An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.16, No.5, pp.1706-1720 (2026)
https://doi.org/10.36922/IJOCTA026150057

RESEARCH ARTICLE

On time optimal control problem in locally convex linear topological space

Ishrat Amin and Gargi Chakraborty*

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamilnadu, India
bhatishratamin@gmail.com, gargichakraborty.math@gmail.com

ARTICLE INFO

Article History:
Received: April 6, 2026
Revised: May 20, 2026
Accepted: June 1, 2026
Published Online: July 27, 2026

Keywords:
Controllable continuous-time system
Adjoint of locally convex linear Topological space
Polar set
Transition mapping
Pontryagin Maximum Principle

AMS Classification 2010:
49J15, 49N05, 46A03

ABSTRACT

This paper explores the minimum-time optimal control problem in a locally convex linear topological space L. To solve this problem, we assume that L is Hausdorff and that the continuous time system is controllable. The system is represented by $\sum = \{I, L, E_\pi, X, Y, \phi, \psi\}$, where $I = [t_0, t]$, X is the state space of state variables and Y is the output space. In this system $\sum$, we take the unit ball in L as the polar set E_π, which characterises the locally convex linear topological space L. By employing the concept of this unit ball and defining the continuous linear operator $T_t : E_\pi \to X$ for each fixed time $t \in I$, we first establish the existence of an optimal control $l \in E_\pi$ by proving the compactness of the attainable set $A(t)$. Next, we obtain the minimum-time optimal control by considering a decreasing sequence $\{t_n\} \to t = t(1)$ and by considering a sequence $\{l_n\}$ in E_π such that $\{l_n\} \to l$. Thus, the concept of this unit ball is a novel technique in our study. Furthermore, we approach Pontryagin's maximum principle for the existence of the minimum-time optimal control l for fixed $t \in I$ minimizing the time cost function J_t by maximizing the Hamiltonian H, considering the differential equation $\dot{u} = Au(t) + Bl(t)$ represented by the dynamical system $\sum$. Additionally, we provide a geometric interpretation for maximizing the Hamiltonian H to obtain the minimum-time optimal control l_{opt} at time $t = t_{opt} = t(1)$. Finally, we provide two examples based on Pontryagin's maximum principle related to our problem.

1. Introduction

Minimum-time optimal control focuses on guiding a dynamical system to a specified target as quickly as possible while satisfying the system equations and all admissible constraints. Unlike other optimal control problems that balance time with energy or cost considerations, the sole criterion here is minimization of the transfer time. Such formulations are particularly relevant in areas that require swift maneuvering or response, such as robotics, aerospace engineering,and automotive control systems, where timing is critical. Nevertheless, the inclusion of bounded controls and state limitations significantly increases the analytical and computational complexities of these problems. In light of this, several authors have studied minimum-time optimal control problems with a range of constraints and system configurations.

Leigh explained the minimum-time optimal control, which uses the unit ball as a norm,

$$\Omega = \{u : \|u\| \leq k\}. \tag{1}$$

*Corresponding Author

where k is the given real number and is concerned with bringing a system to a desired condition in the shortest possible time.

The theory of optimal control depends on an attainable set and its characteristics. The author used the Lyapunov theorem for the range of vector measures and discussed the Pontryagin Maximum Principle.[1] Henry Hermes and Joseph P. Lasalle discussed the general linear time optimal control problem by considering the vector differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

where A is $n \times n$ matrix, B is $n \times r$ matrix. They used the unit ball as

$$\Omega^0[0,t] = \{u; \text{u is measurable}, |u_j(\tau)| = 1, j = 1, \ldots r, \tau \in [0,t]\}. \tag{2}$$

In this theory of time-optimal control, the authors established a reachable set together with its crucial properties, thereby proving the Bang-Bang Principle for the derivation of time optimal control and its uniqueness.[2] Chakraborty solved the optimal control problem using functional analysis for the existence of optimal control in locally convex linear topological spaces governed by linear plants with extensive restrictions on the control variables. The necessary and sufficient conditions were also derived to obtain a clear statement of optimal control.[3] Pontryagin *et al.* introduced the Pontryagin Maximum Principle as the most effective tool for solving optimal control problems. This principle became a paradigm in optimal control theory, and its remarkable properties have been studied from many perspectives.[4,5] Yosida presented lectures at the University of Tokyo on set theory, topological spaces, measure spaces and linear spaces.[6] Jerzy Zabczyk wrote Mathematical Control Theory, which covers advanced concepts such as linear algebra, differential equations, and calculus. Furthermore, the book covers stabilization of nonlinear systems using topological methods, realization theory for nonlinear systems, impulsive control and positive systems, control of rigid bodies, stabilization of infinite dimensional systems, and solution of the minimum energy problem. In addition, this book by Dong, D. E. devotes several chapters to controllability with vanishing energy, boundary control systems, and delayed systems, and it provides additional proofs, theorems, findings, and a more comprehensive index.[7] Prasad and Kumar contributed to the knowledge and development of topological dynamical systems. In addition, the authors explained how variational calculus served as the main analytical tool in the classical study of dynamical system. This approach, based on Hamilton's Principle of Least Action, led to what are now referred to as topologies compatible with the dynamical structure of the system. Finally, the study expanded this investigation to topological dynamical systems in the more general setting oftopological vector spaces, with particular emphasis on locally compact groups and locally convex spaces.[8] The convexity and compactness of attainable sets were explained by Pugh and Hermes, who also provided proofs of the related theorems.[9,10]

Dong *et al.* addressed these limitations by formulating the time-optimal ergodic search problem, aiming to generate fast, adaptive exploration trajectories using Pontryagin's Maximum Principle and directed transcription optimization. The study demonstrated effective minimum time coverage in both simulations and drone experiments. It also integrated constraints such as obstacle avoidance and showcased flexibility across scales. A receding horizon framework further enabled adaptive sampling for tasks such as localization.[11] In Linear Topological Spaces, Kelley *et al.* made a significant contribution by presenting a systematic and rigorous treatment of topological vector spaces. They developed foundational concepts connecting topology and linear algebra, clarified the structure and properties of locally convex spaces, and provided essential tools for understanding continuous linear mappings in modern analysis.[12] Ghosh and Chakraborty contributed to the field of control theory by using mathematical methods to examine robust stability and robustly strictly positive real (SPR) properties with uncertainties for an interval plant family. In their study, they employed a novel technique by constructing a Kharitonov rectangle as a Kharitonov box at a certain frequency. Using this approach, they provided insights into the robust stability and robustly strictly positive real (SPR) properties of an interval plant family in real-world uncertainty scenarios.[13–15]

Recent studies have extensively investigated controllability and optimal control problems for semilinear systems in Hilbert spaces, including second-order, stochastic, retarded, and fractional-order models. Using various fixed point techniques and controllability methods, several authors established existence, uniqueness, approximate controllability, and optimal control results under suitable assumptions.These contributions provide a significant foundation for the development of controllability theory for nonlinear evolution systems and are useful in the preparation of the present work.[16–20]

The works in optimal control theory concerning the existence of optimal controls for infinite-dimensional semilinear systems with admissible controls in nonreflexive spaces are also useful and relevant to the present study.[21] Similarly, studies on infinite-dimensional fractional evolution equations with endpoint state constraints and Pontryagin-type optimality conditions were also relevant to the present work. Their treatment of variational analysis and fractional control systems motivated the development of our results.[22]

In addition, several important contributions related to optimal control theory, Pontryagin-type maximum principles, fractional differential systems, variational analysis, and infinite-dimensional control problems have also been reported in the literature. In particular, the studies on impulsive and hybrid systems, minimal time control problems, mean field and probabilistic control models, convex analytical approaches, and fractional evolution equations provided useful mathematical background and motivation for the formulation and analysis carried out in the present work.[23–37] Moreover, illustrative Example 2 in Section 8.1 is developed by employing certain concepts and analytical techniques from the theory of fractional differential equations presented in.[37–40]

1.1. Comparision

Previous studies on time-optimal control by Leigh[1] and Hermes and LaSalle[2] considered norm-induced unit balls in (1) and (2), respectively. In the present work, the time-optimal control problem is formulated in a locally convex linear topological space L by considering the polar set E_π as the unit ball in (4). The controllable continuous-time system is given by

$$\sum = \{I, L, E_\pi, X, Y, \phi, \psi\}, \tag{3}$$

where $I = [t_0, t]$, X is the state space, Y is the output space, and for each fixed $t \in I$, the state transition mapping is represented by the continuous linear operator $T_t : E_\pi \to X$ in (5).

