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RESEARCH ARTICLE

Ulam's stability of neutral fractional stochastic differential equation with the application of homeostasis circuit

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ABSTRACT

In this paper, Ulam's stability of neutral fractional stochastic differential equations (NFSDEs) with sub-fractional Brownian motion (sfBm) is studied. Existence and stability results are established using fixed point theory, semigroup theory, and fractional calculus in a stochastic setting. We modify unstable impulses into stable impulses in the signal transformation of the cell tissue trajectory. Numerical simulations are provided to verify the obtained results. Fractional components are used to construct a fractional-order analogue circuit model for an artificial tissue homeostasis circuit, which is relevant to the application features. The proposed circuit offers several advantages for framing effective cell populations, including (i) predicting the death of β cells at the microscopic level for type 1 diabetics, (ii) regenerating the dead cells through stem cells, and (iii) achieving stable signal transformation between β cells and stem cells.

1. Introduction

Random noise affects several phenomena in engineering, physics, biomedical, and other fields of study. Thus, the stochastic differential equation model is undeniable (see[1] and[2]). Recently, fractional differential equations have received wide attention; see references.[3–9] Few authors in[10–18] have studied the qualitative properties of fractional stochastic differential equations. Furthermore, electrical networks include a lossless transmission line, which is illuminated by a neutral differential equation. Impulsive differential equations have been an active research area in recent years because of their applications in science, engineering, medicine, and many other fields, with extensive research interest.[19–22] The activity of instantaneous impulses does not clarify certain dynamics of the development procedure of pharmacotherapy. One can interpret the circumstance as an impulsive action that begins suddenly and remains dynamic over a limited period, which can be captured by non-instantaneous impulses. In[23,24] presented the existence result for a class of generalized fractional stochastic differential equations with non-instantaneous impulses driven by fBm. In a few papers,[25–27] the researchers examined stability properties of impulsive differential

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equations. Self-similarity, non-stationary of the increment, and Holder path are the properties[28] of sfBm, which greatly help as a model tool in telecommunication, finance, and many other applications. The β cells are the sub-optimal system of microscopic pancreatic cells. Thus, sfBm is utilized to represent the diffusion part of the cells. In general, insulin slows glycogen production in the liver whenever glycogen metabolism is irregular. Therefore, delay differential equations have been embedded in this study. The delay differential infinite-dimensional system has rich dynamics and is more qualified to consider complex objects through a long life history. Tissue homeostasis is a vital system that manipulates the different cell populations within a tissue.[29] To regenerate the dead β cell due to miscarriage of homeostasis, artificial adult stem cells could be utilized as per the following model of the Hill equation described as

$$\frac{dAI1}{dt} = K_p\frac{C}{C+K_d} - K_{deg}AI1$$

which represents the differentiated cell population, excluding the diffusion of molecules. Numerical demonstrations of different genuine situations are prompted by the modeling of differential equations. Depending on the total number of requested subsidiaries, these customary models may present large errors. The fractional differential equation helps to correct the error by utilizing fractional derivatives and gives an amazing portrayal of the memory and heredity properties of cycles. Hence, it is necessary to upgrade the previous model to a fractional-order delay system. The proposed fractional-order circuits[30,31] are the foundation for identifying the dead cell in a more suitable way than the previous one.

Stability is one of the significant qualitative properties of a dynamical system in that the ability to control is noteworthy in biological applications since deviations in the mathematical model unavoidably lead to estimation errors. Here, the RC circuit model has been utilized to examine stability properties, as it has been taken into consideration by numerous scientists throughout its potential applications. Ulam-Hyers stability supports finding the nearby exact solution. Ulam-Hyers stability results of differential systems have been investigated by many authors; see.[32–34] There is no existing literature discussing Ulam's stability behavior of the proposed model driven by sfBm. Therefore, the study of Ulam's stability is concerned with the system described by

$$\begin{aligned} &{}^{C}D^{q}_{0^+}[x(\ell) - \mathcal{L}(\ell, x(t-\rho_1(\ell)))] \\ &= \left[Ax(\ell) + \hat{\mathfrak{f}}(\ell, x(\ell-\rho_2(\ell)))\right] \end{aligned} \tag{1}$$

$$\begin{aligned} &+ \Lambda(\ell, x(\ell-\rho_3(\ell)))\frac{dS^{\mathbb{H}}_{Q}(\ell)}{d\ell}, \\ &\ell \in (\varsigma_i, \ell i+1],\ i = 0, 1, 2\&, k \\ &x(\ell) = \mathcal{N}i(\ell, x(\ell-\rho_4(\ell)))), \ell \in (\ell_i, \varsigma_i], i = 1, 2, \dots, k, \\ &x_0(\cdot) = \varphi \in \mathcal{BF}0([\hat{m}(0), 0], \mathcal{H}), \end{aligned} \tag{2}$$

