An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.16, No.5, pp.1960-1975 (2026)
https://doi.org/10.36922/IJOCTA026250124

RESEARCH ARTICLE

New solitary wave solutions and modulation stability of the time-fractional coupled Konno-Oono model arising in a magnetic field

Hira Tariq[1], Hira Ashraf[1], Ghazala Akram[2,3], Hafiz Abdul Wajid[4*]

[1]Department of Mathematics, Government College Women University, Sialkot, Pakistan
[2]Institute of Mathematics, University of the Punjab, Lahore, Pakistan
[3]Department of Computer Engineering, Biruni University, Istanbul, Türkiye
[4]Department of Electrical Engineering, Faculty of Engineering, Islamic University of Madinah, Madinah, Saudi Arabia
hiratariq47@gmail.com, hiraashraf1012@gmail.com, toghazala2003@yahoo.com, hawajid@iu.edu.sa

ARTICLE INFO

Article History:
Received: June 19, 2026
Revised: July 09, 2026
Accepted: July 14, 2026
Published Online: September 1, 2026

Keywords:
Time-fractional coupled Konno-Oono equation
Generalized projective Riccati equation method
Modified auxiliary equation method
Soliton solutions
Fractional derivatives
Modulation stability

ABSTRACT

The time-fractional coupled Konno-Oono equation model describes a current-carrying rope placed in an outside magnetic field. This model is important for understanding wave behavior in such physical systems. However, finding a wide range or new soliton solutions of this model remains relatively unexplored. In this paper, we study this model using two analytical methods: the modified auxiliary equation method and the generalized projective Riccati equation method. The effects of different orders of time derivatives on the system are also examined using three types of derivatives: conformable, beta, and M-truncated. The proposed model yields many new solitary wave solutions, including periodic solitons, singular periodic shapes, dark-bright solitons, and anti-bell-shaped solutions. Unlike previous work, this study provides a broader family of solutions using simple and reliable methods. The physical behavior of these solutions is shown using 2D line graphs, 3D surface graphs, and contour plots generated with Wolfram Mathematica 9. We also discuss the stability of the proposed model. These results help us understand various natural and dynamical processes.

1. Introduction

Fractional derivatives and integrals have proven to be powerful tools for constructing. The concept of fractional calculus originated with L'Hopital, who on 30th September 1695, wrote to Leibniz inquiring about the meaning of a derivative of non-integer order.[1]

In recent decades, the study of fractional-order derivation and integration has developed into a broad research field, appearing not only in mathematics but also in physics, chemistry, biology, and mechanical engineering. Many of the "useful consequences" predicted by Leibniz have since been uncovered and confirmed.[2] The extension of ordinary corresponds to to fractional calculus means moving from integer order to any real order $\alpha > 0$, giving rise to new definitions for integrals and derivatives, of which the ordinary definitions become special cases. Numerous scientists and researchers , including Lioville, Laplace, Riemann, Abel, Grunwald, Letnikov, Sonin, Laurent, Nekrassov, Levy, Fourier, and others , have contributed to the development of fractional calculus.[3–6]

Over the past few decades, fractional calculus has proven to be a key source for describing complex physical behaviors in various fields, including signal processing, biological population models, optics, fluid mechanics, and electronic networking. Among the different fractional models studied, finding travelling wave solutions—a particular

*Corresponding Author

type of exact solution—for nonlinear fractional differential equations (NLFDEs) is a crucial topic. Many powerful techniques have been used to obtain explicit solutions for such problems, including the simplest equation method,[7] the modified auxiliary equation method,[8] the (G'/G)-expansion method,[9] the extended tanh-function method,[10] the first integral method,[11] the extended trial equation method,[12] the Hirota bilinear technique,[13] the modified Kudryashov method,[14] the sine-cosine method,[15] the sub-equation method,[16] the simplified bilinear method,[17] the multipliers method,[18] the Jacobi elliptic function method,[19] the Sine-Gordon expansion method,[20] the extended Fan-sub equation method,[21] the new transfer function method,[22] and others.

The modified auxiliary equation approach (MAEA) is particularly suitable for constructing analytical solutions, especially travelling wave solutions of NLFDEs.[23–26] Likewise, the generalized projective Riccati equation approach (GPREA) has been successfully applied to many NLFDEs and is known to be powerful and direct.[27–31] However, most existing studies have applied these two methods separately or using only standard derivatives.

To the best of our knowledge, no previous work has applied both MAEA and GPREA together to the time-fractional coupled Konno-Oono equation model (TFCKOEM) using three different fractional derivatives. Moreover, a broad family of new solitary wave solutions for this model has not yet been reported in the scientific literature.

Therefore, this paper aims to fill this gap by deriving new travelling wave solutions for the TFCKOEM using both GPREA and MAEA. For the wave transformations, we employ three distinct fractional operators: the conformable derivative, the beta derivative, and the M-truncated derivative. These solutions have not previously appeared in the literature.

It should be noted that the conformable, beta, and M-truncated fractional derivatives considered in this work belong to the class of local fractional operators. Each of these operators can be expressed as the classical first-order derivative multiplied by an appropriate time-dependent scaling function. As a result, they lack the integral kernel structure that characterizes nonlocal fractional derivatives such as the Caputo and Riemann–Liouville operators. Consequently, they are not capable of describing memory-dependent or hereditary phenomena, and the differences observed in the corresponding formulations are mainly attributable to time-scaling effects rather than genuine nonlocal behavior. The main focus of this study is not the modeling of systems with memory effects but rather the analytical investigation of the TFCKOEM under different local fractional frameworks. These operators are particularly useful in this context because they retain many properties of classical calculus, such as simplified differentiation rules, which make it possible to derive exact travelling wave solutions in a straightforward manner. Hence, they serve as convenient mathematical tools for constructing analytical solutions of NLPDEs.

In contrast, fractional operators of the Caputo and Riemann-Liouville type incorporate nonlocal memory through integral formulations and are therefore more suitable for describing processes with hereditary characteristics and anomalous diffusion. However, such operators often lead to more complicated analytical structures and require different solution techniques. Since the present study is restricted to local fractional formulations, a detailed comparison with nonlocal models is beyond its scope and may be considered an interesting direction for future investigation.

Solitons are a special class of nonlinear localized wave structures that propagate over long distances while retaining their shape, amplitude, and velocity due to the delicate balance between nonlinearity and dispersion. Unlike ordinary waves, which gradually disperse or dissipate during propagation, solitons exhibit remarkable stability and retain their identities even after interacting with other solitons, emerging from collisions with only a phase shift. Since the pioneering observation of solitary waves by John Scott Russell in 1834, solitons have attracted significant attention in nonlinear science and mathematical physics. Their unique ability to maintain their shape while propagating has made them an important subject of research. Solitons are encountered in numerous physical contexts, including fluid mechanics, plasma physics, nonlinear optical media, condensed matter systems, quantum field theory, and various biological processes.

The mathematical investigation of solitons has significantly advanced the understanding of nonlinear evolution equations, where exact analytical solutions provide valuable insights into the underlying wave dynamics. In recent years, the extension of soliton theory to fractional-order differential equations has attracted considerable attention because fractional calculus incorporates history-dependent and hereditary characteristics that cannot be represented by conventional integer-order models. Consequently, the study of solitary wave solutions in time-fractional nonlinearsystems offers a more realistic description of many complex physical phenomena and contributes to the development of mathematical models with enhanced predictive capability.[32–37]

The remainder structure of this paper is structured as follows. Section 2 presents some basic definitions and results for the derivatives used. Section 3 introduces the governing model. Section 4 describes the proposed methods. In Section 5, we apply these methods to construct novel analytical solutions for the TFCKOEM. Section 6presents the graphical representation of the obtained solutions. Section 7 examines the modulation stability of the model. Section 8 provides the conclusions.

2. Basic definitions

Fractional calculus is closer to reality owing to its flexibility and plays a key role in applied sciences. Here, we present the basic definitions and the properties of three fractional derivatives: the conformable derivative, the beta derivative, and the M-truncated derivative.