Chakraborty[3] applied functional analytic techniques to optimal control problems with restricted controls. Motivated by this framework, the present article establishes the existence of time-optimal controls using the polar set E_π and derives the corresponding optimality conditions through maximizing the Hamiltonian H via the Pontryagin Maximum Principle. The foundational role of the Pontryagin Maximum Principle in optimal control theory was established by Pontryagin *et al.*[4,5]

Topological methods developed by Yosida,[6] Zabczyk,[7] and Kelley *et al.*[12] provide the analytical framework for the present study in locally convex spaces. Furthermore, while Pugh[9] and Hermes[10] discussed convexity and compactness properties of attainable sets, the present work proves these properties for the attainable set $A(t)$ using the polar set E_π in the locally convex setting. Related developments on dynamical systems and robustness properties were also studied in,[8,13] whereas the present work focuses on minimum-time optimal control using the polar set E_π in L.

In contrast to recent studies such as Dong *et al.*,[11] which focused on minimum-time coverage conditions, the present work characterizes the optimal trajectory via the boundary of the attainable set $A(t)$ and relates it to the maximization of the instantaneous projection onto the normal vector $\psi(t)$ to the hyperplane Ω.

1.2. Advantage

By using the polar of the unit ball, the entire space can be characterized. In industrial applicaton, the shape of the unit ball may be expanded or contracted as needed. In a medical context, the affected region expands most rapidly along the outward normal at an extreme point of the unit ball, since any pathogen progresses most rapidly through that direction, as shown in Figure 1 and 2. For this reason, if that point cannot be controlled by medication, vaccination, or other techniques, the pathogen will be unable to spread through either the extreme point or the boundary of the affected region. Thus, the findings of this study, particularly the properties of the unit ball, may have applications in both industrial processes and medicine. In addition, by employing geometric concepts of the unit ball such as polar sets, nets, and Kharitonov-style rectangular or cuboid regions, it is possible to develop control strategies for time-delay multi-agent systems. These techniques can be utilized for various system types, including continuous-time controllable models, linear differential systems, nonlinear and distributed systems, dynamical systems, normal and proper systems, uncertain-interval plants and other appropriate system structures.

1.3. Contribution

In this paper, we present the time-optimal control problem for a controllable continuous time system $\sum$ (3). In this system, we take the unit ball to be the polar set E_π in L of a bounded subset E of L^*, defined as

$$E_\pi = \{l \in L : sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle| \leq 1; f \in E \subset L^*\}\}, \quad (4)$$

where L is a Hausdorff locally convex linear topological space and L^* is the set of all continuous linear functionals defined on L.[3] The unit ball E_π in (4) is a useful technique in our problem and is also referred to as the set of admissible controls. For each fixed time $t \in I$ we define the continuous linear operator as

$$T_t : E_\pi \to X \quad (5)$$

representing the state transition mapping ϕ for fixed t_0 and $u(t_0)$, and we obtain the existence of a minimum-time optimal control $l \in E_\pi$ at time $t = t(1)$, $t \in I$ such that

(1) $u(t(1)) = v(t(1))$
(2) $t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$.

To prove conditions 1 and 2, we assume L is reflexive and then use the compactness of the attainable set $A(t)$ given as

$$A(t) = \{u(t) : u(t) = T_t(l), l \in E_\pi\}, \quad (6)$$

where $A(t)$ is the resulting output space Y.

In addition, we consider a decreasing sequence $\{t_n\} \to t = t(1)$ and a sequence $\{l_n\} \to l$ in E_π such that $T_{t_n}(l_n) = v(t_n)$. Since L is reflexive, the polar set E_π in L is compact by Remark 1.

Next, we apply Pontryagin's Maximum Principle, which is a necessary condition for the existence of the minimum-time optimal control l at $t = t(1)$, by representing the system Σ in (3) by a differential equation,

$$\dot{u} = Au(t) + Bl(t), \quad (7)$$

where A is an $n \times n$ matrix and B is an $n \times r$ matrix. Then, using the concept of the unit ball as a polar set E_π in L, we minimize the time cost function J_t with respect to the optimal control $l \in E_\pi$ at the minimum time $t = t(1)$. The functional is defined via conditions (1) and (2), given by

$$J_t(l) = \sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle|\}, \quad (8)$$

and the minimization is achieved by maximizing the Hamiltonian

$$H = \langle \psi, \dot{u}\rangle, \quad (9)$$

where ψ is an outward normal on H to $\partial A(t)$, the boundary of the attainable set $A(t)$. To gain further geometric insight into the problem, we provide a geometric interpretation of the maximization of the Hamiltonian $H = c$, where c is a constant. Thus we have shown that $l_{opt}(t)$ is the time-optimal control, such that $u(t\,; l_{opt}) \in \partial A(t)$, $\forall t \in [t_0, t_{opt}]$, where $t = t_{opt} = t(1)$. Finally, we provide two examples based on Pontryagin's Maximum Principle related to our problem.

2. Preliminaries

2.1. Inner product space

A Euclidean vector space is the real n-dimensional space $\mathbf{R^n}$ equipped with the standard dot product. Given two vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

with the inner product[2] defined as

$$\begin{aligned}\langle x, y\rangle &= x^\top y \\ &= \sum_{i=1}^{n} x_i y_i \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\end{aligned}$$

2.2. Bounded set

A subset A of a linear topological space is said to be bounded if for each neighborhood U of the origin, there exists a real number α such that A is contained inside αU.[12]

2.3. Barrelled space

In a locally convex linear topological space L, any convex, balanced, and absorbing closed set is called a barrel. L is called a barrelled space if each of its barrels is a neighborhood of zero.[6]

2.4. Theorem

A locally convex linear topological space L is reflexive if and only if it is barrelled and every closed, convex, balanced and bounded set of L is compact in the weak topology of L.[6]

2.5. Characterizations of reflexive space

A locally convex Hausdorff space is reflexive if and only if it is semi-reflexive (bounded weakly closed sets are weakly compact) and evaluable (strongly bounded subsets of the adjoint are equicontinuous).[12]

2.6. Krein–Milman theorem

A non-void compact subset K of a locally convex linear topological space has at least one extreme point.[6]

2.7. Extreme point

In a linear topological space, a point x of a convex set A is an extreme point of A if and only if x is not an interior point of any line segment whose end points are in A.[12]

2.8. Adjoint of a locally convex linear topological space

In a locally convex linear topological space L, the adjoint space, denoted by L^*, comprises all continuous linear functionals defined on L.[12]

2.9. Boundary and interior of polar set ($\boldsymbol{E_\pi}$)

Let L be a locally convex linear topological space. The polar set E_π for every set $E \subseteq L^*$ is then defined in (4).

From (4), the boundary of the polar set E_π is represented by ∂E_π as follows:

$\partial E_\pi = \{l \in L : sup_{l \in E_\pi \subset L}\{|\langle l, f\rangle|, f \in E \subset L^*\} = 1\}$,

and interior of the polar set E_π is represented by $\text{Int}E_\pi$ as

$\text{Int}E_\pi = \{l \in L : sup_{l \in E_\pi \subset L}\{|\langle l, f|\rangle, f \in E \subset L^*\} < 1\}$.

