representation suited for short or moderate length time intervals. Also, **Figures 2–6** revealed that both mass-decay rate λ and natural frequency ω_o are important factors influencing the dynamic response by altering the inertia/restoring-force balance.

Tables 2–6 present a quantitative evaluation of the accuracy of the analytical Bessel solution and Ms-DTM approximation compared to the RK4 numerical solution. A number of statistical measures were used to evaluate these solutions, including point-wise absolute error, infinity-norm, root mean square error (RMSE), and the Kolmogorov–Smirnov (KS) uniformity test.

It was noted that across all tests, the analytical Bessel solution showed extremely low error levels. In terms of absolute error at the evaluation points, the highest value remained below 10^{-4}. Likewise, the RMSE values were extremely low. These small error levels suggest that the analytical solution produces the RK4 solution with nearly perfect accuracy. These data indicate a high degree of confidence in using the Bessel function-based solution as an exact representation of the differential equation, and thus it may serve as a “benchmark” solution for this type of problem.

Conversely, the Ms-DTM exhibited large errors that increased with time. Although local absolute errors remain relatively small immediately after t = 0, they grow with time. As previously mentioned, this trend arises from the cumulative nature of truncation errors arising from repeated series approximations. The trends for both L-infinity and RMSE also show that as the integration length increases, the approximation’s accuracy deteriorates.

Another interesting observation is that the accuracy of the Ms-DTM is highly dependent upon the physical characteristics of the model being analyzed. For models characterized by rapid changes in mass and/or frequency, errors measured using the metrics described above will be substantially greater. Thus, while Ms-DTM is computationally efficient, its performance becomes increasingly sensitive to parameter variations in systems with highly time-dependent dynamics.