2.1. Conformable derivative

Definition 1. *For a function $g : (0,\infty) \to \mathbb{R}$ and $0 < \alpha \leq 1$, the conformable derivative of order α is defined as*

$$D_\alpha(g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{1-\alpha}) - g(t)}{\epsilon}, \quad t > 0.$$

If g is α-differentiable, then $D_\alpha(g)(t) = t^{1-\alpha} g'(t)$ on $(0, a)$.[38]

Theorem 1. *Let $0 < \alpha \leq 1$. Suppose f and g are α-differentiable at a point t. Then the following properties hold:*

(i) Linearity:

$$D_\alpha(qf + wg)(t) = qD_\alpha(f)(t) + wD_\alpha(g)(t), \quad \text{for } q, w \in \mathbb{R}.$$

(ii) Power rule: for any real constant ς,

$$D_\alpha(t^\varsigma) = \varsigma t^{\varsigma - \alpha}.$$

(iii) Product rule:

$$D_\alpha(fg)(t) = f(t)D_\alpha(g)(t) + g(t)D_\alpha(f)(t).$$

(iv) Quotient rule:

$$D_\alpha\left(\frac{f}{g}\right)(t) = \frac{f(t)D_\alpha(g)(t) - g(t)D_\alpha(f)(t)}{[g(t)]^2}.$$

2.2. Beta derivative

Definition 2. *For $0 < \alpha \leq 1$ and a function $g(t)$ defined for $t \geq 0$, the beta derivative of order α is defined as*[39]

$$D_t^\alpha(g(t)) = \lim_{\epsilon \to 0} \frac{g\left(t + \epsilon\left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right) - g(t)}{\epsilon}.$$

The beta derivative satisfies the following properties for α-differentiable functions g and y at a point t:

(i) Linearity: for $a, b \in \mathbb{R}$,

$$D_t^\alpha(ag(t) + by(t)) = aD_t^\alpha g(t) + bD_t^\alpha y(t).$$

(ii) Derivative of a constant: for any constant c,

$$D_t^\alpha(c) = 0.$$

(iii) Product rule:

$$D_t^\alpha(g(t)y(t)) = g(t)D_t^\alpha y(t) + y(t)D_t^\alpha g(t).$$

(iv) Quotient rule:

$$D_t^\alpha\left(\frac{g(t)}{y(t)}\right) = \frac{y(t)D_t^\alpha g(t) - g(t)D_t^\alpha y(t)}{[y(t)]^2}.$$

(v) Relation to ordinary derivative: if g is differentiable on $(0, a)$, then

$$D_t^\alpha(g(t)) = \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha} \frac{dg(t)}{dt}.$$

2.3. M-truncated derivative

Definition 3. *Let $0 < \alpha \leq 1$, $\beta > 0$, and $t \geq 0$. For a function $g(t)$, the M-truncated fractional derivative of order α and parameter β is defined as*[40]

$${}_0D_M^{\alpha,\beta}(g(t)) = \lim_{\epsilon \to 0} \frac{g\left(t\, \mathbb{E}_\beta(\epsilon t^{-\alpha})\right) - g(t)}{\epsilon},$$

where $\mathbb{E}_\beta(\cdot)$ is the Mittag-Leffler function. The subscript 0 indicates that the derivative is taken at $t = 0$.

The M-truncated derivative satisfies the following properties for α-differentiable functions g and h at a point t:

(i) Linearity: for $a, b \in \mathbb{R}$,

$${}_0D_M^{\alpha,\beta}(ag(t) + bh(t)) = a\, {}_0D_M^{\alpha,\beta} g(t) + b\, {}_0D_M^{\alpha,\beta} h(t).$$

(ii) Derivative of a constant: for any constant c,

$${}_0D_M^{\alpha,\beta}(c) = 0.$$

(iii) Product rule:

$${}_0D_M^{\alpha,\beta}(g(t)h(t)) = g(t)\, {}_0D_M^{\alpha,\beta} h(t) + h(t)\, {}_0D_M^{\alpha,\beta} g(t).$$

(iv) Quotient rule:

$${}_0D_M^{\alpha,\beta}\left(\frac{g(t)}{h(t)}\right) = \frac{h(t)\, {}_0D_M^{\alpha,\beta} g(t) - g(t)\, {}_0D_M^{\alpha,\beta} h(t)}{[h(t)]^2}.$$

(v) Relation to ordinary derivative: if g is differentiable on $(0, a)$, then

$${}_0D_M^{\alpha,\beta}(g(t)) = \frac{t^{1-\alpha}}{\Gamma(\beta + 1)} \frac{dg(t)}{dt}.$$

3. The governing model

Recent advances have established fractional calculus as a powerful tool for modeling diverse phenomena across the physical and engineering sciences. It offers a more accurate description than ordinary calculus. This has motivated researchers to study and find new solutions for many models using fractional operators. In this article, three different fractional operators are applied to the time-fractional coupled Konno-Oono equation model (TFCKOEM): the conformable derivative, the M-truncated derivative, and the beta derivative.[41]

In 2022, Elbrolosy and Elmandouh[42] found different types of traveling wave solutions for this model. Their work considered the model as a dynamical system in a magnetic field. In 2023, Yasmin and Aljahdaly[43] studied symmetric solitary wave solutions using two forms of an extended direct algebraic method.

Consider the time-fractional coupled Konno-Oono equation:

$$\begin{aligned} D_t^{\alpha} q(\varpi,t) + 2p(\varpi,t)p_{\varpi}(\varpi,t) = 0,\\ D_t^{\alpha} p(\varpi,t) + p_{\varpi}(\varpi,t) - 2p(\varpi,t)q(\varpi,t) = 0. \end{aligned} \tag{1}$$

Here, $0 < \alpha \leq 1$ and D_t^{α} is a fractional derivative (defined in Section 2). The functions $p(\varpi,t)$ and $q(\varpi,t)$ represent the displacements of two interacting particles in the medium.

4. Framework of the proposed methods

We now describe the two methods used in this paper. Both methods are applied to nonlinear partial differential equations (NLPDEs).

Consider an NLPDE of the form

$$V(p, p_{\varpi}, p_t, p_{\varpi\varpi}, p_{tt}, p_{\varpi t}, \dots) = 0, \tag{2}$$

where V is a polynomial in $p(\varpi,t)$ and its partial derivatives.

We use the wave transformation

$$p(\varpi,t) = P(\zeta), \quad \text{where} \quad \zeta = \varpi \pm ct. \tag{3}$$

Substituting **Equation (3)** into the NLPDE reduces it to an ordinary differential equation (ODE):

$$V(P, P', P'', P''', \dots) = 0. \tag{4}$$

4.1. Method 1: Modified auxiliary equation method

The core workflow consists of the following key steps.

Step 1: We assume that the solution of the ODE has the form

$$P(\zeta) = a_0 + \sum_{i=0}^{M} l_i K^{-if(\zeta)} + \sum_{i=0}^{M} z_i K^{if(\zeta)}, \tag{5}$$

where a_0, l_i, and z_i are constants to be determined later. The function $f(\zeta)$ satisfies the ODE

$$f'(\zeta) = \frac{1}{\ln K}\left(\sigma K^{f(\zeta)} + \alpha K^{-f(\zeta)} + \beta\right). \tag{6}$$

Step 2: We find the positive integer M using the homogeneous balance method, which balances the highest-order derivative term with the highest-power nonlinear term in the ODE.

Step 3: We substitute **Equations (5)** and **(6)** into the ODE and group terms with the same powers of $K^{if(\zeta)}$ for $i = -M, \dots, M$. This yields a system of algebraic equations. Setting each coefficient to zero and solving the system using Wolfram Mathematica 9 gives the values of $\alpha, \beta, \sigma, z_i, l_i$, and the other parameters.

Step 4: We insert these values together with the solution of the **Equation (6)** into **Equation (5)** to obtain analytical solutions of the original NLPDE.

4.2. Method 2: Generalized projective Riccati equation method

The core workflow consists of the following key steps.

Step 1: We assume that the solution of ODE has the following expression:

$$P(\zeta) = A_0 + \sum_{i=1}^{M} \vartheta^{i-1}(\zeta)\left[A_i\vartheta(\zeta) + B_i\tau(\zeta)\right]. \tag{7}$$

Here, A_0, A_i, and B_i are constants to be determined later. The functions $\vartheta(\zeta)$ and $\tau(\zeta)$ satisfy the ODEs

$$\vartheta'(\zeta) = \varepsilon\vartheta(\zeta)\tau(\zeta), \tag{8}$$

$$\tau'(\zeta) = \hbar + \varepsilon\tau^2(\zeta) - m\vartheta(\zeta), \quad \varepsilon = \pm 1. \tag{9}$$

These functions also satisfy the relation:

$$\tau^2(\zeta) = -\varepsilon\left(\hbar - 2m\vartheta(\zeta) + \frac{m^2-1}{\hbar}\vartheta^2(\zeta)\right), \tag{10}$$

where m and $\hbar$ are nonzero constants.

If $\hbar = m = 0$, the solution simplifies to

$$P(\zeta) = \sum_{i=0}^{M} A_i\tau^i(\zeta), \tag{11}$$

and $\tau(\zeta)$ satisfies:

$$\tau'(\zeta) = \tau^2(\zeta). \tag{12}$$

Step 2: We find the positive integer M using the homogeneous balance method, as before.