Remark 1. *The polar set E_π in (4) is a neighborhood of the origin in a Hausdorff locally convex linear topological space L if E is a bounded subset of L^*. In addition, E_π is absorbing, balanced, bounded and convex in L. Since E_π is a barrel, by Section 2.3, E_π is closed in the weak topology of L and hence closed in the strong topology of L. Moreover, if L is reflexive, then E_π is compact by Section 2.4.*

Remark 2. *The locally convex linear topological space L is characterized by the polar set E_π.*

2.10. Extremal

For each nonzero $f \in L^*$, if there exists at least one element $l \in E_\pi \subset L$ such that $f(l) = |\langle l, f\rangle|$ and $sup_{l \in E_\pi \subset L}\{|\langle l, f\rangle|, f \in E \subset L^*\} = 1$, then this element $l \in E_\pi$ is called an extremal of $f \in L^*$.[12]

2.11. Bang-Bang control

Consider the time optimal control for the system $\sum(3)$ where $E_\pi = \{l \in L : sup_{l \in E_\pi \subset L}\{|\langle l, f\rangle| \leq 1; f \in E \subset L^*\}\}$ by (4). In control theory, a control $l \in E_\pi$ for almost all $t \in I = [t_0, t]$ is considered as a bang-bang if it satisfies the condition $sup_{l \in E_\pi \subset L}|\langle l, f\rangle| = 1$.[1]

Remark 3. *The polar set E_π in (4) is referred to as the set of admissible inputs and the attainable set $A(t) \subset X$ where X is the state space of all state variables. In addition, $A(t)$ is the output space Y.*

Remark 4. *Thus the system $\sum$ (3) is controlled. Hence, according to (6), there must be some $t \in I$ for which $v(t) \in A(t)$.*

3. Problem statement

Let $\sum$ be a controllable continuous-time system $\sum = \{I, L, E_\pi, X, Y, \varphi, \psi\}$ where L is a Hausdorff locally convex linear topological space, $I = [t_0, t]$ is the time interval, X is the state space, and Y is the output space. Let the polar set E_π of bounded subset $E \subset L^*$ in L be $E_\pi = \{l \in L : sup_{l \in E_\pi \subset L}\{|\langle l, f\rangle| \leq 1; f \in E \subset L^*\}\}$ where L^* is the dual space of all continuous linear functionals defined on L. To determine the minimum-time optimal control $l \in E_\pi$ at the time $t = t(1)$ with $t \in I$, given an initial state $u(t_0) \in X$ and a continuous function $v(t) \in X$ defined on I, we consider the following conditions:

(1) $u(t(1)) = v(t(1))$.
(2) $t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$.

Here, $A(t) \subset X$ is the attainable set and Y is the output space, where X is the state space of all state variables. For each fixed $t \in I$, the operator $T_t : E_\pi \to X$ is a continuous linear operator which is onto by Remark 7 and satisfies $u(t) = T_t(l)$ with $l = T_t^{-1}(u(t))$, for $u(t) \in A(t)$ as given by (5) and (6). Thus, we ensure that $\sup_{l \in E_\pi \subset L}\{|\langle l, f\rangle|\}$ serves as the time cost functional J_t, which is to be minimized by maximizing the Hamiltonian H at the minimum time $t = t(1)$.

To solve this problem, we exploit the reflexivity of L and determine the minimum-time optimal control using the following theorems:

Theorem 1. *The attainable set is convex and compact.*

Proof. E_π is convex by Remark 1. By the definition of the attainable set $A(t)$ in (6), $T_t : E_\pi \to X$ in (5) is a continuous linear operator, and linear operators preserve convexity. Then in Remark 3, the attainable set $A(t) \subset X$ is convex. In our problem we consider L is Hausdorff and reflexive. Moreover, E_π is a polar set of a bounded subset E of L^* in L. Then, by Remark 1, E_π is compact. Since a continuous linear operator T_t preserves compactness, the attainable set $A(t)$ is compact. □

Theorem 2. *Let $u(t) \in \partial A(t)$ be a point on the boundary of attainable set $A(t)$. Then, $T_t^{-1}(u(t))$ has a minimum element for each fixed $t \in I$ if and only if $u(t) \in A(t)$.*

Proof. Let $u(t) \in \partial A(t) \cap A(t)$. By the definition of the attainable set $A(t)$ in (6), we have $u(t) = T_t(l)$. Subsequently, $T_t^{-1}(u(t))$ contains an element l in E_π for each fixed $t \in I$. Since $u(t) \in \partial A(t)$ and T_t is onto by Remark 7, this element l of E_π must satisfy $sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle|\} = 1$ which implies $l \in \partial E\pi$ for each fixed $t \in I$.

Conversely, we consider that l to be the minimum element of $T_t^{-1}(u(t))$ for each fixed $t \in I$. Because T_t is a linear operator, it follows that αl is the minimum element of $T_t^{-1}(\alpha u(t))$ for any scalar $\alpha > 0$.

Now if $\alpha < 1$, then $\alpha u(t) \in A(t)$ and so $\alpha u(t)$ has a pre-image which satisfy

$$sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle|\} \leq 1,$$

in Section 2, using Definition 2.9.

As α is arbitrary, it follows that $sup_{l\in E\pi\subset L}\{|\langle l, f\rangle|\} \leq 1$. However, $u(t) \in \partial A(t)$. Then $l \in \partial E_\pi$ satisfying $sup_{l\in E_\pi\subset L}\{|\langle l, f\rangle|\} = 1$ and $u(t) = T_t(l) \in A(t)$. □

Remark 5. *By Remark 3, $A(t) \subset X$. From Theorem 2, we can conclude that for each fixed time $t \in I$ and each $u(t) \in X$, $T_t^{-1}(u(t))$ has a minimum element l if and only if $A(t)$ is closed.*

Remark 6. *In our problem, we assume that L is a Hausdorff locally convex linear topological space and that L is reflexive. Then by Section 2.5, L is semi-reflexive and bounded weakly closed sets in L are weakly compact. Consequently, the unit ball E_π , viewed as a polar set in L, is weakly compact. Hence, $T_t^{-1}(u(t))$ has a minimum element l for each $u(t) \in X$ and each fixed $t \in I$.*

Theorem 3. There exists at least one time-optimal control satisfying the following conditions:
(1) $u(t(1)) = v(t(1))$.
(2) $t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$.

Proof. In Section 3, we consider the controllable continuous-time system $\sum = \{I, L, E_\pi, X, Y, \varphi, \psi\}$ (3) and assume that L is reflexive. Then, from the controllability assumption, we have $v(t) \in A(t)$ for some $t \in I$ according to Remark 4. From the problem statement, $v(t) \in A(t)$ for $t > t(1)$ where

$$t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}.$$

By Remark 1, E_π is compact and hence E_π is closed. Now, let $\{t_n\}$ be a decreasing sequence in I such that $\{t_n\} \to t = t(1)$, and let $\{l_n\}$ be a sequence in E_π such that $\{l_n\} \to l$ satisfies

$$T_{t_n}(l_n) = v(t_n), \tag{10}$$

by the definition of the attainable set $A(t)$ in (6). In addition, by (6), we can write

$$T_{t_n}(l_n) = u(t_n). \tag{11}$$

Now from (10) and (11), we have

$$v(t_n) = u(t_n), \tag{12}$$

for some $t_n \in I$. Then, by the triangle inequality,

$$\begin{aligned} |v(t(1)) - u(t(1), l_n)| &= |v(t(1)) - v(t_n) + v(t_n) - u(t(1), l_n)| \\ &\leq |v(t(1)) - v(t_n)| + |v(t_n) - u(t(1), l_n)| \\ &= |v(t(1)) - v(t_n)| + |u(t(1), l_n) - v(t_n)|, \end{aligned} \tag{13}$$

and by the assumption $\{t_n\} \to t(1)$, we can write as

$$\begin{aligned} &u(t(1), l_n) \leq u(t_n), \\ &\implies T_{t(1)}(l_n) \leq T_{t_n}(l_n), \text{ using (11).} \end{aligned}$$