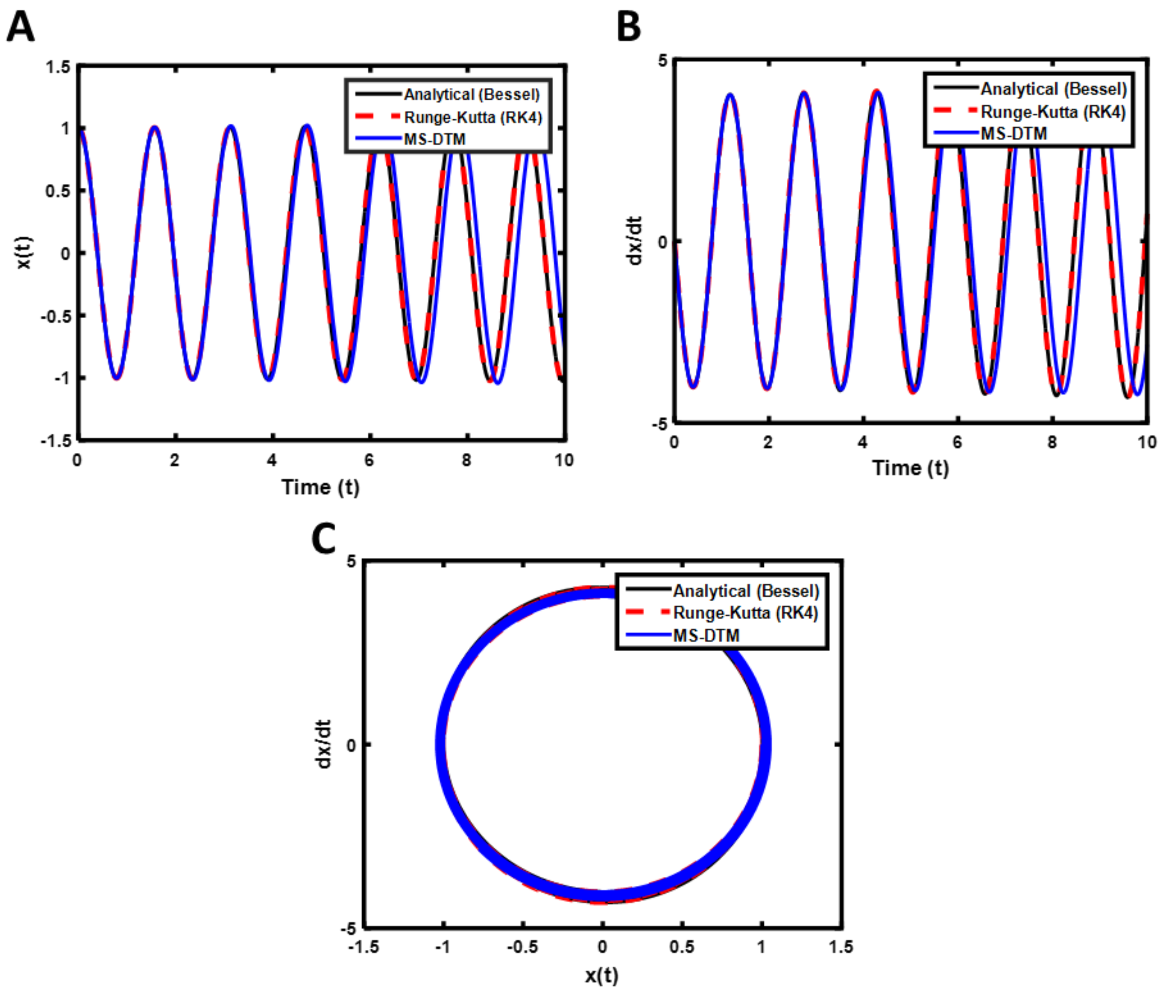

Figure 5. Graphical representation for the time-dependent mass of (A) $x(t)$ vs t , (B) $\frac{dx}{dt}$ vs t, and (C) $\frac{dx}{dt}$ vs $x(t)$ for case four.

Table 2. Absolute error between the RK4 method and Bessel's, and the difference between the RK4 method and Ms-DTM when $\lambda = 0.1$ and ω_o= 2

Time (t)	**Bessel**	**RK4**	**Ms-DTM**	**Absolute error |RK4-Bessel|**	**Absolute error |RK4-Ms-DTM|**
1	−0.484451	−0.484459	−0.484459	9.00×10^{-06}	0.00
2	−0.498420	−0.498441	−0.635028	2.10×10^{-05}	1.37×10^{-01}
3	1.056093	1.056153	1.151673	6.10×10^{-05}	9.55×10^{-02}
4	−0.937580	−0.937647	−0.415826	6.70×10^{-05}	5.22×10^{-01}
5	0.415937	0.415968	−0.869257	3.10×10^{-05}	1.29
6	0.153764	0.153781	1.303129	1.70×10^{-05}	1.15
7	−0.577677	−0.577729	−0.303944	5.20×10^{-05}	2.74×10^{-01}
8	0.821683	0.821744	−1.145218	6.10×10^{-05}	1.97
9	−0.915901	−0.915943	1.447286	4.20×10^{-05}	2.36
10	0.871364	0.871364	−0.138857	0.00	1.01
Max absolute error (L-inf)	–	–	–	6.70×10^{-05}	2.36
Root mean square error	–	–	–	4.22×10^{-05}	1.18
Uniform distribution fitting (p-value)	–	–	–	0.5231	0.0114

Abbreviations: Ms-DTM: Multistep-Differential Transform Method; RK4: Runge–Kutta.