Step 3: We substitute **Equations (8)–(10)** into the ODE and collect terms with the same powers of $\vartheta^j\tau^i$ (where $j = 0, 1, 2, \dots$ and $i = 0, 1$)

or powers of τ^j (where $j = 0, 1, 2, \ldots$). Setting each coefficient to zero gives a system of algebraic equations. We solve this system using Wolfram Mathematica 9 or Maple to obtain the values of $\hbar, c, m, A_0, A_i, B_i$, and other parameters.

Step 4: The system admits the following known solutions:[44]

(1) If $\varepsilon = -1$ and $\hbar \neq 0$:

$$\begin{aligned} \vartheta_1(\zeta) &= \frac{\hbar\,\mathrm{sech}(\sqrt{\hbar}\,\zeta)}{m\,\mathrm{sech}(\sqrt{\hbar}\,\zeta)+1}, \quad \tau_1(\zeta) = \frac{\hbar\tanh(\sqrt{\hbar}\,\zeta)}{m\,\mathrm{sech}(\sqrt{\hbar}\,\zeta)+1}, \\ \vartheta_2(\zeta) &= \frac{\hbar\,\mathrm{csch}(\sqrt{\hbar}\,\zeta)}{m\,\mathrm{csch}(\sqrt{\hbar}\,\zeta)+1}, \quad \tau_2(\zeta) = \frac{\hbar\coth(\sqrt{\hbar}\,\zeta)}{m\,\mathrm{csch}(\sqrt{\hbar}\,\zeta)+1}. \end{aligned} \tag{13}$$

(2) If $\varepsilon = 1$ and $\hbar \neq 0$:

$$\begin{aligned} \vartheta_3(\zeta) &= \frac{\hbar\sec(\sqrt{\hbar}\,\zeta)}{m\sec(\sqrt{\hbar}\,\zeta)+1}, \quad \tau_3(\zeta) = \frac{\hbar\tan(\sqrt{\hbar}\,\zeta)}{m\sec(\sqrt{\hbar}\,\zeta)+1}, \\ \vartheta_4(\zeta) &= \frac{\hbar\csc(\sqrt{\hbar}\,\zeta)}{m\csc(\sqrt{\hbar}\,\zeta)+1}, \quad \tau_4(\zeta) = \frac{\hbar\cot(\sqrt{\hbar}\,\zeta)}{m\csc(\sqrt{\hbar}\,\zeta)+1}. \end{aligned} \tag{14}$$

(3) If $\hbar = m = 0$:

$$\vartheta_5(\zeta) = \frac{G}{\zeta}, \quad \tau_5(\zeta) = \frac{1}{\varepsilon\zeta}, \tag{15}$$

where G is a nonzero constant.

Step 5: We substitute the values of $A_0, A_i, B_i, c, m, \hbar$ together with the solutions from **Equations (13)**, **(14)**, or **(15)** into **Equation (7)** to obtain the analytical solutions of **Equation (1)**.

In this study, we take the case $\varepsilon = 1$. (The case $\varepsilon = -1$ can be treated analogously.)

5. Implementation of methods to the time-fractional Konno-Oono equation

5.1. Analysis of solutions using MAEA

We now apply the modified auxiliary equation method (MAEA) to the time-fractional coupled Konno-Oono equation model (TFCKOEM) to obtain new and distinct solitary wave solutions.

To find exact analytical solutions of **Equation (1)**, we use the transformation

$$p(\varpi, t) = P(\xi), \quad q(\varpi, t) = Q(\xi), \tag{16}$$

where c is the wave velocity. This transformation converts the system of NLPDEs in **Equation (1)** into a system of ODEs. The appropriate transformations for the three different derivatives are given below.

(a) For the conformable derivative:

$$\xi(\varpi, t) = kx - c\frac{t^\alpha}{\alpha}. \tag{17}$$

(b) For the beta derivative:

$$\xi(\varpi, t) = kx - \frac{c}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha. \tag{18}$$

(c) For the M-truncated derivative:

$$\xi(\varpi, t) = kx - \frac{ct^\alpha\Gamma(1+\beta)}{\alpha}. \tag{19}$$

Using these transformations, **Equation (1)** is converted into the following coupled ODE system:

$$\begin{aligned} -cQ' + 2kPP' &= 0, \\ -ckP'' - 2PQ &= 0. \end{aligned} \tag{20}$$

Integrating the first equation of **(20)** with respect to ξ and simplifying gives

$$c^2kP'' + 2kP^3 + 2Ph = 0, \tag{21}$$

where h is the integration constant.

Using the homogeneous balance method, we find $M = 1$. The ODE solution then takes the form

$$P(\xi) = a_0 + a_1K^{f(\xi)} + b_1K^{-f(\xi)}, \tag{22}$$

where

$$f'(\xi) = \frac{1}{\ln K}\left(\alpha K^{-f(\xi)} + \beta + \sigma K^{f(\xi)}\right). \tag{23}$$

Substituting **Equation (22)** and its derivatives into **Equation (21)** and collecting terms with the same powers of $K^{if(\xi)}$ for $i = -M, \ldots, M$, and equating the coefficients to zero, we obtain the following algebraic system:

$$\begin{aligned}
K^{f}: &\quad c^2ka_1\beta^2 + 2ka_1c^2\alpha\sigma + 6ka_0^2a_1 + 6ka_1^2b_1 + 2ha_1 = 0,\\
K^{2f}: &\quad c^2a_1k\beta\sigma + 2a_1kc^2\beta\sigma + 6a_0ka_1^2 = 0,\\
K^{3f}: &\quad 2a_1\sigma^2kc^2 + 2ka_1^3 = 0,\\
K^{-f}: &\quad c^2b_1k\beta^2 + 2kb_1c^2\alpha\sigma + 6kb_1^2a_1 + 6a_0^2kb_1 + 2hb_1 = 0,\\
K^{-2f}: &\quad c^2kb_1\beta\alpha + 2kb_1c^2\beta\alpha + 6ka_0b_1^2 = 0,\\
K^{-3f}: &\quad 2b_1k\alpha^2c^2 + 2kb_1^3 = 0,\\
\text{Constant}: &\quad c^2ka_1\alpha\beta + b_1kc^2\beta\sigma + 2ka_0^3 + 12ka_0a_1b_1 + 2a_0h = 0.
\end{aligned} \tag{24}$$

Solving this system with Wolfram Mathematica 9 yields the following solution families discussed in following subsections one by one.

5.1.1. Family 1

The Family 1 solutions are given by obtained from **Equation (24)**.

$$a_0 = \frac{ic\beta}{2}, \quad a_1 = ic\sigma, \quad b_1 = 0, \quad \alpha = \frac{-4h + c^2\beta^2k}{4\sigma kc^2}. \tag{25}$$

The corresponding solitary wave solutions are:

Case 1: $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{ic\beta}{2} + ic\sigma\left(\frac{-\beta + \sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)}{2\sigma}\right), \tag{26}$$

or

$$P(\xi) = \frac{ic\beta}{2} + ic\sigma\left(\frac{-\beta + \sqrt{4\alpha\sigma - \beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)}{2\sigma}\right). \tag{27}$$

Case 2: $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{ic\beta}{2} + ic\sigma\left(\frac{-\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)}{2\sigma}\right), \tag{28}$$

or

$$P(\xi) = \frac{ic\beta}{2} + ic\sigma\left(\frac{-\beta + \sqrt{\beta^2 - 4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)}{2\sigma}\right). \tag{29}$$

Case 3: $\beta^2 - 4\alpha\sigma = 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{ic\beta}{2} + ic\sigma\left(\frac{-2 + \beta\xi}{2\sigma\xi}\right). \tag{30}$$

Here, ξ is given by the wave transformation in **Equations (17)**, **(18)**, or **(19)**, depending on the derivative used.