Thus, we have

$$u(t(1), l_n) - u(t_n) \leq T_{t(1)}(l_n) - T_{t_n}(l_n). \tag{14}$$

From (13), using (12) and (14), we obtain

$$\begin{aligned} |v(t(1)) - u(t(1), l_n)| \leq\ & |v(t(1)) - v(t_n)| \\ & + |T_{t(1)}(l_n) - T_{t_n}(l_n)|. \end{aligned} \tag{15}$$

In Section 3, from the problem statement and the definition of the attainable set in (6), both v and T_t are continuous. Then we have

$$\begin{aligned} &v(t_n) \to v(t(1)) \\ &\implies |v(t_n) - v(t(1))| = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &\text{and } T_{t_n}(l_n) \to T_{t(1)}(l_n) \\ &\implies |T_{t_n}(l_n) - T_{t(1)}(l_n)| = 0. \end{aligned} \tag{17}$$

Then, from (15), using (16) and (17) we can write

$$\begin{aligned} &|v(t(1)) - u(t(1), l_n)| = 0 \\ &\implies v(t(1)) = u(t(1)). \end{aligned} \tag{18}$$

where $t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$ is the minimum time and $\{l_n\} \to l$ is the optimal control. Thus, it is clear that there exists at least one optimal control $l \in E_\pi$ at the minimum time $t = t(1)$ that satisfies conditions 1 and 2 in our problem. □

Remark 7. *For a controllable continuous time system $\sum$, the continuous linear operator T_t : $E_\pi \to X$ in (5) is onto by [,[1] Theorem 7.9], and we define the time-optimal control as $l = l(t)$ at the time $t \in I$. Thus, using (6) and (5), we can write the minimum-time optimal control at $t = t(1)$ as*

$$l(t(1)) = T_t^{-1}(u(t(1))). \tag{19}$$

Remark 8. *From Theorem 3, it follows that an optimal control exists. Then by,[1] there exists a bang-bang optimal control satisfying $sup_{l\in E_\pi\subset L}\{|\langle l, f\rangle|, f \in E \subset L^* = 1\}$.*

Remark 9. *According to Theorem 3, L is reflexive and by Subsections 2.4 and 2.5, the polar set E_π in L is compact. Then, by the Krein–Milman theorem, (Section 2.6), $l \in E_\pi$ is an extreme point and is unique. Thus by Subsection 2.11, it follows that l is necessarily a bang-bang control on the time interval $I \in [t_0, t(1)]$ and is of the form*

$$l(t) = k\,\mathrm{sgn}(f(t)).$$

4. Application of Pontryagin Maximum Principle: Minimize the cost function J using the maximization of Hamiltonian and the form of optimal control

Let us consider

$$\dot{u} = Au(t) + Bl(t),\ u(t) \in X \text{ and } l \in E_\pi, t \in I$$

be the form of differential equation in (7), which describes the dynamical system $\sum$(3). In this setting, L is the locally convex linear topological space, $I = [t_0, t]$ is the time interval and E_π is the polar set in L. we can then determine at least one optimal control l in minimum time t by minimizing the cost function $J_t(l) = \sup_{l \in E_\pi \subset L} \{|\langle l, f\rangle|\}$ in (8), while the solution of the differential equation (26) satisfies $u(t) = T_t(l)$ via (5) and (6) in Remark 11.

Now we define an additional adjoint variable ψ and costate variable $\dot{\psi}$ such that

$$\dot{\psi} = -\partial H / \partial u, \tag{20}$$

where $\psi \in L^*$, L^* is the adjoint space of the locally convex linear topological space L, and H represents the Hamiltonian of the system given by $H = \langle \psi, \dot{u}\rangle$ in (9).

Now, we determine the optimal control l_{opt} in minimum time $t \in I$ using (9) and (7) to minimize (8), and we have

$$H = \langle \psi, Au(t) + Bl(t)\rangle. \tag{21}$$

Thus by (20) and (21), we have

$$\begin{aligned} &\dot{\psi} = -\frac{\partial}{\partial u}\langle \psi, Au(t) + Bl(t)\rangle \\ \Longrightarrow\ &\dot{\psi} = -\langle \psi, A\rangle \\ \Longrightarrow\ &\dot{\psi} = -\psi^\top A, \quad \text{by the definition 2.1} \\ \Longrightarrow\ &\psi(t) = ce^{-At}, \quad \text{c is a constant} \\ &\psi(t) = \psi(0)Q(t), \end{aligned} \tag{22}$$

where $\psi(0) = c$ at time $t = 0$.

Remark 10. *By the property of the transition matrix $Q(t)$, we have $e^{-At} = Q(t)$.*

By the Pontryagin Maximum Principle for the time-optimal control problem, we maximize the Hamiltonian H given in (21) to minimize the cost functional $J_t(l) = \sup_{l(t) \in E_\pi \subset L} \{|\langle l, f\rangle|\}$ in (8).

Thus, we define

$$H(\psi, u(t), t, l_{opt}) = P \quad \text{for almost all } t,$$

and then by (21), we can write

$$H = \langle \psi, Au(t) + Bl(t)\rangle = \langle \psi, Au(t)\rangle + \sup_{l \in E_\pi \subset L} \langle \psi, Bl(t)\rangle.$$

Now we have

$$P = \langle \psi, Au(t)\rangle + \sup_{l \in E_\pi \subset L} \langle \psi, Bl(t)\rangle, \tag{23}$$

to maximize $\langle \psi, Bl(t)\rangle$ at all times where l_{opt} is chosen. Based on the compactness of the attainable set $A(t)$ established in Theorem 1, and the existence of a minimum-time optimal control in Theorem 3, where $T_t : E_\pi \to X$ is the continuous linear operator that is onto by Remark 7 , we obtain an optimal control $l \in E_\pi$ satisfying $u(t(1)) = v(t(1))$ where $t = t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$. Thus the control function $l(t)$ serves as a minimum-time optimal control if it maximizes the Hamiltonian H at almost all points on the boundary $\partial A(t(1))$. Moreover, since L is reflexive in our problem, it follows from Remarks 7, 8, and 9 that $l(t)$ is an extreme point. Thus, we can write the control function $l(t)$ as

$$l_{opt}(t) = \mathrm{sgn}(\psi(t)), \tag{24}$$

where $l_{opt} \in \partial E_\pi$ satifies $sup_{l \in E_\pi \subset L}\{|\langle l_{opt}, f\rangle|\} = 1$ at the minimum time $t = t(1) = t_{opt}$ and by (22), $\psi(t) = \psi(0)Q(t)$. Setting $\psi(0) = \lambda$ with λ an arbitrary constant, we can rewrite (22) as $\psi(t) = \lambda Q(t)$, which leads to (24) in the form of the optimal control

$$l_{opt}(t) = \mathrm{sgn}(\lambda Q(t)), \tag{25}$$

at minimum time $t = t_{opt}$.