where ${}^{C}D^{q}0^{+}$ describes the Caputo differentiation of order $q \in (0, 1)$. Choose A be an operator on $D(A) \subset \mathcal{H} \to \mathcal{H}$. $x(\cdot)$ takes values in the Hilbert space $\mathcal{H}$ with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. $(K, |\cdot| K, \langle \cdot, \cdot \rangle)$ be an another separable Hilbert space. We use the same $|\cdot|$ and $\langle \cdot, \cdot \rangle$ symbols for K norm and inner product respectively. Let $\hat{J} := [0, b]$. The mappings $\mathcal{L}, \hat{\mathfrak{f}} : \hat{J} \times \mathcal{H} \to \mathcal{H}$, $\hat{\Lambda} : \hat{J} \times \mathcal{H} \to L2^0(K, \mathcal{H})$ are appropriate functions to be defined in the sequel. Let $K_0 = Q^{\frac{1}{2}}K$, also $L^0_2(K_0, \mathcal{H})$ signifies the space of all Q-Hilber-Schmidt operators from K_0 to $\mathcal{H}$ and $\mathcal{N}i : [\ell_i, \hat{\hat{\zeta}} i] \times \mathcal{H} \to \mathcal{H}$ $(i = 1, 2, \cdots, k, k \in \mathbb{N})$ are all Borel measurable functions. Also, assume that $\hat{J}_i = (\hat{\hat{\zeta}}_i, \ell_{i+1}]$, $\overline{\hat{J}}_i = [\hat{\hat{\zeta}} i, \ell_{i+1}]$, $\hat{T}i = (\hat{\hat{\zeta}}_i, \ell_{i+1}]$, $\overline{\hat{T}}_i = [\hat{\hat{\zeta}}_i, \ell_{i+1}]$ and $\ell_i, \hat{\hat{\zeta}}_i$ are fixed numbers assuring $0 = \ell_0 = \hat{\hat{\zeta}}_0 < \ell_1 \le \hat{\hat{\zeta}}_1 < \cdots < \ell \le \hat{\hat{\zeta}}_i < \ell_{i+1} = b < \infty$. Assume $\rho_j(\ell) \in \tilde{C}(\mathbb{R}^+, \mathbb{R}^+)(j = 1, 2, 3, 4)$ succeed $t - \rho_j(t) \to \infty$ as $\ell \to \infty$ and $\hat{m}(0) \ = \ max\inf \hat{\hat{\zeta}} \ge 0(\hat{\hat{\zeta}} - \rho_j(\hat{\hat{\zeta}}))$, $j = 1, 2, 3, 4$ and $\mathcal{BF}0([\hat{m}(0), 0])$ stands for the family of all (almost surely) a.s. bounded, $\mathcal{F}0$-measurable, continuous random variables $\varphi(\ell) : [\hat{m}(0), 0] \to \mathcal{H}$ along the following norm

$$|\varphi| \mathcal{B} = \sup \hat{m} \le \ell \le 0E\, |\varphi(\ell)|\, \mathcal{H}.$$

The novelty and importance of the study are highlighted as follows:

- The non-instantaneous impulsive condition is imposed on NFSDEs for investigation.
- The Banach fixed point theorem is used to derive the existence and stability result by relaxing the compactness conditions on the semigroup of a bounded linear operator $T_q(\ell), \ell \in \hat{J}$.
- Sufficient conditions are established for Hyers-Ulam stability of NFSDEs, driven by sfBm.
- Results are derived through the p^{th} moment norm.

In the application aspects,

(1) Hyers-Ulam stability results and non-instantaneous impulses of NFSDEs are used to identify the exact location of the dead β-cells.

(2) Generate a stable signal transformation from the new stem cells of the trajectory tracking to β cells. Then the feedback control (homeostasis) mechanism of β cells becomes activated.

The paper contains five sections: Section 1, provides the literature review. Section 2 explains basic concepts, notations, and lemmas. In Section 3, we derive the necessary conditions for the existence, uniqueness, and stability of NFSDEs with non-instantaneous impulses. In Section 4, numerical simulations are offered to establish the results. Section 5 is devoted to the conclusion.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the normal filtration $\mathcal{F} = \{\mathcal{F}_\ell\}_{\ell \geq 0}$. Let $S_Q^{\mathbb{H}}(\ell) = \{S(\ell); \ell \in \hat{J}\}$ denote a sub-fractional Brownian motion (sfBm) with Hurst parameter $\mathbb{H} \in (\frac{1}{2}, 1)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

The space of all $\mathcal{F}_0$-adapted processes $\tilde{\tilde{\Psi}}(t, w) : [\hat{m}(0), \infty) \times \Omega \to \mathcal{R}$ that are almost surely continuous in t for fixed $w \in \Omega$ satisfy $\tilde{\tilde{\Psi}}(\hat{\hat{\zeta}}, w) = \varphi(\hat{\hat{\zeta}})$ for $\hat{\hat{\zeta}} \in [\hat{m}(0), 0]$ and $E\left\|\tilde{\tilde{\Psi}}(\ell, w)\right\|_{\mathcal{H}}^p < \infty$. Define $\mathbb{Y} = C(\hat{J}, L_P(\Omega, \mathcal{H}))$ is a Banach space with the norm

$$\left\|\tilde{\tilde{\Psi}}\right\|_{\mathbb{Y}}^p = \sup_{t \geq 0} E\left\|\tilde{\tilde{\Psi}}(\ell)\right\|_{\mathcal{H}}^p < \infty.$$

Denote $J_0 = [0, \ell_1]$, $\hat{J}_i = (\hat{\hat{\zeta}}_i, \ell_{i+1}], i = 1, 2, .., k$. Take $\tilde{Z} = PC(\hat{J}, L_P(\Omega, \mathcal{H})) = \{x : x \in (\hat{J}, L_P(\Omega, \mathcal{H})), x(\ell_i^-), x(\ell_i^+)$ exists and $x(\ell_i^-) = x(\ell_i^+), i = 1, 2, ...k\}$ with $\|\cdot\|_{\tilde{Z}} = \sup_{\ell \in \hat{J}}\{E\|x(\ell)\|^p\}^{\frac{1}{p}}$.