5.1.2. Family 2

The solutions of 2nd family are given by obtained from Equation (24).

$$a_0 = 0, \quad a_1 = ic\sigma, \quad b_1 = \frac{ih}{4ck\sigma}, \quad \alpha = \frac{h}{-4\sigma kc^2}. \tag{31}$$

The corresponding solitary wave solutions are:

Case 1: $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$:

$$\begin{aligned}
P(\xi) = ic\sigma&\left(\frac{\sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)}{2\sigma}\right)\\
&+\frac{ih}{4ck\sigma}\left(\frac{2\sigma}{\sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)}\right),
\end{aligned} \tag{32}$$

or

$$P(\xi) = ic\sigma\left(\frac{\sqrt{4\alpha\sigma-\beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma-\beta^2}\,\xi\right)}{2\sigma}\right) + \frac{ih}{4ck\sigma}\left(\frac{2\sigma}{\sqrt{4\alpha\sigma-\beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma-\beta^2}\,\xi\right)}\right). \tag{33}$$

Case 2: $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$:

$$P(\xi) = ic\sigma\left(\frac{\sqrt{\beta^2-4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}{2\sigma}\right) + \frac{ih}{4ck\sigma}\left(\frac{2\sigma}{\sqrt{\beta^2-4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}\right), \tag{34}$$

or

$$P(\xi) = ic\sigma\left(\frac{\sqrt{\beta^2-4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}{2\sigma}\right) + \frac{ih}{4ck\sigma}\left(\frac{2\sigma}{\sqrt{\beta^2-4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}\right). \tag{35}$$

Case 3: $\beta^2 - 4\alpha\sigma = 0$ and $\sigma \neq 0$:

$$P(\xi) = ic\sigma\left(\frac{-2+\beta\xi}{2\sigma\xi}\right). \tag{36}$$

5.1.3. Family 3

The third solution family is given by:

$$a_0 = \frac{\sqrt{h}}{\sqrt{2k}}, \quad a_1 = ic\sigma, \quad b_1 = 0, \quad \alpha = \frac{-3h}{2c^2k\sigma}, \quad \beta = \frac{i\sqrt{2h}}{c\sqrt{k}}. \tag{37}$$

The corresponding solitary wave solutions are: **Case 1:** $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{\sqrt{h}}{\sqrt{2k}} + ic\sigma\left(\frac{-\beta+\sqrt{4\alpha\sigma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma-\beta^2}\,\xi\right)}{2\sigma}\right), \tag{38}$$

or

$$P(\xi) = \frac{\sqrt{h}}{\sqrt{2k}} + ic\sigma\left(\frac{-\beta+\sqrt{4\alpha\sigma-\beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma-\beta^2}\,\xi\right)}{2\sigma}\right). \tag{39}$$

Case 2: $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{\sqrt{h}}{\sqrt{2k}} + ic\sigma\left(\frac{-\beta+\sqrt{\beta^2-4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}{2\sigma}\right), \tag{40}$$

or

$$P(\xi) = \frac{\sqrt{h}}{\sqrt{2k}} + ic\sigma\left(\frac{-\beta+\sqrt{\beta^2-4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\,\xi\right)}{2\sigma}\right). \tag{41}$$

Case 3: $\beta^2 - 4\alpha\sigma = 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{\sqrt{h}}{\sqrt{2k}} + ic\sigma\left(\frac{-2+\beta\xi}{2\sigma\xi}\right). \tag{42}$$

5.1.4. Family 4

Now, the solutions of fourth family are given below:

$$a_0 = \frac{i\sqrt{h}}{\sqrt{k}}, \quad a_1 = 0, \quad b_1 = ic\alpha, \quad \beta = \frac{2\sqrt{h}}{c\sqrt{k}}. \tag{43}$$

The corresponding solitary wave solutions are:

Case 1: $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{i\sqrt{h}}{\sqrt{k}} + ic\alpha \left(\frac{2\sigma}{-\beta + \sqrt{4\alpha\sigma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)} \right), \tag{44}$$

or

$$P(\xi) = \frac{i\sqrt{h}}{\sqrt{k}} + ic\alpha \left(\frac{2\sigma}{-\beta + \sqrt{4\alpha\sigma - \beta^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\sigma - \beta^2}\,\xi\right)} \right). \tag{45}$$

Case 2: $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{i\sqrt{h}}{\sqrt{k}} + ic\alpha \left(\frac{2\sigma}{-\beta + \sqrt{\beta^2 - 4\alpha\sigma}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)} \right), \tag{46}$$

or

$$P(\xi) = \frac{i\sqrt{h}}{\sqrt{k}} + ic\alpha \left(\frac{2\sigma}{-\beta + \sqrt{\beta^2 - 4\alpha\sigma}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)} \right). \tag{47}$$

Case 3: $\beta^2 - 4\alpha\sigma = 0$ and $\sigma \neq 0$:

$$P(\xi) = \frac{i\sqrt{h}}{\sqrt{k}} + ic\alpha \left(\frac{2\sigma\xi}{-2 + \beta\xi} \right). \tag{48}$$

5.2. Analysis of solutions using GPREA

We now apply the generalized projective Riccati equation method (GPREA) to **Equation (1)** to construct analytical solutions.

Using the transformations given in **Equations (17)–(19)**, **Equation (1)** is reduced to the ODE

$$c^2kP'' + 2kP^3 + 2Ph = 0. \tag{49}$$

We assume the solution has the form:

$$P(\xi) = A_0 + \sum_{i=1}^{M} \vartheta^{i-1}(\xi)\left[A_i\vartheta(\xi) + B_i\tau(\xi)\right]. \tag{50}$$

By the help of the homogeneous balance method, we get $M = 1$. Therefore, **Equation (50)** reduces to:

$$P(\xi) = A_0 + A_1\vartheta(\xi) + B_1\tau(\xi). \tag{51}$$

Subsituting **Equation (51)** together with **Equations (8)–(10)** into **Equation (49)** and collecting terms with the same powers of ϑ and τ, then setting each coefficient to zero, yields the following algebraic system, where R is an arbitrary constant:

$$\begin{aligned}
\vartheta^3\tau^0 &: \quad -c^2kA_1(m^2-1) - c^2k(m^2-1) + 2kA_1^3 - 6kA_1B_1^3(m^2-1) = 0,\\
\vartheta^2\tau^0 &: \quad 2c^2RA_1km + Rmkc^2 + 6RkA_0A_1^2 + 12mkRA_1B_1^2 - 6kA_0B_1^2(m^2-1) = 0,\\
\vartheta^1\tau^0 &: \quad -c^2kA_1\hbar^2 + 6RkA_0^2A_1 - 6kR^2A_1B_1^2 + 12mkRA_0B_1^2 + 2RhA_1 = 0,\\
\vartheta^2\tau^1 &: \quad -2kc^2B_1(m^2-1) - 2kB_1^3(m^2-1) + 6RkA_1^2B_1 = 0,\\
\vartheta^1\tau^1 &: \quad 2c^2RkB_1m - c^2kRB_1m + 4mRkB_1^3 + 12RkA_0A_1B_1 = 0,\\
\vartheta^0\tau^1 &: \quad -2\hbar^2kB_1^3 + 6RkA_0^2B_1 + 2RhB_1 = 0,\\
\text{Constant} &: \quad 2RkA_0^3 - 6kR^2A_0B_1^2 + 2RhA_0 = 0.
\end{aligned} \tag{52}$$

Solving this system with Wolfram Mathematica 9 yields the following solution cases.

5.2.1. Case 1

The first solution family obtained by solving system on mathematica 9 is given below:

$$\begin{gathered} A_0 = 0, \quad A_1 = 1, \quad B_1 = -\frac{ic}{2}, \\ \hbar = -\frac{4h}{kc^2}, \quad m = \frac{\sqrt{c^4k - 16h}}{\sqrt{k}c^2}. \end{gathered} \tag{53}$$

Using **Equations (13)–(15)**, and **(52)**, we obtain the following analytical solutions.

For $\varepsilon = 1$ and $\hbar \neq 0$:

$$P(\xi) = \frac{\hbar \sec(\sqrt{\hbar}\,\xi)}{m \sec(\sqrt{\hbar}\,\xi) + 1} - \frac{ic}{2} \cdot \frac{\hbar \tan(\sqrt{\hbar}\,\xi)}{m \sec(\sqrt{\hbar}\,\xi) + 1}, \tag{54}$$

$$P(\xi) = \frac{\hbar \csc(\sqrt{\hbar}\,\xi)}{m \csc(\sqrt{\hbar}\,\xi) + 1} - \frac{ic}{2} \cdot \frac{\hbar \cot(\sqrt{\hbar}\,\xi)}{m \csc(\sqrt{\hbar}\,\xi) + 1}. \tag{55}$$

Here, ξ is given by the wave transformation in **Equations (17)**, **(18)**, or **(19)**, depending on the derivative used.

5.2.2. Case 2

The second solution family obtained by solving system on mathematica 9 is given below:

$$\begin{gathered} A_0 = 0, \quad A_1 = 1, \quad B_1 = \frac{ic}{2}, \\ \hbar = -\frac{4h}{kc^2}, \quad m = -\frac{\sqrt{c^4k - 16h}}{\sqrt{k}c^2}. \end{gathered} \tag{56}$$

Using **Equations (13)–(15)**, and **(52)**, we obtain:

$$P(\xi) = \frac{\hbar \sec(\sqrt{\hbar}\,\xi)}{m \sec(\sqrt{\hbar}\,\xi) + 1} + \frac{ic}{2} \cdot \frac{\hbar \tan(\sqrt{\hbar}\,\xi)}{m \sec(\sqrt{\hbar}\,\xi) + 1}, \tag{57}$$

$$P(\xi) = \frac{\hbar \csc(\sqrt{\hbar}\,\xi)}{m \csc(\sqrt{\hbar}\,\xi) + 1} + \frac{ic}{2} \cdot \frac{\hbar \cot(\sqrt{\hbar}\,\xi)}{m \csc(\sqrt{\hbar}\,\xi) + 1}. \tag{58}$$

5.2.3. Case 3

The third solution case obtained by solving system on mathematica 9 is given below:

$$A_0 = 0, \quad A_1 = -1, \quad B_1 = 0, \quad \hbar = 0. \tag{59}$$

For $\hbar = m = 0$, using **Equation (15)**, we obtain:

$$P(\xi) = -\frac{G}{\xi}. \tag{60}$$

6. Graphical representation of extracted solutions

In this section, we present the graphical interpretation of our results. We successfully applied GPREA and MAEA using three fractional derivatives to obtain novel solitary wave solutions of the TFCKOEM.