Remark 11. *In this paper, L is the locally convex linear topological space and L^* is its adjoint space. The set E_π is the polar set in L of a bounded set $E \subset L^*$ in Section 1.3, (4), and $I = [t_0, t]$ is the time interval in Section 1.1. For each fixed $t \in I$, $T_t : E_\pi \to X$ is a continuous linear operator, where X is the the state space of state variables $u(t)$, as in Section 1.1. To prove Theorem 1, we assume that L is Hausdorff and reflexive. Then, by Remarks* 1 *and 2 , the polar set E_π is compact in L . Thus, $T_t(E_\pi)$ is compact and hence the attainable set $A(t) \subset X$ is compact. In our problem, we consider $\sum$ as a controllable continuous dynamical time system , where X is infinite-dimensional. According to*[1]*, the controllability concepts for infinite-dimensional systems coincide with those for finite-dimensional systems. Consequently, the attainable set $A(t) \subset X$ is compact in this infinite dimensional space under a continuous linear operator T_t.*

Again, using Pontryagin's maximum principle, we consider the differential equation $\dot{u}(t) = Au(t) + Bl(t)$ in (7) *from Section 1.3 which is represented by the system $\sum$, where $l \in E_\pi$ for some time $t \in I$ and $u(t) \in A(t) \subset X$. The solution of* *the differential equation can be written as*

$$u(t) = u(t_0)\Phi(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$
$$\Longrightarrow u(t) - u(t_0)\Phi(t,t_0) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \quad (26)$$

using integrating factor $e^{\int A(t)dt}$ *and consider the transition matrix* $\Phi(t,t_0) = e^{A(t)-A(t_0)} = \Phi(t-t_0)$.

To investigate problems of optimal transfer from some initial state to a final state it is sufficient to study the problem $u(t) = T_t(l)$ *for each fixed* $t \in I$, *which represents* (26). *With this concept, we use the matrix form* $\dot{u}(t) = Au(t) + Bl(t)$ *in the system of Hamiltonian by Equation* (9) *as*

$$H = \langle \psi, \dot{u} \rangle$$
$$= \langle \psi, Au(t) + Bl(t) \rangle \text{ by Equation (21)}$$

to obtain the optimal control l *in minimum time* t *while minimizing the time cost function as* $J_t(l) = \sup_{l\in E_\pi}\{|\langle l, f\rangle|\}$ *in* (8) *and* $P = \langle \psi, Au(t)\rangle + \sup_l \langle \psi, Bl(t)\rangle$ *by* (23).

Remark 12. *In Section 4,* $u(t\ ;\ l_{opt})$ *is on the boundary of* $A(t)$ *for each* $l = l_{opt}$ *where* $l_{opt} = T_t^{-1}(u(t)) \in \partial E_\pi$ *as* T_t *is onto mapping using Remark 7, and for each* $t \in [t_0, t(1)]$, $t = t_{opt} = t(1)$.

Remark 13. *Pontryagin's maximum principle in Section 4 states that* J_t *in (8) is minimized when* l *is chosen at all times to maximize the Hamiltonian in (21).*

5. Geometric interpretation

A hyperplane is represented by the equation $H = \langle \psi, \dot{u} \rangle = c$, where c is a constant and H is the Hamiltonian. Let this hyperplane be denoted by $\mathbf{\Omega}(t)$; in this case, $\psi(t)$ is normal to Ω for all t. It is evident that choosing l in a way that maximizes the projection of the vector $\dot{u}(t)$ onto $\psi(t)$ is geometrically equivalent to maximizing the Hamiltonian. In Remark 12, l_{opt} is the time-optimal control and $u(t\ ;\ l_{opt}) \in \partial A(t)$ for every $t \in [t_0, t_{opt}]$ as shown in **Figure 1**. Assuming Ω is the hyperplane passing through $u(t_{opt})$, Ω supports $A(t)$. Its normal is $\psi(t)$, and the projection of the motion $\dot{u}$ is given by the Gram-Schmidt process:

$$m = \langle \dot{u}, \psi(t)/sup_{l\in E_\pi \subset L}|\langle l, \psi(t)\rangle|^2 \rangle \psi(t). \quad (27)$$

Thus, the normal $\psi(t)$ onto the hyperplane Ω that lies on the boundary of $A(t)$, is maximum, and the optimal trajectory passes through the boundary of the attainable set $A(t)$ to maximize the instantaneous projection of motion onto the normal $\psi(t)$ as shown in **Figure 2**.

6. Figures for geometric interpretations

In **Figure 3**, E_π is the polar set as a unit ball characterizing the locally convex linear topological space L by Remark 2 associated with the attainable set $A(t)$ which is the output space Y by Remark 3. The time-optimal control $l(t)$ lying on the boundary of E_π produces the trajectory $u(t) = T_t(l)$ which in turn reaches the boundary of $A(t)$ at a minimum time $t = t(1)$. A supporting hyperplane Ω of the attainable set $A(t)$ with outward normal ψ at $u(t(1))$ such that $u(t(1)) \in \partial A(t) \cap \Omega$ indicates the extremal direction of the optimal trajectory. Because $T_t : E_\pi \to X$ is a continuous linear operator and is onto by Remark 7, then the optimal control satisfies $l(t(1)) \in \partial E_\pi$ such that $l(t(1)) = T_t^{-1}(u(t(1)))$.

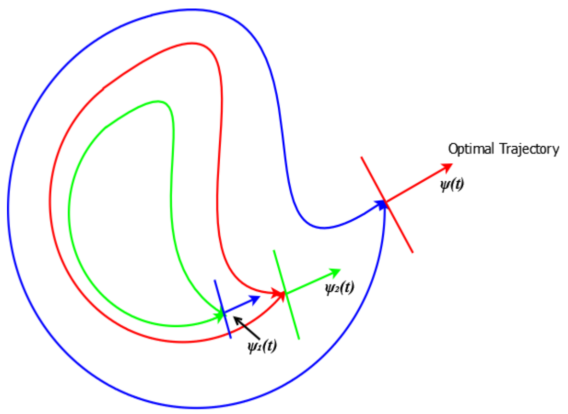

Figure 1. Geometric representation of the time optimal trajectory corresponding to the attainable set $A(t)$ associated with the control system.

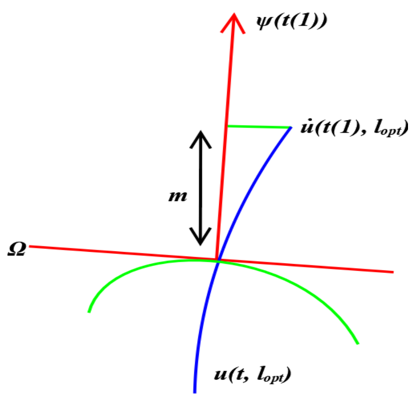

Figure 2. Relationship among the time-optimal trajectory, the boundary of the attainable set $A(t)$, and the adjoint variable $\psi(t)$ with the projection m in the equation (27) for motion $\dot{u}(t(1))$ where Ω is the supporting hyperplane of the attainable set $A(t)$ passing through $u(t(1))$ and whose normal is $\psi(t)$.

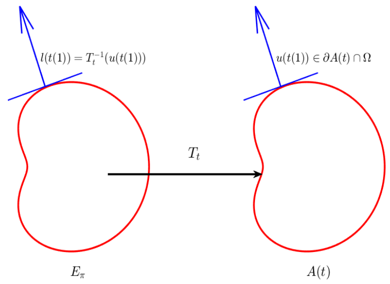

Figure 3. Geometric interpretation of the output $u(t(1))$ in the attainable set $A(t)$ and the associated optimal control $l \in E_\pi$ at $t = t(1)$, where $T_t : E_\pi \to X$ is a continuous linear operator.

.

7. Example 1

Determine the form of the minimum-time optimal control for the following systems:

$$\dot{u}_1 = u_2,$$
$$\dot{u}_2 = -u_1 + l,$$

with $u(t_0) = 0$ and $\sup_{l \in E_\pi \subset L} \{|\langle l, f\rangle|\} \leq 1$ by the method of Pontryagin's maximum principle.