Lemma 1. *([35]) For A, $R_q(\ell)$ and $T_q(\ell$, one has*

$$AT_q(\ell)x = A^{1-\alpha}T_q(\ell)A^\alpha x,$$
$$|A^\theta T_q(\ell)| \leq \frac{qC_\theta}{\ell^{q\theta}} \frac{\Gamma(2-\theta)}{\Gamma(1+q(1-\theta))}, \ \ell \in \hat{J}. \quad (3)$$

Remark 1. *([36])*

$$E\left\|\int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(t - \hat{\hat{\zeta}}) \tilde{\Phi}(\hat{\hat{\zeta}}) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|_{\mathbb{Y}}^p$$
$$\leq \tilde{K} C_{\mathbb{H}}^{\frac{p}{2}} \ell^{p\mathbb{H} - \frac{p}{2}} \hat{M}_T^p t^{\frac{2pq-p-1}{2}} \Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}$$
$$\times \int_0^\ell \left\|\tilde{\Phi}(\hat{\hat{\zeta}})\right\|_{L_p^0}^p d\hat{\hat{\zeta}}.$$

Lemma 2. *A $\mathcal{H}$-valued $\mathcal{F}_\ell$-adapted stochastic process $\{x(\ell) : \ell \in \hat{J},\ 0 \leq b < \infty\}$ is called a mild solution of system $(1) - (3)$, if for every $x_0 = \varphi \in \mathcal{B}_{\mathcal{F}_0}([\hat{m}(0), 0], \mathcal{H})$, $x(\ell)$ has cadlag paths on $\hat{J}$ a.s. $\forall\ \ell \in \hat{J}$, the function $(\ell - \hat{\hat{\zeta}})^{q-1} AT_q(\ell - \hat{\hat{\zeta}}) \mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})))$ is integrable and satisfies the following integral equations*

$$x(\ell) = \begin{cases} R_q(\ell)[\phi(0) - \mathcal{L}(0, \varphi(0 - \rho_1(0)))] \\ +\mathcal{L}(\ell, x(\ell - \rho_1(\ell))) + \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} \\ \times AT_q(\ell - \hat{\hat{\zeta}}) \mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}})) d\hat{\hat{\zeta}} \\ + \int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \\ \times \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}),\ \ell \in [0, \ell_1] \\ \mathcal{N}_i(\ell, x(\ell - \rho_4(\ell))),\ \ell \in (\ell_i, \hat{\hat{\zeta}}_i], \\ i = 1, \cdots, k \end{cases} \quad (4)$$

$$\begin{cases} R_q(\ell - \hat{\hat{\zeta}}_i)\Big[\mathcal{N}_i(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_4(\hat{\hat{\zeta}}_i))) \\ -\mathcal{L}(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_1(\hat{\hat{\zeta}}_i)))\Big] + \mathcal{L}(\ell, x(\ell - \rho_1(\ell))) \\ + \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} AT_q(\ell - \hat{\hat{\zeta}}) \mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})) d\hat{\hat{\zeta}} \\ + \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\Lambda}(\hat{\hat{\zeta}}, \\ x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}})) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}),\ \ell \in (\hat{\hat{\zeta}}_i, \ell_i + 1] \end{cases}$$

where $R_q(\ell)x$ & $T_q(\ell)x$ is specified in.[36]

The following inequalities are used to derive the Hyers-Ulam stability result.
Choose $\epsilon > 0$ satisfying

$$\begin{cases} E\Big\|{}^C D_{0+}^q [\hat{y}(\ell) - \mathcal{L}(\ell, \hat{y}(\ell - \rho_1(\ell)))] \\ -\big[A\hat{y}(\ell) - \hat{\mathfrak{f}}(t, \hat{y}(\ell - \rho_2(\ell))) \\ -\Lambda(t, \hat{y}(\nu - \rho_3(t))) \dfrac{dS_Q^{\mathcal{H}}(\ell)}{dt}\big]\Big\|^p \leq \epsilon, \\ \ell \in (\nu_i, \ell_{i+1}] \\ E\|[{}^C D_{0+}^q \hat{y}(\ell) - \mathcal{L}(t, \hat{y}(\ell - \rho_1(\ell)))] \\ -\mathcal{N}_i(\ell, \hat{y}(\ell - \rho_4)))\|^p \leq \epsilon, \ell \in (\ell_i, \nu_i], \\ i = 1, 2, \ldots, k \end{cases} \quad (5)$$

Definition 1. *([37]) One can say that system $(1) - (3)$ is Hyers-Ulam stable if $\exists$ a solution $\hat{y}$ that satisfies the above inequalities (6) with the initial value $\hat{y}_{\ell_0} = x_{\ell_0} = \varphi, \varphi \in ([\hat{m}(0), 0], \mathcal{H})$and if $x(t)$ satisfies the system $(1) - (3)$, then $\exists$ a constant $\mathcal{G}$ has the property such that*

$$\|\hat{y}(\ell) - x(\ell)\|^p \leq \mathcal{G}\epsilon,\ \forall \ell \in \hat{J}.$$

Definition 2. *([38]) One can say that system $(1) - (3)$ is Hyers-Ulam stable if $\exists$ a solution $\hat{y}$ that satisfies the above inequalities (7) with the initial value $\hat{y}_{\ell_0} = x_{\ell_0} = \varphi, \varphi \in ([\hat{m}(0), 0], \mathcal{H})$and if $x(t)$ satisfies the system $(1) - (3)$, then $\exists$ a constant $\mathcal{G}$ has the property such that*

$$\|\hat{y}(\ell) - x(\ell)\|^p \leq \mathcal{G}\epsilon,\ \forall \ell \in \hat{J}.$$

The following assumptions are imposed to describe the stability of the problem

(A_1): The function $f : \hat{J} \times \mathcal{H} \to \mathcal{H}$ satisfies, for some $\hat{C}_1, \hat{C}_2 \geq 0$

$$E\|f(\ell, x) - |f(\ell, y)\|^p \leq \hat{C}_1\|x - y\|_{\tilde{Z}}^p$$
$$E\|f(\ell, x)\|^p \leq \hat{C}_2\|x\|_{\tilde{Z}}^p$$

(A_2): The function $\hat{\Lambda} : \hat{J} \times \mathcal{H} \to L_p^0$ satisfies, for some $\hat{C}_3, \hat{C}_4 \geq 0$

$$E\|\hat{\Lambda}(\ell, x) - \hat{\Lambda}(\ell, y)\|^p \leq \hat{C}_3(\|x - y\|^p_{\tilde{Z}})$$
$$E\|\hat{\Lambda}(\ell, x)\|^p \leq \hat{C}_4(\|x\|^p_{\tilde{Z}}).$$