6.1. Key findings

The main findings of this work are as follows:

- To the best of our knowledge, several classes of solutions reported here are new. These include periodic solitons, singular periodic shapes, dark-bright solitons, and anti-bell-shaped solutions, none of which were found in earlier studies on this model.
- The three derivatives (conformable, beta, and M-truncated) produce different wave behaviors, which we illustrate through graphs for various fractional orders.
- The obtained solutions help explain natural phenomena in applied sciences, especially those involving magnetic fields. For example, hyperbolic tangent functions appear in the calculation of magnetic moment and in special relativity. These solutions are also useful for all-optical switches, optical amplifiers, and mode-locked lasers.
- Both methods are simple and reliable, providing a wide family of solutions with low computational effort.

6.2. Graphical results

We now present graphs of selected solutions. All plots were generated using Wolfram Mathematica 9. **Figure 1** shows the dark-bright periodic soliton from **Equation (28)** with three-dimensional surface graphs, contour graphs, and two-dimensional line graphs for different fractional orders. The parameter values are $c = 0.02$, $t = 0.1$, $\beta = 0.6$, and $\sigma = 0.5$.

Figure 2 shows the dark-bright soliton from **Equation (33)** using the same parameter values.

Figure 3 shows real and the imaginary parts of the periodic soliton from **Equation (40)**. The parameter values are $c = 0.02$, $t = 0.1$, $\hbar = 3$, $m = 2$.

6.3. Implications of the results

The obtained solutions provide a reliable basis for new experiments. They can be used by researchers with appropriate initial conditions. The fractional-order results help understand physical changes and are also useful for controlling laboratory experiments.

These new solutions are valuable for theoretical analysis and experimental work in all-optical switches, optical amplifiers, and mode-locked lasers.

7. Modulation stability of the coupled Konno-Oono equation model

Many nonlinear natural phenomena exhibit instability behavior arising from the interplay between nonlinear and dispersive effects. In this section, we study the modulation instability of the TFCKOEM using linear stability analysis.

We begin with steady-state solutions of the TFCKOEM of the form

$$\begin{aligned} p(\varpi,t) &= \sqrt{P} + \phi_1(\varpi,t)\exp(\psi(t)), \\ q(\varpi,t) &= \sqrt{P} + \phi_2(\varpi,t)\exp(\psi(t)), \end{aligned} \tag{61}$$

$$\text{where} \quad \psi(t) = P(\alpha_1 + P\alpha_2\epsilon)t.$$

Here, P represents the normalized optical power, and ϵ is a small perturbation parameter.

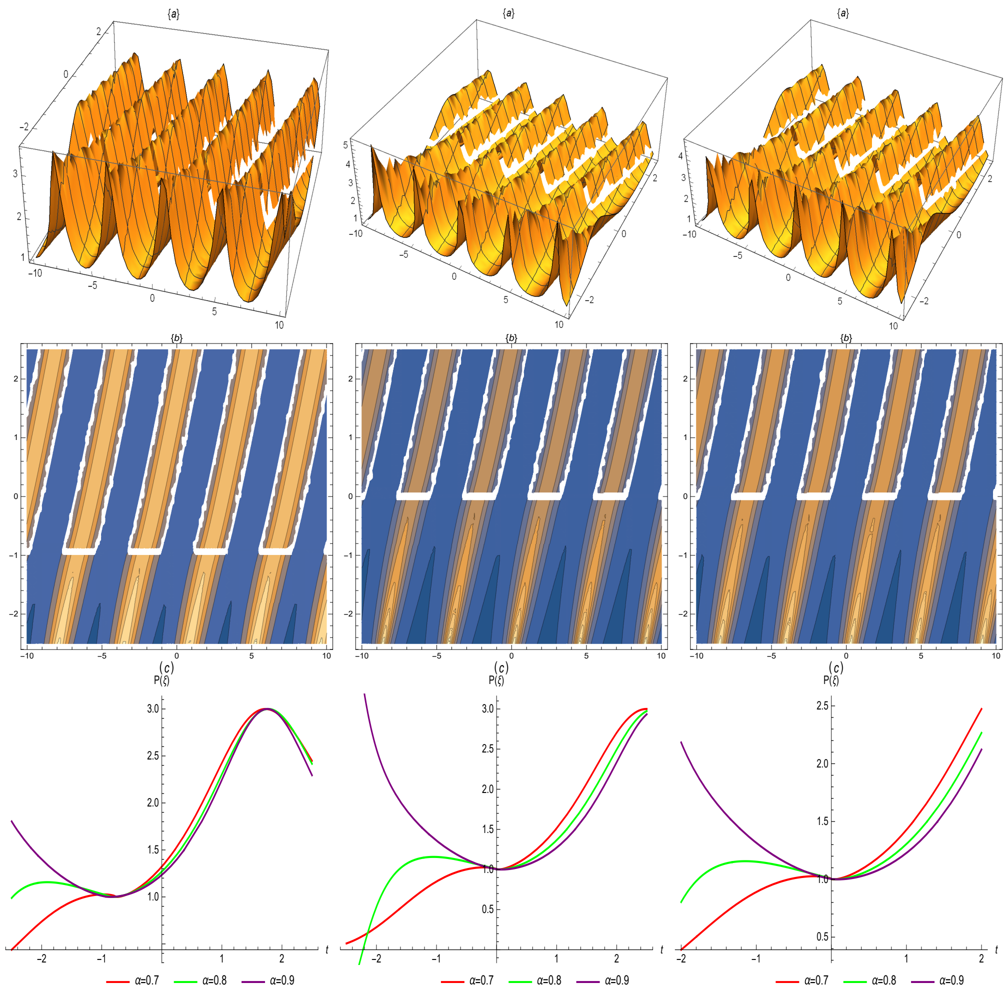

Figure 1. (a) represents the three dimensional plot for all three derivatives (b) represents the contour plot for all three derivatives (c) represents the parametric plot for all three derivatives that are beta, conformable ,and M-truncated fractional derivatives respectively for different values of α. Parameters: $c = 0.02$, $t = 0.1$, $\beta = 0.6$, $\sigma = 0.5$.

Substituting **Equation (61)** into **Equation (1)** and linearizing the resulting equation gives

$$\begin{aligned} \frac{\partial \phi_2}{\partial t} + P\alpha_1\phi_2 + P^2\alpha_2\epsilon\phi_2 + 2\sqrt{P}\frac{\partial \phi_1}{\partial \varpi} &= 0, \\ \phi_1 + \phi_2 &= 0. \end{aligned} \tag{62}$$

We now assume solutions of **Equation (62)** take specific forms depending on the fractional derivative used.

7.1. Perturbation solutions for each derivative

(a) For the conformable derivative:

$$\begin{aligned} \phi_1(\varpi,t) &= \beta_1 \exp i\left(\nu\varpi + \frac{\omega c t^\alpha}{\alpha}\right), \\ \phi_2(\varpi,t) &= \beta_2 \exp i\left(\nu\varpi + \frac{\omega c t^\alpha}{\alpha}\right). \end{aligned} \tag{63}$$

(b) For the beta derivative:

$$\phi_1(\varpi,t)=\beta_1 \exp i\left(\nu kx-\frac{\omega c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right),$$
$$\phi_2(\varpi,t)=\beta_2 \exp i\left(\nu kx-\frac{\omega c}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right). \tag{64}$$

(c) For the M-truncated derivative:

$$\phi_1(\varpi,t)=\beta_1 \exp i\left(\nu kx-\frac{\omega c t^{\alpha}\Gamma(1+\beta)}{\alpha}\right),$$
$$\phi_2(\varpi,t)=\beta_2 \exp i\left(\nu kx-\frac{\omega c t^{\alpha}\Gamma(1+\beta)}{\alpha}\right). \tag{65}$$

Here, ω is the perturbation frequency and ν is the normalized wave number.