Now to determine the form of the time-optimal control for the given systems we have

$$\begin{aligned}\dot{u} &= Au(t) + Bl(t)\\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} l, \qquad (28)\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and the Hamiltonian H using (28), is given by

$$H = \langle \psi, \dot{u} \rangle$$
$$H = \langle\ \psi, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} l\ \rangle,$$

and the adjoint variable $\dot{\psi}$ is given by

$$\begin{aligned}\dot{\psi} &= -\partial H/\partial u, \ \text{ by (20)}\\
\implies \frac{d\psi}{dt} &= -\langle \psi, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle\\
\implies \frac{d\psi}{dt} &= -\psi^\top \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{by the definition 2.1}\\
\implies \psi^\top(t) &= ce^{-\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t\right]}\\
&= \psi(0)e^{-\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t\right]} \qquad (29)\end{aligned}$$

where $c = \psi(0)$ at time $t = 0$. By (23) in Section 4, we have

$$P = \langle \psi, Au(t)\rangle + sup_{l \in E_\pi \subset L} \langle \psi, Bl(t)\rangle$$
$$\implies P = \langle \psi, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rangle + sup_{l \in E_\pi \subset L} \langle \psi, \begin{bmatrix} 0 \\ 1 \end{bmatrix} l \rangle$$

so that l_{opt} must maximize $\langle \psi, Bl(t)\rangle$ at all times. Also, from (25) we have

$$l_{opt}(t) = sgn(\lambda Q(t)).$$

Now,

$$\begin{aligned}\langle \psi, Bl(t)\rangle &= \psi^\top Bl(t) \quad \text{by the definition 2.1}\\
&= \psi(0)e^{-\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t\right]} Bl\ , \quad \text{by (29)}\\
&= \psi(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} l - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t\psi(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} l + \frac{1}{2!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 t^2\psi(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} l + \cdots\\
&+ \frac{1}{k^n!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n t^n\psi(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} l\\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \psi(0)l - \begin{bmatrix} 1 \\ 0 \end{bmatrix} t\psi(0)l + \frac{1}{2} \begin{bmatrix} 0 \\ -1 \end{bmatrix} t^2\psi(0)l - \frac{1}{6} \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^3\psi(0)l\\
&+ \frac{1}{24} \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^4\psi(0)l - \frac{1}{120} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^5\psi(0)l + \frac{1}{720} \begin{bmatrix} 0 \\ t \end{bmatrix} t^6\psi(0)l + \cdots\\
&= \psi(0)l \begin{bmatrix} -(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots) \\ 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots \end{bmatrix}\\
&= \psi(0)l \begin{bmatrix} -sint \\ cost \end{bmatrix}, \qquad (30)\end{aligned}$$

which satisfies (25) as the control function $l(t)$ exists at the minimum time $t = t_{opt} = t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$, where $Q(t) = \begin{bmatrix} -sint \\ cost \end{bmatrix}$ and $\lambda = \psi(0)$.

Thus

$$l_{opt}(t) = sgn\left(\psi(0)\begin{bmatrix} -sint \\ cost \end{bmatrix}\right), \tag{31}$$

where $l_{opt} = T_t^{-1}(u(t)) \in \partial E_\pi$ by Remark 12 and l_{opt} satisfies $sup_{l\in E\pi\subset L}\{|\langle l, f\rangle|, f \in E \subset L^*\} = 1$.

Now choose $\psi(0) = [\psi_1(0), \psi_2(0)]$ where $-\psi_1(0) = cos\delta$ and $\psi_2(0) = sin\delta$. Thus (31) becomes

$$\begin{aligned} l_{opt}(t) &= -\psi_1(0)sin(t) + \psi_2(0)cos(t), \quad \psi_1^2(0) + \psi_2^2(0) \neq 0 \\ &= cos\delta sint + sin\delta cost \\ &= sin(\delta + t)\ ,\ -\pi \le \delta \le \pi, \end{aligned} \tag{32}$$

which is the form of the minimum-time optimal control for the given system related to our problem in Section 3.

8. Application to a fractional control system

In this section, we illustrate the applicability of the proposed optimal control framework by considering a fractional differential equation with lower-order terms. When modeling dynamical systems with memory and inherited characteristics, the Caputo fractional derivative is often employed. Let $0 < \alpha < 1$, and let $x(t)$ be a suitably smooth function. The definition of the order α Caputo fractional derivative is

$${}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x'(s)}{(t-s)^{\alpha}}\,ds, \quad 0 < \alpha < 1.$$

where the Gamma function is denoted by $\Gamma(\cdot)$. This formulation is especially appropriate for control and physical applications since it permits the use of classical initial conditions.[37]

Consider the following linear fractional control system described by the Caputo derivative

$${}^{C}D_t^{\alpha}X(t) = AX(t) + Bl(t), \quad 0 < \alpha < 1, \tag{33}$$

subject to initial condition $X(0) = X_0$, where A and B are matrices of the proper dimensions, l is the control input, and $X(t) \in \mathbb{R}^n$ is the state vector. These types of systems are frequently employed in fractional optimal control problems.[38,39]

8.1. Example 2

Consider a Caputo fractional differential equation of order $2q$ together with additional lower-order terms.[40]

$${}^{C}D_{0+}^{2q}u(t) - 3{}^{C}D_{0+}^{q}u(t) + 2u(t) = 0, \tag{34}$$

with fractional initial conditions $u_1(0) = u_{01}, {}^{C}D_{0+}^{q}u_1(0) = u_{02}$. The system state vector is expressed as

$$X(t) = \begin{bmatrix} u(t) \\ {}^{C}D_{0+}^{q}u(t) \end{bmatrix}.$$

Then

$${}^{C}D_{0+}^{q}u(t) = u_2(t) \tag{35}$$

$${}^{C}D_{0+}^{q}u_1(t) = u_2(t). \tag{36}$$

By differentiating (35) we have

$${}^{C}D_{0+}^{2q}u_2(t) = {}^{C}D_{0+}^{2q}u(t). \tag{37}$$

From (35) and (37) we obtain

$${}^{C}D_{0+}^{2q}u_2(t) = 3u_2(t) - 2u_1(t). \tag{38}$$

Employing (36), (38), the System (34) can be represented in matrix form as

$$\begin{aligned} {}^{C}D_{0+}^{q}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= AX(t), \end{aligned} \tag{39}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and the initial condition

$$X(0) = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}.$$

To write (39) as a control, introduce a control input $l(t)$

$${}^{C}D_{0+}^{q}X(t) = AX(t) + Bl(t), \text{ by (33).} \tag{40}$$

choose $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and (40) becomes

$${}^{C}D_{0+}^{q}X(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} l(t).$$

The Hamiltonian H is defined as

$$\begin{aligned} H(X, l, \psi) &= \langle \psi(t), AX(t) + Bl(t)\rangle + 1 \\ &= \langle \psi(t), AX(t)\rangle + \langle \psi(t), Bl(t)\rangle + 1 \end{aligned} \tag{41}$$

The adjoint system corresponding to the controlled Caputo fractional state equation is derived using the fractional Pontryagin Maximum Principle together with the fractional integration-by-parts formula applied to the Hamiltonian variational equation. The adjoint dynamics involves the adjoint operator associated with the state operator A (which reduces to the transpose matrix A^T in the present finite-dimensional setting), while the control operator B (33) appears through the Hamiltonian maximization condition and the switching function [37, pp. 20–22].

Now, the switching function is defined as

$$\phi(t) = \langle \psi(t), B\rangle. \tag{42}$$

Then, (41) becomes

$$H = \langle \psi(t), AX(t)\rangle + \phi(t)l(t) + 1$$

According to the Pontryagin Maximum Principle, the optimal control maximizes the Hamiltonian H

$$H(X, l_{opt}(t), \psi(t)) = max_{sup_{l\in E_\pi \subset L}|\langle l,f\rangle|\leq 1} H(X(t), l, \psi(t)). \tag{43}$$

Assume that the control satisfies $sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle|\}$ ≤ 1, using Definition 2.9. Then by Section 4 and (43), we can statethat $l_{opt} \in \partial E_\pi$ is the optimal control that satisfies $sup_{l\in E_\pi \subset L}\{|\langle l, f\rangle|\} = 1$ at the minimum time $t = t_{opt} = t(1) = \inf\{t : u(t) = v(t), t \in [t_0, \infty)\}$.