(A_3): (i) The mapping $\mathcal{L} : [0, \infty) \times \mathcal{H} \to \mathcal{H}$ is continuous in the p^{th} mean sense and satisfies, for any $x, y \in \mathcal{H}$ and $\ell \geq 0$, $\mathcal{L}(\ell, x(\ell - \rho_1)) \in D(A^\alpha)$, $\exists$ a function $\hat{C}_5 \geq 0$ such that

$$\lim_{t \to \hat{\hat{\zeta}}} E\left\|(A)^\alpha \mathcal{L}(\ell, x(\ell - \rho_1)) - (A)^\alpha \mathcal{L}(\hat{\hat{\zeta}}, x(\ell - \rho_1))\right\|^p_{\tilde{Z}} = 0,$$
$$E\|(A)^\alpha \mathcal{L}(\ell, x(\ell - \rho_1))\|^p_{\mathcal{H}} \leq \hat{C}_5 \|x\|^p_{\tilde{Z}}$$

where $\alpha \in (0, 1]$ and satisfies $p\alpha > 1$, p is an integer $p \geq 2$. Choose $\mathcal{L}(\ell, 0) = 0$, $\ell \geq 0$.

(A_4): (i) The function $\mathcal{N}_i : [0, \infty) \times \mathcal{H} \to \mathcal{H}$ are completely continuous and $\exists$ constants $\hat{C}_{\mathcal{N}_i}, i = 1, \cdots, k$ such that $E\|\mathcal{N}_i(\ell, x)\|^p_{\tilde{Z}} \leq \hat{C}_{\mathcal{N}_i} E\|x\|^p_{\tilde{Z}}$, for each $x \in \mathcal{H}$ and $\mathcal{N}_i(\ell, 0) = 0$.

(A_5): [39] $\exists$ a constant $\mathfrak{K} < 1$ where $\mathfrak{K} = \hat{C}_5\|A^\alpha\|^p + \hat{C}_5\|A^{1-\alpha}\|^p \hat{K}(q, \alpha)\ell_1^{pq-pq\alpha-1} \left(\frac{p-1}{pq-pq\alpha-1}\right)^{p-1} \hat{C}_5 \hat{+} \hat{C}_{\mathcal{N}_i} + \hat{M}_T^p(\ell - \hat{\hat{\zeta}}_i)^{pq-1}\left(\frac{p-1}{pq-1}\right)^{p-1}\hat{C}_1 + \tilde{K} C_{\mathbb{H}}^{\frac{p}{2}} \ell^{p\mathbb{H} - \frac{p}{2}} \hat{M}_T^p(\ell - \hat{\zeta}_i)^{\frac{2pq-p-1}{2}} \left(\frac{p-2}{2pq-p-2}\right)^{\frac{p-2}{2}} \hat{C}_3$

3. Main result

Theorem 1. *Assume that hypotheses $(A_1)-(A_4)$ hold. Then there exists a mild solution of the **Equations (1)-(3)**.*

Proof. The operator $\hat{\Psi} : \tilde{Z} \to \tilde{Z}$ is defined as $(\hat{\hat{\Psi}}x)(\ell) = \tilde{\varrho}(\ell)$ for $\ell \in [m(0), 0]$, $\ell \geq 0$ and

$$(\hat{\hat{\Psi}}x)(\ell) = \begin{cases} R_q(\ell)[\phi(0) - \mathcal{L}(0, \varphi(0 - \rho_1(0)))] \\ +\mathcal{L}(\ell, x(\ell - \rho_1(t))) + \int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} \\ \times T_q(\ell - \hat{\hat{\zeta}}) A\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \\ \times \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \\ \times \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\zeta}), \\ \ell \in [0, \ell_1] \\ N_i(\ell, x(\ell - \rho_4(\ell))), \\ R_q(\ell - \hat{\hat{\zeta}}_i)\Big[\mathcal{N}_i(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_4(\hat{\hat{\zeta}}_i))) \\ -\mathcal{L}(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_1(\hat{\hat{\zeta}}_i)))\Big] \\ +\mathcal{L}(\ell, x(t - \rho_1(t))) \\ + \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \\ A\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, \\ x(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\ + \int_{\hat{\hat{\zeta}}_i}^t (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \\ \hat{\Lambda}(\hat{\hat{\zeta}}, \\ x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}), \ell \in (\hat{\hat{\zeta}}_i, \ell_i + 1] \end{cases} \quad (6)$$

The proof is divided into the following three steps. □

Step 1. $\hat{\hat{\Psi}}$ maps bounded sets into bounded sets in $\tilde{Z}$.

Indeed, it is sufficient to show that for every $\hat{r} > 0$, $\exists$ a constant $\theta > 0$ such that for every $x \in B_{\hat{r}} = \{x : \|x\|^p_{\tilde{Z}} \leq \hat{r}\}$, one can get $\left\|\hat{\hat{\Psi}}x\right\|^p_{\mathcal{H}} \leq \theta$. From $(A_1) - (A_4)$ and utilizing the Holder's inequality for $\ell \in [0, \ell_1]$, we obtain

$$\sup_{\ell \in [0, \ell_1]} E\|x(\ell)\|^p$$
$$\leq 5^{p-1} \sup_{[0, \ell_1]} \Bigg\{ E\left\|R_q(\ell)\Big[\varphi(0) - \mathcal{L}(0, \varphi(-\rho_1(0)))\Big]\right\|^p$$
$$+ E\|\mathcal{L}(\ell, x(\ell - \rho_1(\ell)))\|^p$$
$$+ E\left\|\int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q((\ell - \hat{\hat{\zeta}}) A\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}\right\|^p$$
$$+ E\left\|\int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q((\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}\right\|^p$$
$$+ E\left\|\int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3)) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p \Bigg\}$$
$$= 5^{p-1} \sum_{j=1}^5 \hat{\pi}_j$$