7.2. Dispersion relation

Substituting **Equations (63)–(65)** into **Equation (62)**yields the following dispersion relation:

$$\beta_2=-\beta_1,$$
$$P=\frac{2k^2\nu^2+\alpha_1 c\omega-2\sqrt{k^4\nu^4+\alpha_1 ck^2\nu^2\omega}}{\alpha_1^2}. \tag{66}$$

To analyze the modulation stability behavior of the system, we construct the stability region in **Figure 4** based on the dispersion relation given in **Equation (66)**.

7.3. Stability condition

The derived dispersion relation shows that steady-state stability depends on both the normalized wave number ν and the normalized optical power P.

The steady state is considered stable against perturbations if the following conditions hold:

- The dispersion ω remains real for all wave numbers ν.
- $c \neq 0$.
- $\sqrt{k^4\nu^4+\alpha_1 ck^2\nu^2\omega}>0$.

When these conditions are satisfied, the system is stable. If any condition fails, instability may occur.

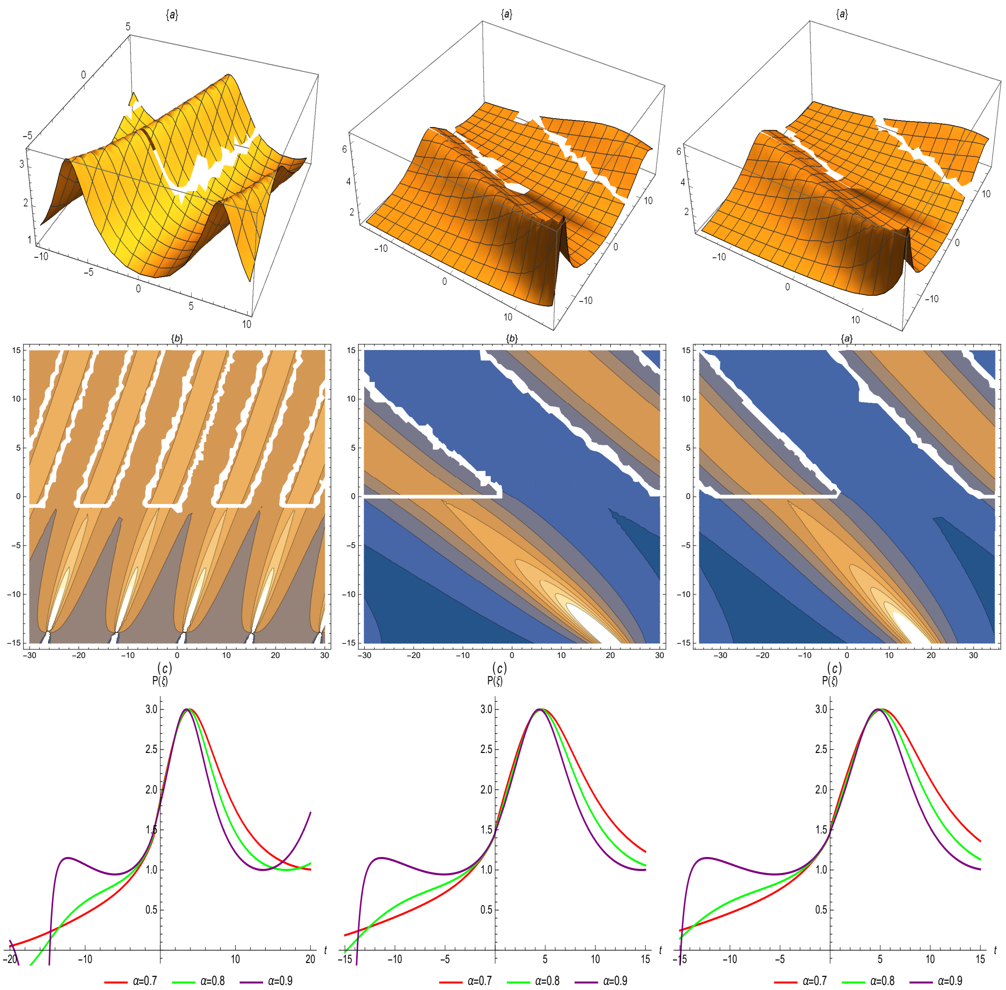

Figure 2. (a) represents the three dimensional plot for all three derivatives (b) represents the contour plot for all three derivatives (c) represents the parametric plot for all three derivatives that are beta, conformable ,and M-truncated fractional derivatives respectively for different values of α. Parameters: $c=0.02$, $t=0.1$, $\beta=0.6$, $\sigma=0.5$.

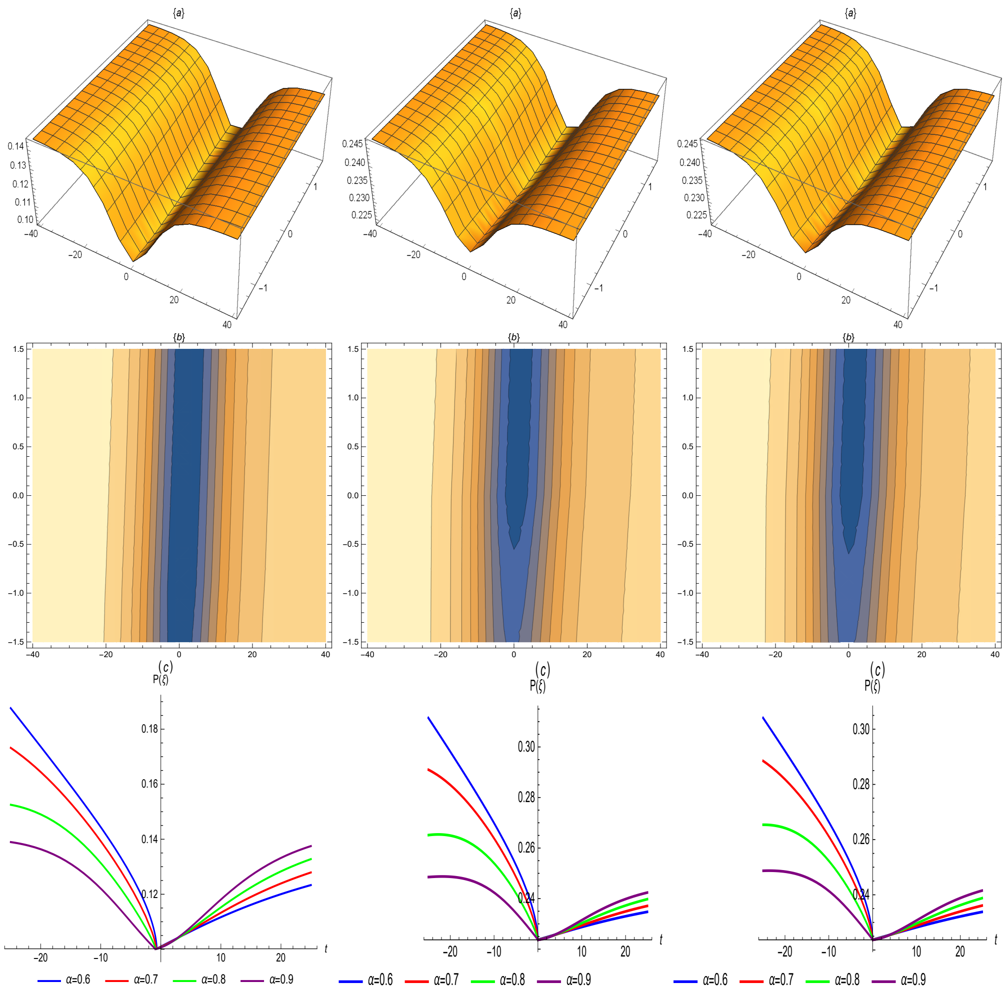

Figure 3. (a) represents the three dimensional plot for all three derivatives (b) represents the contour plot for all three derivatives (c) represents the parametric plot for all three derivatives that are beta, conformable ,and M-truncated fractional derivatives respectively for different values of α. Parameters: $c = 0.02$, $t = 0.1$, $\beta = 0.6$, $\sigma = 0.5$.

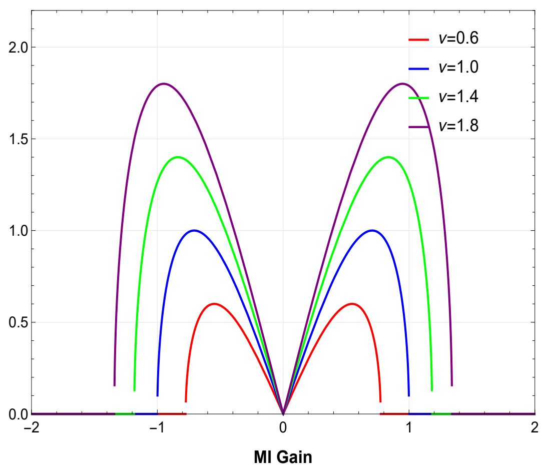

Figure 4. Gain spectrum of modulation instability (MI) for distinct values of ν.