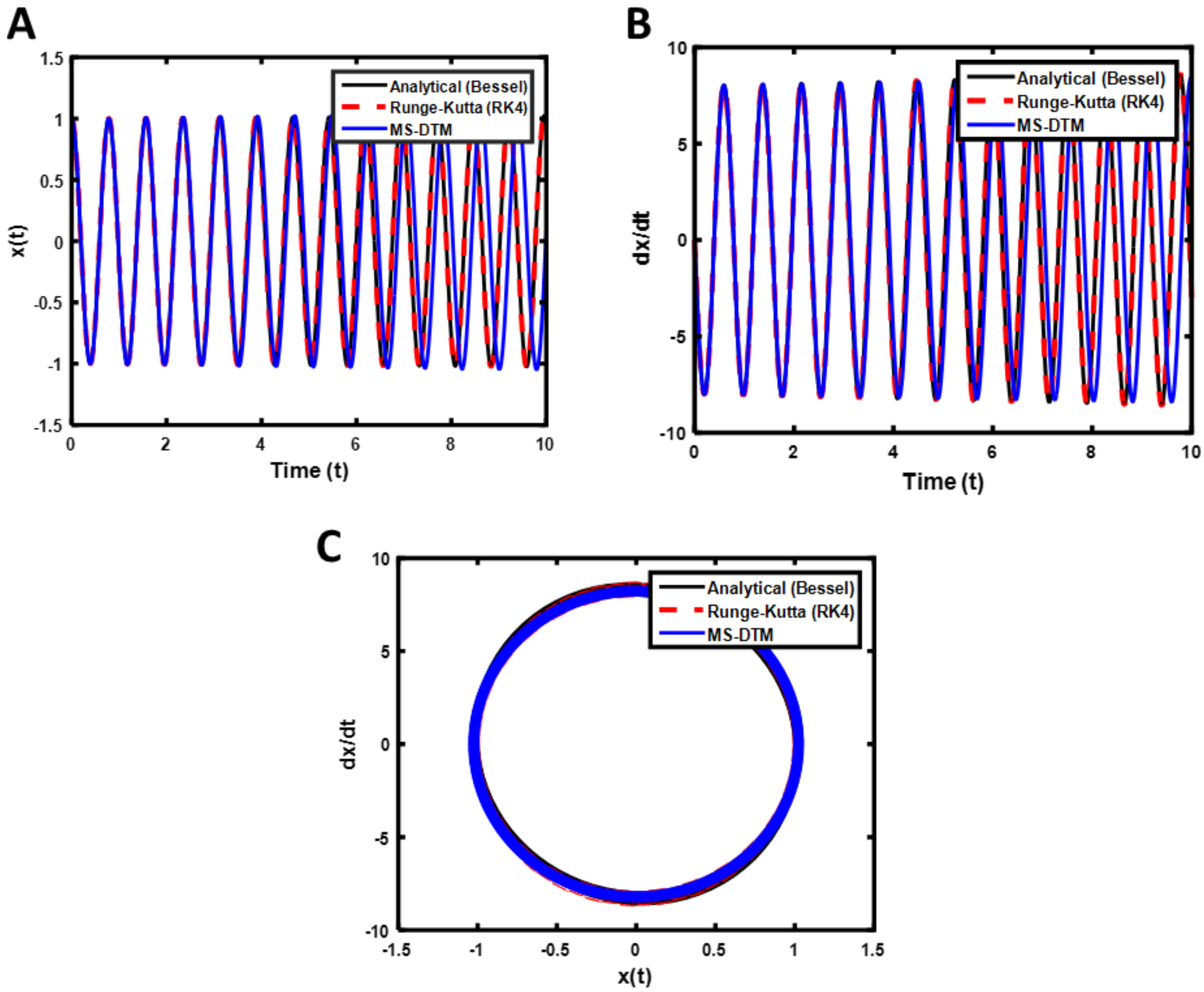

Figure 6. Graphical representation for the time-dependent mass of (a) $x(t)$ vs t , (b) $\frac{dx}{dt}$ vs t and (c) $\frac{dx}{dt}$ vs $x(t)$ for case five.

Table 3. Absolute error between the RK4 method and Bessel's, and the difference between the RK4 method and Ms-DTM when $\lambda = 0.01$ and $\omega_o = 2$

Time (t)	Bessel	RK4	Ms-DTM	Absolute error |RK4-Bessel|	Absolute error |RK4-Ms-DTM|
1	-0.422874	-0.422881	-0.422881	7.00×10^{-06}	0.00
2	-0.640545	-0.640566	-0.652460	2.10×10^{-05}	1.19×10^{-02}
3	0.979388	0.979431	0.978854	4.30×10^{-05}	5.77×10^{-04}
4	-0.228080	-0.228091	-0.168704	1.10×10^{-05}	5.94×10^{-02}
5	-0.772791	-0.772832	-0.846035	4.10×10^{-05}	7.32×10^{-02}
6	0.941391	0.941439	0.885809	4.80×10^{-05}	5.56×10^{-02}
7	-0.113668	-0.113673	0.105495	5.00×10^{-06}	2.19×10^{-01}
8	-0.831736	-0.831765	-0.983918	2.90×10^{-05}	1.52×10^{-01}
9	0.926536	0.926555	0.725449	1.90×10^{-05}	2.01×10^{-01}
10	-0.091429	-0.091429	0.380365	0.00	4.72×10^{-01}
Max absolute error (L-inf)	–	–	–	4.30×10^{-05}	4.72×10^{-01}
Root mean square error	–	–	–	2.762245×10^{-05}	1.86×10^{-01}
Uniform distribution fitting (p-value)	–	–	–	0.472851	0.0384

Abbreviations: Ms-DTM: Multistep-Differential Transform Method; RK4: Runge–Kutta.