Using assumptions (A_1) and $(A_4)(i)$, we have

$$\hat{\pi}_1 = E\left\|R_q(\ell)\Big[\varphi(0) - \mathcal{L}(0, \varphi(-\rho_1(0)))\Big]\right\|^p$$
$$\leq \left(\int_0^\infty \hat{\eta}(\hat{\theta}) e^{-q(\ell^q \hat{\theta})} d\hat{\theta}\right)^p \Big[\varphi(0) - \|(A)^{-\alpha}\|^p \times \Big(\hat{C}_5(0)\Big)(\phi(\rho_1(0)))\Big]$$

$$\hat{\pi}_2 = E\|\mathcal{L}(\ell, x(\ell - \rho_1(\ell)))\|^p_{\mathcal{H}} \leq \left\|(A)^{-\alpha}\right\|^p_{\mathcal{H}} \times \Big(\hat{C}_5\Big)(\|x(\ell - \rho_1(\ell))\|)$$

$$\hat{\pi}_3 = E\left\|\int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} \times T_q((\ell - \hat{\hat{\zeta}}) A\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}\right\|^p \leq$$
$$\left[\int_0^\ell \left(A^{1-\alpha}(\ell - \hat{\hat{\zeta}})^{q-1} A^\alpha T_q((\ell - \hat{\hat{\zeta}})\right)^{\frac{p}{p-1}} d\hat{\hat{\zeta}}\right]^{p-1}$$
$$\times \int_0^t E\left\|x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))\right\|^p d\hat{\hat{\zeta}} \leq \|A^{1-\alpha}\|^p$$
$$\times \hat{K}(q, \alpha) \ell_1^{pq - pq\alpha - 1} \Big(\frac{p-1}{pq - pq\alpha - 1}\Big)^{p-1} \hat{C}_5 \hat{\theta}_5$$

where $\hat{K}(q, \alpha) = \dfrac{q^p \Gamma(1 + \alpha)^p (C^p_{1-\alpha})}{\Gamma^p(1 + q\alpha)}$

By Holder's inequality and hypothesis $(A_2)(ii)$, we get

$$\hat{\pi}_4 = E\left\|\int_0^\ell (\ell-\hat{\hat{\zeta}})^{q-1}T_q((\ell-\hat{\hat{\zeta}})\right.$$
$$\left.\times \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}}))d\hat{\hat{\zeta}}\right\|^p$$
$$\leq \left[\int_0^\ell \left((\ell-\hat{\hat{\zeta}})|^{q-1}T_q((\ell-\hat{\hat{\zeta}})\right)^{\frac{p}{p-1}}d\hat{\hat{\zeta}}\right]^{p-1}$$
$$\times \int_0^t E\left\|\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right\|^p d\hat{\hat{\zeta}}$$
$$\leq \hat{M}_T^p \ell^{pq-1}\Big(\frac{p-1}{pq-1}\Big)^{p-1}\hat{C}_2\hat{\theta}_2.$$

From $(A_3)(ii)$, we get

$$\hat{\pi}_5 = E\left\|\int_0^\ell (\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\right.$$
$$\left.\times \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_3)))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p$$
$$\leq \hat{K}\left(\left(\int_0^t (\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\right)^2\right.$$
$$\left.\times \left\|\int_0^\zeta \hat{\Lambda}(\zeta, x(\hat{\hat{\zeta}}-\rho_3)))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^2\right)^{\frac{p}{2}}$$
$$\leq \tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p t^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}$$
$$\times E\left\|\int_0^{\hat{\hat{\zeta}}} \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_3)))\right\|^p d\hat{\hat{\zeta}}$$
$$\leq \tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p t^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3\hat{\theta}_3$$

Hence

$$\sup_{\ell\in[0,\ell_1]} E\|x(t)\|_{\mathcal{H}}^p$$
$$\leq 5^{p-1}\Big\{(\int_0^\infty \hat{\eta}(\hat{\theta})e^{-q(\ell^q\hat{\theta})}d\hat{\theta})^p\Big[\varphi(0)$$
$$+\left\|(A)^{-\alpha}\right\|^p\hat{C}_5\hat{\theta}_5\Big]$$
$$+\left\|(A)^{-\alpha}\right\|^p\hat{C}_5(\hat{\theta}_5(\|t, x(\ell-\rho_1(\ell))\|)$$
$$+\left\|A^{1-\alpha}\right\|^p\hat{K}(q,\alpha)\ell_1^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}$$
$$\times \hat{C}_5\hat{\theta}_5+\hat{M}_T^p\ell_1^{pq-1}\Big(\frac{p-1}{pq-1}\Big)^{p-1}\hat{C}_2\hat{\theta}_2$$
$$+\hat{M}_T^p\ell_1^{p\mathbb{H}+pq-p/2-1}\Big(\frac{p-1}{pq-1}\Big)^{p-1}\hat{K}C_{\mathbb{H}}^{\frac{p}{2}}\hat{C}_3\hat{\theta}_3\Big\} := \hat{\theta}_0.$$

For $\ell\in(\ell_i,\hat{\hat{\zeta}}_i]$, $i=1,\cdots,k$, we obtain

$$E\|\mathcal{N}_i(\ell, x(\ell-\rho_4(\ell)))\|^p \leq \hat{C}_{\mathcal{N}_i}E\|x(\ell-\rho_4(\ell))\|^p$$
$$\max_{i\in[1,\cdots,k]}\hat{C}_{\mathcal{N}_i}\|x\|^p := \overline{\hat{\theta}_0}.$$