8. Comparison

The main novelty of this work lies in the derivation of novel exact wave structures for the TFCK-OEM using the GPREA and the MAEA together with three time-fractional derivatives. Unlike previous studies, which mainly reported conventional traveling wave solutions, the present analysis provides a richer class of exact solutions, including anti-bell-shaped solitons, dark-bright solitons, and singular periodic solitons. To the best of our knowledge, the solution profiles obtained for the TFCK-OEM within the adopted fractional framework are reported here for the first time. In particular, the anti-bell-shaped solitons, dark-bright solitons, and singular periodic solitons enrich the known solution space of the model. Anti-bell-shaped solitons represent localized inverted wave profiles, while dark-bright solitons capture the coexistence of localized depressions and elevations arising from wave interactions. The singular periodic solitons reveal complex recurring wave patterns with singular characteristics, thereby broadening the physical interpretation of nonlinear wave propagation in magnetic field environments and enriching the analytical solution space of the model.

9. Conclusion

In this paper, we have studied the time-fractional coupled Konno-Oono equation model using the MAEA and GPREA with three fractional derivatives (conformable, beta, and M-truncated). We obtained several new solitary wave solutions, including periodic solitons, singular periodic shapes, dark-bright solitons, and anti-bell-shaped solutions. These solutions have not been previously reported. We have demonstrated the physical behavior of these solutions using three-dimensional surface plots, contour diagrams, and wo-dimensional line graphs. The modulation stability analysis reveals that stability depends on the normalized wave number and optical power. Both methods have proven to be simple, reliable, and efficient. Our findings are useful for applications in all-optical switches, optical amplifiers, and mode-locked lasers. In future work, we will apply these methods to other fractional nonlinear systems.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for supporting this research.

Fundings

None.

Conflict of interest

The authors declare they have no competing interests.

Authors contributions

Investigation: All authors
Methodology: Hira Tariq, Hira Ashraf, Ghazala Akram
Formal analysis: All authors
Conceptualization: Hira Tariq, Hira Ashraf, Ghazala Akram
Software: All authors
Validation: All authors
Visualization: All authors
Resources: All authors
Supervision: All authors
Project administration: All authors
Writing - original draft: Hira Tariq, Hira Ashraf, Ghazala Akram
Writing - review and editing: All authors

Availability of data

Not applicable.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

References

1. Hilfer R, ed. *Applications of Fractional Calculus in Physics.* World Scientific; 2000. Accessed March 2, 2000. https://tinyurl.com/ycxhmn25
2. Machado JT, Kiryakova V, Mainardi F. Recent history of fractional calculus. *Commun Nonlinear Sci Numer Simul.* 2011;16(3):1140-1153. https://doi.org/10.1016/j.cnsns.2010.05.027
3. Loverro A. Fractional calculus: history, definitions and applications for the engineer. Technical report. University of Notre Dame, Department of Aerospace and Mechanical Engineering; 2004:1-28. Accessed October 20, 2011. https://www.researchgate.net/profile/Joseph-Shomberg/post/What_is_the_non-local_behaviour_of_fractional_calculus/attachment/65f0b89f1d0f563db3082f0e/AS%3A11431281228985261%401710274719361/download/FracCalc-Loverro.pdf
4. Akram G, Arshed S, Sadaf M. Soliton solutions of generalized time-fractional Boussinesq-like equation via three techniques. *Chaos Solitons Fract.* 2023;173:113653. https://doi.org/10.1016/j.chaos.2023.113653
5. Zainab I, Akram G. Effect of β-derivative on time fractional Jaulent-Miodek system under modified auxiliary equation method and exp(-g(Ω))-expansion method. *Chaos Solitons Fract.* 2023;168:113147. https://doi.org/10.1016/j.chaos.2023.113147
6. Chu YM, Arshed S, Sadaf M, Akram G, Maqbool M. Solitary wave dynamics of thin-film ferroelectric material equation. *Results Phys.* 2023;45:106201. https://doi.org/10.1016/j.rinp.2022.106201
7. Kudryashov NA. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons Fract.* 2005;24(5):1217-1231. https://doi.org/10.1016/j.chaos.2004.09.109
8. Kudryashov NA. A note on the G/G-expansion method. *Appl Math Comput.* 2009;217(4):1755-1758. https://doi.org/10.1016/j.amc.2010.03.071
9. Mahak N, Akram G. The modified auxiliary equation method to investigate solutions of the perturbed nonlinear Schrödinger equation with Kerr law nonlinearity. *Optik.* 2020;207:164467. https://doi.org/10.1016/j.ijleo.2020.164467

10. Samir I, Ahmed HM. Retrieval of solitons and other wave solutions for stochastic nonlinear Schrödinger equation with non-local nonlinearity using the improved modified extended tanh-function method. *J Opt.* 2024;55(1):103-112. https://doi.org/10.1007/s12596-024-01776-3

11. Behera S. Analysis of traveling wave solutions of two space-time nonlinear fractional differential equations by the first-integral method. *Mod Phys Lett B.* 2024;38(04):2350247. https://doi.org/10.1142/S0217984923502470

12. Akram G, Sadaf M, Arshed S, Latif R, Inc M, Alzaidi AS. Exact traveling wave solutions of (2+1)-dimensional extended Calogero-Bogoyavlenskii-Schiff equation using extended trial equation method and modified auxiliary equation method. *Opt Quantum Electron.* 2024;56(3):424. https://doi.org/10.1007/s11082-023-05900-8

13. Wang S. Novel soliton solutions of CNLSEs with Hirota bilinear method. *J Opt.* 2023;52(3):1602-1607. https://doi.org/10.1007/s12596-022-01065-x

14. Jawarneh Y, Mukhtar S, Islam S, Amadou Y. Comparative analytical study of the (2+1)-dimensional Heisenberg spin chain equation using the modified Kudryashov and unified Riccati methods. *Sci Rep.* 2026. https://doi.org/10.1038/s41598-026-52543-2

15. Hamad QS, Saleh SAM, Suandi SA, et al. A Review of Enhancing Sine Cosine Algorithm: Common Approaches for Improved Metaheuristic Algorithms. *Arch Comput Methods Eng.* 2025;32:2549-2606. https://doi.org/10.1007/s11831-024-10218-z

16. Khan MI, Farooq A, Nisar KS, Shah NA. Unveiling new exact solutions of the unstable nonlinear Schrödinger equation using the improved modified Sardar sub-equation method. *Results Phys.* 2024;59:107593. https://doi.org/10.1016/j.rinp.2024.107593

17. Wang H, Zhu M, Hong W, Wang C, Li W, Tao G, Wang Y. Network-wide traffic signal control using bilinear system modeling and adaptive optimization. *IEEE Trans Intell Transp Syst.* 2022;24(1):79-91. https://doi.org/10.1109/TITS.2022.3215537

18. Zhang H, Dong Y, Xu X, Liu Z, Liu P. A novel framework of the alternating direction method of multipliers with application to traffic assignment problem. *Transp Res C Emerg Technol.* 2024;169:104843. https://doi.org/10.1016/j.trc.2024.104843

19. Hussain A, Chahlaoui Y, Zaman FD, Parveen T, Hassan AM. The Jacobi elliptic function method and its application for the stochastic NNV system. *Alex Eng J.* 2023;81:347-359. https://doi.org/10.1016/j.aej.2023.09.017

20. Mamun AA, Lu C, Ananna SN, Uddin MM. Rational Sine-Gordon expansion method to analyze the dynamical behavior of the time-fractional phi-four and (2+1) dimensional CBS equations. *Sci Rep.* 2024;14(1):9473. https://doi.org/10.1038/s41598-024-60156-w

21. Belemechri F, Kadem A. A systematic approach to exact solutions of nonlinear wave equations via the extended fan sub-equation method. *J Comput Anal Appl.* 2025;34(4). https://doi.org/10.48047/jocaaa.2024.34.04.2

22. Leonard BV, Sooriamoorthy D. A study on transfer function to estimate the central aortic blood pressure waveform. *J Phys.* 2023;2523(01):012022. https://doi.org/10.1088/1742-6596/2523/1/012022

23. Han Y, Zhang Y. A Modified Auxiliary Method for Efficient Solutions to the (2+1)-Dimensional Variable-Coefficient Burgers' Equation. *Axioms.* 2025;14(12):882. https://doi.org/10.3390/axioms14120882

24. Akram G, Sadaf M, Zainab I. The dynamical study of Biswas-Arshed equation via modified auxiliary equation method. *Optik.* 2022;255:168614. https://doi.org/10.1016/j.ijleo.2022.168614