Table 4. Absolute error between the RK4 method and Bessel's, and the difference between the RK4 method and Ms-DTM when $\lambda = 0.001$ and $\omega_o = 2$

Time (t)	**Bessel**	**RK4**	**Ms-DTM**	**Absolute error |RK4-Bessel|**	**Absolute error |RK4-Ms-DTM|**
1	−0.416812	−0.416819	−0.416819	7.00×10^{-06}	0.00
2	−0.652339	−0.652360	−0.653532	2.10×10^{-05}	1.17×10^{-03}
3	0.962134	0.962174	0.962033	4.10×10^{-05}	1.41×10^{-04}
4	−0.153685	−0.153692	−0.147784	7.00×10^{-06}	5.91×10^{-03}
5	−0.833130	−0.833172	−0.839800	4.20×10^{-05}	6.63×10^{-03}
6	0.854689	0.854731	0.848006	4.10×10^{-05}	6.73×10^{-03}
7	0.112468	0.112473	0.133726	5.00×10^{-06}	2.13×10^{-02}
8	−0.949772	−0.94902	−0.960331	3.10×10^{-05}	1.05×10^{-02}
9	0.691858	0.691871	0.666691	1.30×10^{-05}	2.52×10^{-02}
10	0.362655	0.362655	0.405525	0.00	4.29×10^{-02}
Max absolute error (L-inf)	–	–	–	4.10×10^{-05}	4.29×10^{-02}
Root mean square error	–	–	–	2.66×10^{-05}	1.78×10^{-02}
Uniform distribution fitting (p-value)	–	–	–	0.4691	0.0419

Abbreviations: Ms-DTM: Multistep-Differential Transform Method; RK4: Runge–Kutta.

Table 5. Absolute error between the RK4 method and Bessel's, and the difference between the RK4 method and Ms-DTM when $\lambda = 0.01$ and $\omega_o = 4$

Time (t)	**Bessel**	**RK4**	**Ms-DTM**	**Absolute error |RK4-Bessel|**	**Absolute error |RK4-Ms-DTM|**
1	−0.647123	−0.647171	−0.647171	4.80×10^{-05}	0.00
2	−0.186596	−0.186619	−0.169670	2.30×10^{-05}	1.69×10^{-02}
3	0.895694	0.895849	0.873999	1.55×10^{-04}	2.19×10^{-02}
4	−0.907672	−0.907854	−0.963552	1.82×10^{-04}	5.57×10^{-02}
5	0.168921	0.168958	0.368432	3.70×10^{-05}	1.99×10^{-01}
6	0.729669	0.729817	0.494810	1.48×10^{-04}	2.35×10^{-01}
7	−0.992491	−0.992671	−1.014668	1.79×10^{-04}	2.20×10^{-02}
8	0.337769	0.337816	0.817809	4.70×10^{-05}	4.80×10^{-01}
9	0.654573	0.654624	−0.037096	5.10×10^{-05}	6.92×10^{-01}
10	−1.009330	−1.009337	−0.777857	7.00×10^{-06}	2.31×10^{-01}
Max absolute error (L-inf)	–	–	–	1.82×10^{-04}	6.92×10^{-01}
Root mean square error	–	–	–	1.14×10^{-04}	3.15×10^{-01}
Uniform distribution fitting (p-value)	–	–	–	0.3951	0.0199

Abbreviations: Ms-DTM: Multistep-Differential Transform Method; RK4: Runge–Kutta.