Remark 14. *As H is linear in $l(t)$, the maximum occurs when $l(t)$ has the same sign as the switching function.*

Thus, we have

$$\begin{aligned} l_{opt}(t) &= sgn(\phi(t)) \\ &= sgn(\langle \psi(t), B\rangle) \text{ by (42)} \\ &= sgn(\lambda Q(t)) \text{ by (25)}. \end{aligned}$$

$$l_{\mathrm{opt}}(t) = \begin{cases} 1, & \text{if}\phi(t) > 0, \\ -1, & \text{if}\phi(t) < 0. \end{cases}$$

for $\phi(t) = \lambda Q(t)$ and it is the bang-bang control. Since the Hamiltonian is linear with respect to the control variable and the control is bounded, the optimal control takes values at the boundary of the admissible control set, resulting in a bang–bang control law. Consequently, the obtained control represents the minimum-time optimal control.

9. Results

The existence of minimum-time optimal control for a controllable continuous-time system in a Hausdorff locally convex linear topological space L is established. By utilizing the polar set E_π as a unit ball, which is a compact neighborhood of the origin in L, the attainable set $A(t)$ is shown to be compact using the continuous linear operator T_t as $T_t : E_\pi \to X$ for each fixed time $t \in I$, which guarantees the existence of an optimal control on the boundary of E_π in L and the minimum time is obtained as the limit of a decreasing sequence of admissible times with convergent controls in E_π. Moreover, the necessary optimality conditions are derived using the Pontryagin maximum principle to obtain the form of the optimal control $l_{opt}(t) = \mathrm{sgn}(\lambda Q(t))$ in (25) at the minimum time $t = t_{opt}$, and the Hamiltonian maximization is interpreted geometrically. An example illustrates the theoretical results when $l_{opt} = sin(\delta + t)$, $-\pi \leq \delta \leq \pi$ (32) .

10. Discussion

The results show that minimum-time optimal control problems in Hausdorff locally convex linear topological spaces can be effectively handled using the polar set as a unit ball E_π. This method enables the identification and description of optimal controls, without enforcing restrictive norm assumptions. The structure of the optimal controls is further clarified by the geometric interpretation of Hamiltonian maximization, which also emphasizes the analytical benefits of the suggested formulation. These results suggest that a wider class of infinite-dimensional control issues could benefit from the proposed strategy.

11. Conclusion

Thus using the polar set E_π in L, we can conclude from Remark 13 that the geometric interpretation for maximizing the Hamiltonian $H = \langle \psi, \dot{u}\rangle = c$, where $\psi(t)$ is an outward normal on H to the boundary of the attainable set $A(t)$ and we have shown that $l_{opt}(t)$ is the time-optimal control such that $u(t\ ; l_{opt}(t)) \in \partial A(t)$, $\forall t \in [t_0, t_{opt}]$ and $\dot{u} = Au(t) + Bl(t)$, where A is an $n \times n$ matrix, B is an $n \times r$ matrix, u is an element in X, and c is constant. In future work, we are interested in extending the time-optimal control problem using the bang-bang principle. In addition, by using the ideas of a unit ball such as a polar set, nets, and Kharitonov-type rectangles or cuboids, controllers can be designed for multi-agent systems with time delays. These concepts can be applied to a wide range of system classes, including controllable continuous-time systems, linear differential equation systems, nonlinear systems, dynamical systems, distributed and networked systems, normal and proper systems, interval-uncertain plants, and other suitable system formulations. Again, the present study opens several directions for further investigation. One possible extension is the study of nonlinear and semilinear fractional control systems in locally convex topological vector spaces. The analysis may also be generalized by incorporating state constraints, delayed controls, and nonconvex admissible control sets. Another important direction is the investigation of robust and stochastic formulations in the presence of uncertainties and perturbations. Further research can also be devoted to infinite-dimensional dynamical systems governed by partial differential equations, delay equations, and evolution inclusions. The geometric properties of the attainable set, including its boundary structure and supporting functionals, may provide deeper insight into the characterization of optimal trajectories and controls related to the shape and size of the constructed unit ball in structural engineering. In addition, the development of computational techniques for approximating attainable sets and optimal controls remains an important practical problem. The obtained theoretical results may further find applications in quantum control, engineering models, and other systems described by fractional dynamics. It would also be interesting to establish generalized Pontryagin-type optimality conditions under weaker topological assumptions and for more general classes of control systems.

Acknowledgements

The authors would like to thank the esteemed authors cited in the references for their invaluable contributions to time optimal control problems, which have greatly improved the tools for the quality of this research. The first author expresses gratitude to Dr. Gargi Chakraborty and VIT University, Vellore, for their support and encouragement.

Funding

None.

Conflict of interest

The authors have no relevant interests to declare.

Author contributions

Conceptualization: All authors
Formal analysis: All authors
Methodology: All authors
Resources: Ishrat Amin
Validation: Ishrat Amin
Visualization: Ishrat Amin
Investigation: Gargi Chakraborty
Supervision: Gargi Chakraborty
Writing – original draft: Ishrat Amin
Writing – review & editing: All authors

Availability of data

No datasets were generated or analyzed during the current study.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this Manuscript.

References

1. Leigh JR. *Functional Analysis and Linear Control Theory*. Academic Press; 1982. Accessed March 28,2026. https://books.google.co.in/books?id=LfwU3PuBoj0C

2. Hermes H, LaSalle JP. *Functional Analysis and Time Optimal Control*. Academic Press; 1969. Accessed March 28,2026. https://books.google.co.in/books?id=U2Opf6C83B8C

3. Chakraborty G. On the existence for a class of optimal control problems. *Indian J Pure Appl Math.* 1989;20:761–767. Accessed March 28,2026. https://portal.mardi4nfdi.de/wiki/Publication:1123388

4. Pontryagin LS, Boltyanski VG, Gamkrelidze RV, Mishchenko EF. *The Mathematical Theory of Optimal Processes*. Interscience Publishers, John Wiley & Sons; 1962:1–90. Accessed March 28,2026. https://api.pageplace.de/preview/DT0400.9781351433075_A37629368/preview-9781351433075_A37629368.pdf

5. Barbero-Liñán M, Muñoz-Lecanda MC. Geometric approach to Pontryagin's maximum principle. *Acta Appl Math.* 2009;108(2):429–485. https://doi.org/10.1007/s10440-008-9320-5

6. Yosida K. *Functional Analysis*. Springer-Verlag; 1965:40–390. Accessed March 28,2026. https://books.google.co.in/books?id=QqNpbTQwKXMC

7. Zabczyk J. *Mathematical Control Theory: An Introduction*. 2nd ed. Birkhäuser/Springer; 2020:40–230. https://doi.org/10.1007/978-3-030-44778-6

8. Prasad K, Kumar AN. A short description on locally convex and polar topologies. *Int J Eng Sci Invent.* 2020;9:1–5. https://doi.org/10.35629/6734-0907020105

9. Pugh CC. *Real Mathematical Analysis*. 2nd ed. Springer; 2015:80–220. https://doi.org/10.1007/978-3-319-17771-7

10. Hermes H. On the closure and convexity of attainable sets in finite and infinite dimensions. *SIAM J Control.* 1967;5(3):409–417. https://doi.org/10.1137/0305025

11. Dong DE, Berger H, Abraham I. Time-optimal ergodic search: multiscale coverage in minimum time. *Int J Robot Res.* 2024;44(10-11):1664-1683. https://doi.org/10.1177/02783649241273597

12. Kelley JL, Namioka I, Donoghue WF, *et al. Linear Topological Spaces*. Springer-Verlag; 1976. https://doi.org/10.1007/978-3-662-41914-4

13. Ghosh B, Chakraborty G. Application of mathematics for robust stability and for robustly strictly positive real on an uncertain interval plant. *Int J*