25. Akram G, Sadaf M, Khan MAU. Abundant optical solitons for Lakshmanan-Porsezian-Daniel model by the modified auxiliary equation method. *Optik.* 2022;251:168163. https://doi.org/10.1016/j.ijleo.2021.168163

26. Wang X, Ehsan H, Abbas M, Akram G, Sadaf M, Abdeljawad T. Analytical solitary wave solutions of a time-fractional thin-film ferroelectric material equation involving beta-derivative using modified auxiliary equation method. *Results Phys.* 2023;48:106411. https://doi.org/10.1016/j.rinp.2023.106411

27. Rezazadeh H, Korkmaz A, Eslami M, Vahidi J, Asghari R. Traveling wave solution of conformable fractional generalized reaction Duffing model by generalized projective Riccati equation method. *Opt Quantum Electron.* 2018;50:1-13. https://doi.org/10.1007/s11082-018-1416-1

28. Akram G, Arshed S, Sadaf M, Sameen F. The generalized projective Riccati equations method for solving quadratic-cubic conformable time-fractional Klien-Fock-Gordon equation. *Ain Shams Eng J.* 2022;13(4):101658. https://doi.org/10.1016/j.asej.2021.101658

29. Akram G, Sadaf M, Arshed S, Sameen F. Bright, dark, kink, singular and periodic soliton solutions of Lakshmanan-Porsezian-Daniel model by generalized projective Riccati equations method. *Optik.* 2021;241:167051. https://doi.org/10.1016/j.ijleo.2021.167051

30. Yao SW, Akram G, Sadaf M, Zainab I, Rezazadeh H, Inc M. Bright, dark, periodic and kink solitary wave solutions of evolutionary Zoomeron equation. *Results Phys.* 2022;43:106117. https://doi.org/10.1016/j.rinp.2022.106117

31. Xu S, Huang L. Analytical Solutions to a Novel Integrable System Based on Painlevé Property and Projective Riccati Equations. *Qual Theory Dyn Syst.* 2026;25:76. https://doi.org/10.1007/s12346-026-01502-3

32. Ullah MS, Amin MR, Usman T, Mahbub MAA. Soliton dynamics, overlapping phenomena, multistability, and chaotic nature of a nonlinear model in relativistic wave mechanics. *Sci Rep.* 2026. https://doi.org/10.1038/s41598-026-51442-w

33. Hossain MM, Amin MR, Akter S, Roshid MM, Ullah MS. Soliton dynamics and overlapping phenomena of a nonlinear fractional pseudo-parabolic model in fluids with stability analysis. *Mod Phys Lett B.* 2026;40(11):2650071. https://doi.org/10.1142/S0217984926500715

34. Amin MR, Ullah MS, Murtaza Talukder MG. Analytical soliton solutions and dynamic behaviors of a fractional nonlinear model in relativistic wave mechanics. *AIP Adv.* 2025;15(10). https://doi.org/10.1063/5.0299718

35. Amin MR, Hakim MA, Ullah MS. Analysis of soliton behavior and overlap phenomena in the integrable beta-fractional Akbota equation with stability evaluation of equilibrium points. *AIP Adv.* 2025;15(12). https://doi.org/10.1063/5.0308937

36. Ullah MS, Usman T, Akter M, Amin MR. Qualitative analysis, optical soliton solutions, overlapping phenomena, and image encryption of a nonlinear model arising in mathematical physics. *Mod Phys Lett B.* 2026. https://doi.org/10.1142/S0217984926501563

37. Uddin MS, Begum M, Ullah MS, Abdeljabbar A. Soliton solutions of a (2+1)-dimensional nonlinear time-fractional Bogoyavlenskii equation model. *Partial Differ Equ Appl Math.* 2023;8:100591. https://doi.org/10.1016/j.padiff.2023.100591

38. Has A, Yilmaz B, Baleanu D. On the Geometric and Physical Properties of Conformable Derivative. *Math Sci Appl E-Notes.* 2024;12(2):60-70. https://doi.org/10.36753/mathenot.1384280

39. Atangana A, Alqahtani RT. Modelling the spread of river blindness disease via the Caputo fractional derivative and the beta-derivative. *Entropy.* 2016;18(2):40. https://doi.org/10.3390/e18020040

40. Wang K. New perspective to the fractal Konopelchenko-Dubrovsky equations with M-truncated fractional derivative. *Int J Geom Methods Mod Phys.* 2023;20(05):2350072. https://doi.org/10.1142/S021988782350072X

41. Özkan A, Özkan EM, Yildirim O. On exact solutions of some space-time fractional differential equations with M-truncated derivative. *Fractal Fract.* 2023;7(3):255. https://doi.org/10.3390/fractalfract7030255

42. Elbrolosy ME, Elmandouh AA. Dynamical Behaviour of Conformable Time Fractional Coupled Konno Oono Equation in Magnetic Field. *Math Probl Eng.* 2022;2022:3157217. https://doi.org/10.1155/2022/3157217

43. Yasmin H, Aljahdaly NH, Saeed AM, Shah R. Investigating symmetric soliton solutions for the fractional coupled konno-onno system using improved versions of a novel analytical technique. *J Math.* 2023;11(12):2686. https://doi.org/10.3390/math11122686

44. Li B, Chen Y. General projective Riccati equation method and exact solutions for generalized KdV-type and KdV-Burgers-type equations with nonlinear terms of any order. *Chaos Solitons Fract.* 2004;19(4):977-984. https://doi.org/10.1016/S0960-0779(03)00250-9

Hira Tariq *is an Assistant Professor in the Department of Mathematics at Government College Women University, Sialkot, Pakistan. Since May 2021, she has also served as the Incharge of the Department of Mathematics, where she contributes to academic leadership and departmental administration. Her research focuses on the development and application of analytical techniques for solving differential equations and related mathematical models. Her areas of interest include exact solutions of differential equations, the Residual Power Series Method, spline function theory, analytical methods, and fractional calculus. Dr. Tariq has published research in these fields and continues to pursue studies addressing theoretical and applied problems in mathematics. She is actively involved in teaching undergraduate and postgraduate courses while promoting research activities within the department. Her academic work reflects a commitment to advancing mathematical knowledge through rigorous analysis and the development of efficient solution techniques for complex differential equations.*

https://orcid.org/0000-0003-4080-3628

Hira Ashraf *is working as a Lecturer in the Department of Mathematics at Government Mian Rehmat Ali Graduate College of Home Economics, Gujranwala, Pakistan. She completed her M.S. in Mathematics from Government College Women University, Sialkot, during the session 2021–2023, after earning her B.S. in*

Mathematics from the same university during the session 2017–2021. Her research interests include applied mathematics, nonlinear evolution equations, fractional differential equations, soliton theory, and mathematical modeling. She has published several research articles in peer-reviewed international journals, focusing on exact analytical solutions of nonlinear partial differential equations and their physical significance. She began her academic career as a Lecturer after completing her postgraduate studies and is actively engaged in teaching, research, and mentoring undergraduate students. Her current research focuses on analytical methods for nonlinear fractional models and the construction of novel solitary wave solutions arising in mathematical physics.

https://orcid.org/0009-0004-4795-583X

Ghazala Akram *is a Professor of Mathematics at the University of the Punjab, Lahore. She earned her M.Sc. with a gold medal (1998) and completed her Ph.D. in 2007 on spline-based solutions of boundary value problems. An active researcher in computational mathematics, she has published 245 international journal papers, including 12 impact-factor papers during her Ph.D. Her work spans differential equations, fractional calculus, solitons, and geometric design. She has supervised 10 Ph.D. and 47 M.Phil scholars, with several currently under supervision. Her research impact includes an h-index of 35, i10-index of 141, and 4554 citations. Stanford University ranked her among the top 2% scientists globally for 2024 and 2025. She received the Best Teacher Award (2014) and serves as Chief Editor of the Punjab University Journal of Mathematics.*

https://orcid.org/0000-0003-0288-9299

Hafiz Abdul Wajid *is a Professor of Applied Mathematics in the Electrical Engineering Department at the Islamic University of Madinah, where he contributes to high-quality teaching, supervision, and academic development aligned with FoE policies and ABET accreditation standards. His research spans numerical scheme construction, automated CQI systems, and sustainability-focused mathematical modeling. He works extensively on Education for Sustainable Development and the UN SDGs, particularly SDG-4, SDG-6, SDG-11, and SDG-13. With a strong publication record and interdisciplinary impact, Dr. Wajid's recent work advances smart water management, carbon-offset modeling, and environmental simulations that support national and global sustainability priorities.*

https://orcid.org/0000-0003-1690-0823

An International Journal of Optimization and Control: Theories & Applications (https://accscience.com/journal/ijocta)