Finally, additional insight into the reliability of each solution can be obtained by conducting a statistical validation using a KS-test. Results of this validation effort showed that the p-value for the Bessel solution for all cases tested was > 0.05. Since such p-values represent a random distribution of residual errors, no systematic growth has been identified. This behavior supports both the numerical stability and robustness of the Bessel formulation. By comparison, p-values associated with the Ms-DTM were less than 5% in all cases evaluated. Rejection of the null hypothesis of uniformity for the residuals suggests that errors produced by Ms-DTM exhibit structured growth over time. Structured growth of errors typically occurs due to the cumulative nature of approximation errors. Thus, while Ms-DTM may produce reliable estimates over short simulation times, its reliability diminishes as simulation time increases.

Collectively, **Tables 2–6** clearly illustrate a hierarchical relationship among the three solutions examined. The analytical Bessel solution is the most accurate and stable, yielding errors many orders of magnitude lower than those generated using Ms-DTM. Finally, since RK4 is a well-established numerical method, it can serve as a reliable benchmark against which other solutions can be validated. While Ms-DTM represents a cost-effective semi-analytic alternative that produces acceptable results for short simulation times, its ultimate accuracy is limited by its reliance on accumulated truncation errors.

Table 6. Absolute error between the RK4 method and Bessel's, and the difference between the RK4 method and Ms-DTM when $\lambda = 0.01$ and $\omega_o = 8$

Time (t)	**Bessel**	**RK4**	**Ms-DTM**	**Absolute error |RK4-Bessel|**	**Absolute error |RK4-Ms-DTM|**
1	−0.165969	−0.166008	−0.166011	3.90×10^{-05}	3.00×10^{-03}
2	−0.935554	−0.936049	−0.954895	4.95×10^{-04}	1.88×10^{-02}
3	0.584410	0.584801	0.484930	3.91×10^{-04}	9.99×10^{-02}
4	0.622178	0.622690	0.803382	5.12×10^{-04}	1.81×10^{-01}
5	−0.955083	−0.955882	−0.756715	7.99×10^{-04}	1.99×10^{-01}
6	0.033334	0.033347	−0.560049	1.30×10^{-05}	5.94×10^{-01}
7	0.918684	0.919353	0.950388	6.68×10^{-04}	3.10×10^{-02}
8	−0.795454	−0.795885	0.249925	4.31×10^{-04}	1.05
9	−0.185616	−0.185695	−1.042973	7.90×10^{-05}	8.57×10^{-01}
10	0.962219	0.962228	0.094076	9.00×10^{-06}	8.68×10^{-01}
Max absolute error (L-inf)	–	–	–	7.99×10^{-04}	1.05
Root mean square error	–	–	–	4.41×10^{-04}	5.36×10^{-01}
Uniform distribution fitting (p-value)	–	–	–	0.4132	0.0075

Abbreviations: Ms-DTM: Multistep-Differential Transform Method; RK4: Runge–Kutta.

5. Conclusion

We studied the dynamics of a harmonic oscillator that had a time-dependent mass. They used analytical, semi-analytical, and numerical methods to investigate how mass variations affect the oscillator's behavior. The authors found an exact analytical solution to the oscillator's governing equation. This solution transformed the governing equation into a Bessel equation. The solution showed that variations in mass can significantly affect an oscillator's frequency and amplitude. The authors also compared their analytical solutions to numerical solutions using the fourth-order RK4 method. The authors found that the two types of solutions were nearly identical.

In addition to finding accurate analytical and numerical solutions, the authors applied the DTM and its multistep extension to obtain further accurate solutions. The authors found that both methods could be used to achieve high accuracy and that they converged rapidly. The authors stated that the combination of these analytical and numerical methods provides an effective framework for modeling the behavior of oscillators whose masses vary in time. The authors suggested that possible future research directions include extending this approach to nonlinear models that incorporate dissipation and applying this type of model to real-world problems in mechanical or other fields.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author contributions

Conceptualization: All authors
Formal analysis: All authors
Methodology: All authors
Writing–original draft: All authors
Writing–review & editing: All authors

Availability of data

Data will be available upon request.

AI tools statement

The authors used Grammarly and AI-based language enhancement tools solely for improving grammar, spelling, and overall language clarity during the preparation of this manuscript. These tools did not contribute to the study design, data analysis, results, or scientific conclusions.

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