Robust Nonlinear Control. 2024;35:1463–1472. https://doi.org/10.1002/rnc.7732

14. Ghosh B, Chakraborty G. Necessary conditions for stability of Kharitonov polynomials using characteristic polynomial of uncertain interval matrices. *Syst Sci Control Eng.* 2026;14(1):2646370. https://doi.org/10.1080/21642583.2026.2646370

15. Ghosh B, Chakraborty G. Mathematical approach for robust stability and for robustly strictly positive real on an uncertain plant family of complex polynomials. *Int J Robust Nonlinear Control.* 2026;36(7):4089–4105. https://doi.org/10.1002/rnc.70403

16. Shukla A, Arora U, Sukavanam N. Approximate controllability of retarded semilinear stochastic system with non local conditions. *J Appl Math Comput.* 2015;49(1–2):513–527. https://doi.org/10.1007/s12190-014-0851-9

17. Shukla A, Sukavanam N. Interior approximate controllability of second-order semilinear control systems. *Int J Control.* 2024;97(3):615–624. https://doi.org/10.1080/00207179.2022.2161013

18. Shukla A, Patel R. Existence and optimal control results for second-order semilinear system in Hilbert spaces. *Circ Syst Signal Process.* 2021;40(9):4246–4258. https://doi.org/10.1007/s00034-021-01680-2

19. Patel R, Shukla A, Jadon SS. Existence and optimal control problem for semilinear fractional order $(1, 2]$ control system. *Math Methods Appl Sci.* 2024;47(13):10940–10951. https://doi.org/10.1002/mma.6662

20. Kumar S, Tajinder. Existence of solutions and optimal control for semilinear PDE systems with nonlinear interactions. *Complex Anal Oper Theory.* 2026;20:46. https://doi.org/10.1007/s11785-026-01900-0

21. El Boukhari N. Existence of optimal controls for semilinear systems with a nonreflexive control space. *Stat Optim Inf Comput.* 2026;15:1024–1035. https://doi.org/10.19139/soic-2310-5070-2541

22. Oh Y, Moon J. The infinite-dimensional Pontryagin maximum principle for optimal control problems of fractional evolution equations with endpoint state constraints. *AIMS Math.* 2024;9(3):6109–6144. https://doi.org/10.3934/math.2024299

23. Camacho O, Castillo RE, Leiva H. Optimal control governed impulsive neutral differential equations. *Results Control Optim.* 2024;17:100505. https://doi.org/10.1016/j.rico.2024.100505

24. Almi S, Durastanti R, Solombrino F. A Pontryagin maximum principle for agent-based models with convex state space. *ESAIM Control Optim Calc Var.* 2025;31:37. https://doi.org/10.1051/cocv/2025025

25. El Idrissi R, Lafhim L, Kalmoun EM, Ouakrim Y. Optimality conditions for bilevel optimal control problems with non-convex quasi-variational inequalities. *RAIRO Oper Res.* 2024;58(2):1789–1805. https://doi.org/10.1051/ro/2024065

26. Van Cuong D, Mordukhovich B, Nam NM, Wells M. Convex analysis of minimal time and signed minimal time functions. *arXiv.* 2020. https://doi.org/10.48550/arXiv.2010.10736

27. Bonnet B, Frankowska H. Necessary optimality conditions for optimal control problems in Wasserstein spaces. *arXiv.* 2021. https://doi.org/10.48550/arXiv.2101.10668

28. Fernández-Cara E, Marín Gayte I. Analysis and numerical solution of some minimal time control problems. *Results Appl Math.* 2025;26:100582. https://doi.org/10.1016/j.rinam.2025.100582

29. Reis T, Schaller M. Linear-quadratic optimal control for infinite-dimensional input-state-output systems. *ESAIM Control Optim Calc Var.* 2025;31:36. https://doi.org/10.1051/cocv/2025026

30. Fresse-Colson O, Guérin S, Chen X, Sugny D. Application of the Pontryagin maximum principle to the robust time-optimal control of two-level quantum systems. *Phys Rev Res.* 2025;112:022618. https://doi.org/10.1103/rn5j-kjq3

31. Caillau JB, Ferretti R, Trélat E, Zidani H. An algorithmic guide for finite-dimensional optimal control problems. *arXiv.* 2022. https://doi.org/10.48550/arXiv.2212.03157

32. Bayen T, Bouali A, Bourdin L. The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for local solutions. *SIAM J Control Optim.* 2024;62(4):2412–2432. https://doi.org/10.1137/23M155311X

33. Averboukh Y, Khlopin D. Pontryagin maximum principle for the deterministic mean field type optimal control problem via the Lagrangian approach. *J Differ Equ.* 2025;430:113205. https://doi.org/10.1016/j.jde.2025.02.076

34. Van Ackooij W, Henrion R, Zidani H. Pontryagin's principle for some probabilistic control problems. *Appl Math Optim.* 2024;90(1):5. https://doi.org/10.1007/s00245-024-10151-4

35. Chertovskih R, Ribeiro VM, Gonçalves R, Aguiar AP. Sixty years of the maximum principle in optimal control: historical roots and content classification. *Symmetry.* 2024;16(10):1398. https://doi.org/10.3390/sym16101398

36. De Bernardi CA, Veselý L. On support points and support functionals of convex sets. *Isr J Math.* 2009;171(1):15–27. https://doi.org/10.1007/s11856-009-0037-6

37. Diethelm K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type.* Springer; 2010. https://doi.org/10.1007/978-3-642-14574-2

38. Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Academic Press; 1999. https://doi.org/10.1016/S0076-5392(99)80027-7

39. Agrawal OP. A general formulation and solution scheme for fractional optimal control problems. *Nonlinear Dyn.* 2004;38(1–4):323–337. https://doi.org/10.1007/s11071-004-3764-6

40. Vatsala AS, Pageni G. Solution of linear Caputo fractional differential equations with fractional initial conditions. *Appl Math.* 2025;5(4):157. https://doi.org/10.3390/appliedmath5040157

Ishrat Amin *is a Ph.D. researcher in Mathematics at Vellore Institute of Technology (VIT), Vellore, Tamil Nadu, India.*
https://orcid.org/0009-0009-8583-412X

Gargi Chakraborty *is a professor of Mathematics at Vellore Institute of Technology (VIT), Vellore, Tamil Nadu, India.*
https://orcid.org/0009-0007-3400-0788

An International Journal of Optimization and Control: Theories & Applications (https://accscience.com/journal/ijocta)

Appendix

Nomenclature

$\sum$	Controllable continuous time system
L	Locally convex linear topological space
L^*	Adjoint of all continuous linear functionals defined on L
I	Time interval $I = [t_0, t]$
E	Bounded subset of L^*
E_π	Polar set as a unit ball pological space L
∂E_π	Boundary of polar set E_π
$\text{int}E_\pi$	Interior of polar set E_π
f	$f \in E \subset L^*$
X	State space of state variables
Y	Output Space
T_t	Continuous linear operator at each fixed time $t \in I$
ϕ	State transition mapping represented by T_t
$u(t)$	State variable u(t) for some time t
$v(t)$	Continuous function defined on I
$l(t)$	Control variable l at each fixed time $t \in I$
$l_{opt}(t)$	Optimal control at time $t = t_{opt} = t(1)$
$t(1)$	Minimum time, $t = t_{opt} = t(1)$
$A(t)$	Attainable set
$\partial A(t)$	Boundary of attainable set
H	Hamiltonian
Ω	Supporting Hyperplane
ψ	Outward normal to the attainable set $A(t)$
$\psi^\top$	Transpose of outward normal ψ
$Q(t)$	Transition matrix
e^{-At}	State Transition Matrix
λ	Arbitrary constant
α	Arbitrary Scalar
δ	Arbitrary angle, $-\pi \leq \delta \leq \pi$
A	$n \times n$ matrix
B	$n \times r$ matrix