Similarly, for every $\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]$, $i=1,\cdots,k$, we have

$$\sup_{\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]} E\|x(\ell)\|^p$$
$$\leq 6^{p-1}\sup_{\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]}\Big\{E\left\|R_q(\ell-\hat{\hat{\zeta}}_i)\mathcal{N}_i(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i-\rho_4(\hat{\hat{\zeta}}_i)))\right\|^p$$
$$-E\left\|R_q(\ell-\hat{\hat{\zeta}}_i)\mathcal{L}_i(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i-\rho_1(\hat{\hat{\zeta}}_i)))\right\|^p$$
$$+E\|\mathcal{L}(\ell, x(\ell-\rho_1(\ell)))\|_{\mathcal{H}}^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell (\ell-\hat{\hat{\zeta}})^{q-1}T_q((\ell-\hat{\hat{\zeta}})A\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell (\ell-\hat{\hat{\zeta}})^{q-1}T_q((\ell-\hat{\hat{\zeta}})\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, (\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^t (\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_3))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\Big\}$$
$$\leq 6^{p-1}\sup_{\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]}\Big\{\hat{M}^p\int_0^\infty \hat{\eta}(\hat{\theta}))e^{-q(\ell^q\hat{\theta})}d\hat{\theta})^p$$
$$\times \max_{i\in[1,\cdots,k]}\hat{C}_{\mathcal{N}_i}\hat{\theta}_7(\hat{r})-\left\|(A)^{-\alpha}\right\|^p$$
$$\times \hat{M}^p\int_0^\infty \hat{\eta}(\hat{\theta}))e^{-q(\ell^q\hat{\theta})}d\hat{\theta})^p\Big(\hat{C}_5(t-\hat{\hat{\zeta}}_i)\Big)\hat{\theta}_5+\left\|A^{1-\alpha}\right\|^p$$
$$\times \hat{K}(q,\alpha)(t-\hat{\hat{\zeta}}_i)^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}\hat{C}_5\hat{\theta}_5$$
$$+\hat{M}_T^p(\ell-\hat{\hat{\zeta}}_i)^{pq-1}\Big(\frac{p-1}{pq-1}\Big)^{p-1}\hat{C}_2\hat{\theta}_2+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}$$
$$\times \hat{M}_T^p(\ell-\hat{\hat{\zeta}}_i)^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3\hat{\theta}_3\Big\}$$
$$:= \hat{\theta}_i, i=1,\cdots,k.$$

Choose $\hat{\theta} = \max_{1\leq i\leq k}\{\hat{\theta}_0+\overline{\hat{\theta}_0}+\hat{\theta}_i\}, \forall \ell\in J$. Then $E\left\|(\hat{\hat{\Psi}}x)(\ell)\right\|^p \leq \hat{\theta}$. Hence, for every $x\in\tilde{Z}$, we have $\left\|(\hat{\hat{\Psi}}x)(\ell)\right\|^p \leq \hat{\theta}$.

Step 2. $\hat{\hat{\Psi}}$ is continuous

Choose $\{x^n(\ell)\}_{n=1}^\infty \subseteq B_{\hat{r}}$ for $\lim_{n\to\infty} x^n \to x \in B_{\hat{r}}$. As, the functions $\hat{\mathfrak{f}}, \hat{\Lambda}, \mathcal{L}$ are continuous, $\forall\epsilon>0\ \exists\ N$ satisfying, $\forall n\in N$, we get

$E\left\|\mathcal{L}(\hat{\hat{\zeta}}, x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))-\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right\|^p<\epsilon$

$E\left\|\int_o^t A\mathcal{L}(\hat{\hat{\zeta}}, x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))-\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p<\epsilon$

$E\left\|\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x^n(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))-\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right\|_{\mathcal{H}}^p<\epsilon$

$E\left\|\hat{\Lambda}(\hat{\hat{\zeta}}, x^n(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))-\hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))\right\|^p<\epsilon.$

For every $\ell\in J$, we get

$$E\left\|\mathcal{L}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}}))-\mathcal{L}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right\|^p$$
$$\leq 4^{p-1}\left\|(A)^{-\alpha}\right\|^p\hat{C}_6$$
$$\times E\left\|x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}}))-y(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}}))\right\|^p_{\mathcal{H}}$$
$$\leq 4^{p-1}\left\|(A)^{-\alpha}\right\|^p\hat{C}_6\hat{\theta}_6E\left\|\int_0^t A\mathcal{L}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right.$$
$$\left.-\mathcal{L}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$\leq 4^{p-1}\left\|(1-A)^{-\alpha}\right\|^p\hat{C}_6$$
$$\times E\left\|x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}}))-y(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}}))\right\|^p$$
$$\leq 4^{p-1}K(\alpha,q)\hat{C}_6\hat{\theta}_6$$
$$E\left\|\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))-\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right\|^p_{\mathcal{H}}$$
$$\leq 4^{p-1}\hat{C}_1\hat{\theta}_1$$
$$E\left\|\hat{\Lambda}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))-\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right\|^p$$
$$\leq 4^{p-1}\hat{K}C_{\mathbb{H}}^{\frac{p}{2}}t^{p\mathbb{H}-\frac{p}{2}}\hat{C}_4\hat{\theta}_4)$$

Hence, for $\ell\in[0,\ell_1]$ by the dominated convergence theorem,

$$\sup_{\ell\in\hat{j}}E\left\|(\hat{\hat{\Psi}}x^n)(\ell).-(\hat{\hat{\Psi}}x)(\ell)\right\|^p$$
$$\leq 4^{p-1}\sup_{\ell\in\hat{J}}\left\{E\left\|\mathcal{L}(\ell,x^n(\ell-\rho_1(\ell)))\right.\right.$$
$$\left.-\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}AT_q(\ell-\hat{\hat{\zeta}})\mathcal{L}(\ell,x^n(\ell-\rho_1(\ell)))\right.$$
$$\left.-\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right.$$
$$\left.-\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\Lambda}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))\right.$$
$$\left.\left.-\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\right\}$$

$\to 0$ as $n\to\infty$. For $\ell\in(\ell_i,\hat{\hat{\zeta}}_i], i=1,\cdots,k$

$$E\left\|(\hat{\hat{\Psi}}x^n)-(\hat{\hat{\Psi}}x)(\ell)\right\|^p\leq E\left\|\mathcal{N}_i(\ell,x^n(\ell-\rho_4(\ell)))\right.$$
$$\left.-\mathcal{N}_i(\ell,x(\ell-\rho_4(\ell)))\right\|^p\to 0 \text{ as } n\to\infty.$$

For each $\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]$, $i=1,\cdots,k$, we get

$$\sup_{\ell\in(\ell_i,\ell_{i+1}]}E\left\|(\hat{\hat{\Psi}}x^n)-(\hat{\hat{\Psi}}x)(\ell)\right\|^p$$
$$\leq 6^{p-1}\sup_{\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]}\left\{E\left\|R_q(\ell-\hat{\hat{\zeta}}_i)[\mathcal{N}_i(\hat{\hat{\zeta}}_i,x^n(\hat{\hat{\zeta}}_i-\rho_4(\hat{\hat{\zeta}}_i)))\right.\right.$$
$$\left.-\mathcal{N}_i(\hat{\hat{\zeta}}_i,x(\hat{\hat{\zeta}}_i-\rho_4(\hat{\hat{\zeta}}_i)))]\right\|^p$$
$$-E\left\|R_q(\ell-\hat{\hat{\zeta}}_i)[\mathcal{L}(\hat{\hat{\zeta}}_i,x^n(\hat{\hat{\zeta}}_i-\rho_1(\hat{\hat{\zeta}}_i)))\right.$$
$$\left.-\mathcal{L}(\hat{\hat{\zeta}}_i,x(\hat{\hat{\zeta}}_i-\rho_1(\hat{\hat{\zeta}}_i)))]\right\|^p$$
$$+E\left\|[\mathcal{L}(\ell,x^n(\ell-\rho_1(t)))\right.$$
$$\left.-\mathcal{L}(t,x(t-\rho_1(t)))]\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell(\ell-\hat{\hat{\zeta}})^{q-1}AT_q(\ell-\hat{\hat{\zeta}}_i)[\mathcal{L}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right.$$
$$\left.-\mathcal{L}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))]d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell[(\ell-\hat{\hat{\zeta}})^{q-1}T_q(t-\hat{\hat{\zeta}}_i)[\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right.$$
$$\left.-\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))]d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\Lambda}(\hat{\hat{\zeta}},x^n(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))\right.$$
$$\left.\left.-\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\right\}$$

$\to 0$ as $n\to\infty$, which proves the continuity.

Step 3: The map $\hat{\hat{\Psi}}$ is a contraction. Case 1: When $\ell\in[0,\ell_1]$

$$\sup_{\ell\in\hat{j}}E\left\|(\hat{\hat{\Psi}}x)(\ell)-(\hat{\hat{\Psi}}y)(\ell)\right\|^p$$
$$\leq 4^{p-1}\sup_{\ell\in\hat{J}}\left\{E\left\|\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))\right.\right.$$
$$\left.-\mathcal{L}(\ell,y(\ell-\rho_1(\ell)))\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}AT_q(\ell-\hat{\hat{\zeta}})\right.$$
$$\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))$$
$$\left.-\mathcal{L}(\ell,y(\ell-\rho_1(\ell)))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right.$$
$$\left.-\hat{\mathfrak{f}}(\hat{\hat{\zeta}},y(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))\right.$$
$$\left.\left.-\hat{\Lambda}(\hat{\hat{\zeta}},y(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\right\}\leq\mathfrak{K}\left\|E(x-y)\right\|^p$$

For $\ell\in(\ell_i,\hat{\hat{\zeta}}_i], i=1,\cdots,k$

$$E\left\|(\hat{\hat{\Psi}}x)-(\hat{\hat{\Psi}}xy)(\ell)\right\|^p\leq E\left\|\mathcal{N}_i(\ell,x(\ell-\rho_4(\ell)))\right.$$
$$\left.-\mathcal{N}_i(\ell,y(\ell-\rho_4(\ell)))\right\|^p\leq\mathfrak{K}\left\|E(x-y)\right\|^p$$

For each $\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]$, $i=1,\cdots,k$, we get

$$\sup_{\ell\in(\ell_i,\ell_{i+1}]}E\left\|(\hat{\hat{\Psi}}x)-(\hat{\hat{\Psi}}y)(\ell)\right\|^p\leq 6^{p-1}\sup_{\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]}$$

$$\times \Bigg\{ E \left\| R_q(\ell - \hat{\hat{\zeta}}_i)[\mathcal{N}_i(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_4(\hat{\hat{\zeta}}_i)))\right.$$
$$\left. -\mathcal{N}_i(\hat{\hat{\zeta}}_i, y(\hat{\hat{\zeta}}_i - \rho_4(\hat{\hat{\zeta}}_i)))] \right\|^p$$
$$- E \left\| R_q(\ell - \hat{\hat{\zeta}}_i)[\mathcal{L}(\hat{\hat{\zeta}}_i, x(\hat{\hat{\zeta}}_i - \rho_1(\hat{\hat{\zeta}}_i)))\right.$$
$$\left. -\mathcal{L}(\hat{\hat{\zeta}}_i, y(\hat{\hat{\zeta}}_i - \rho_1(\hat{\hat{\zeta}}_i)))] \right\|^p$$
$$+ E \left\| [\mathcal{L}(\ell, x(\ell - \rho_1(t)))\right.$$
$$\left. -\mathcal{L}(t, y(t - \rho_1(t)))] \right\|^p$$
$$+ E \left\| \int_{\hat{\hat{\zeta}}_i}^{\ell} (\ell - \hat{\hat{\zeta}})^{q-1} A T_q(\ell - \hat{\hat{\zeta}}_i)\right.$$
$$\times [\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})))$$
$$\left. -\mathcal{L}(\hat{\hat{\zeta}}, y(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})))] d\hat{\hat{\zeta}} \right\|^p$$
$$+ E \left\| \int_{\hat{\hat{\zeta}}_i}^{\ell} [(\ell - \hat{\hat{\zeta}})^{q-1} T_q(t - \hat{\hat{\zeta}}_i)\right.$$
$$[\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})))$$
$$\left. -\hat{\mathfrak{f}}(\hat{\hat{\zeta}}, y(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}})))] d\hat{\hat{\zeta}} \right\|^p$$
$$+ E \left\| \int_{\hat{\hat{\zeta}}_i}^{\ell} (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}})\right.$$
$$\hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}})))$$
$$\left. -\hat{\Lambda}(\hat{\hat{\zeta}}, y(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}) \right\|^p \Bigg\}$$
$$\leq \mathfrak{K} \left\| E(x - y) \right\|^p$$

$\forall \ell, E \left\| (\hat{\hat{\Psi}} x - \hat{\hat{\Psi}} y)(\ell) \right\|^p \leq \mathfrak{K} \|x - y\|^p$. Therefore, for every $x \in \tilde{Z}$, the NFSDEs have obtained the mild solution of $(1) - (3)$.

4. Ulam-Hyers stability

To prove Ulam-Hyers stability, we need the following additional assumption.

(A_6): The mapping $\mathcal{L} : [0, \infty) \times \mathcal{H} \to \mathcal{H}$ is continuous in the p^{th} mean sense and satisfies, for any $x, \hat{y} \in \mathcal{H}$ and $\ell \geq 0$, $\mathcal{L}(\ell, x(\ell - \rho_1)) \in D(A^\alpha)$, such that

$$\lim_{t \to \hat{\hat{\zeta}}} E \left\| (A)^\alpha \mathcal{L}(\ell, x(\ell - \rho_1)) - (A)^\alpha \mathcal{L}(\hat{\hat{\zeta}}, \hat{y}(\ell - \rho_1)) \right\|^p$$
$$\leq \hat{C}_5 E \left\| x(\ell) - \hat{y}(\ell) \right\|^p,$$

Theorem 2. *If the hypotheses of Theorem 3.1. and (A_6) are hold, then $(1) - (3)$ is Ulam-Hyers stable.*

Proof. We now consider the Ulam stability of the solution

If $\exists$ a function $f_1(\ell, \hat{y}(\ell - \rho_2(\ell)))$ such that $\|f_1(\ell, x(\ell - \rho_2(\ell))) - f(\ell, \hat{y}(\ell - \rho_2(\ell)))\| < \epsilon$

Consider the following equation

$$^C D_{0^+}^q [\hat{y}(\ell) - \mathcal{L}(\ell, \hat{y}(\ell - \rho_1(\ell)))]$$
$$= \left[A\hat{y}(\ell) + \hat{\mathfrak{f}}_1(\ell, \hat{y}(\ell - \rho_2(\ell))) \right]$$
$$+ \Lambda(\ell, \hat{y}(\ell - \rho_3(\ell))) \frac{dS_Q^{\mathbb{H}}(\ell)}{d\ell} + V(\ell) \;\; \ell \in (\hat{\hat{\zeta}}_i, \ell_{i+1}]$$
$$\hat{y}(\ell) = \mathcal{N}_i(\ell, \hat{y}(\ell - \rho_4))), \ell \in (\ell_i, \hat{\hat{\zeta}}_i], \;\; i = 1, 2, \ldots, k,$$
$$\hat{y}_0(\cdot) = \varphi \in \mathcal{B}_{\mathcal{F}_0}([\hat{m}(0), 0], \mathcal{H}).$$

□

Using the solution of **Equation (5)**, we obtain

$$(\hat{y})(\ell) = R_q(\ell)[\phi(0) - \mathcal{L}(0, \varphi(\ell - \rho_1(0)))]$$
$$+ \mathcal{L}(\ell, \hat{y}(\ell - \rho_1(\ell)))$$
$$+ \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}})$$
$$A\mathcal{L}(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}$$
$$+ \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell$$
$$- \hat{\hat{\zeta}}) \hat{\mathfrak{f}}_1(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}$$
$$+ \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell)$$
$$- \hat{\hat{\zeta}}) \hat{\Lambda}(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}), \;\; \ell \in [0, \ell_1]$$
$$\mathcal{N}_i(\ell, \hat{y}(\ell - \rho_4(\ell))),$$
$$\ell \in (\ell_i, \hat{\hat{\zeta}}_i], \;\; i = 1, \cdots, k$$

$$R_q(\ell - \hat{\hat{\zeta}}_i) \Big[\mathcal{N}_i(\hat{\hat{\zeta}}_i, \hat{y}(\hat{\hat{\zeta}}_i - \rho_4(\hat{\hat{\zeta}}_i)))$$
$$- \mathcal{L}(\hat{\hat{\zeta}}_i, \hat{y}(\hat{\hat{\zeta}}_i - \rho_1(\hat{\hat{\zeta}}_i))) \Big] + \mathcal{L}(\ell, \hat{y}(\ell - \rho_1(t)))$$
$$+ \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) A\mathcal{L}(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}$$
$$+ \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}_1(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}}$$
$$+ \int_{\hat{\hat{\zeta}}_i}^\ell (\ell - \hat{\hat{\zeta}})^{q-1} t T_q(\ell - \hat{\hat{\zeta}})$$
$$\times \hat{\Lambda}(\hat{\hat{\zeta}}, \hat{y}(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}), \;\; \ell \in (\hat{\hat{\zeta}}_i, \ell_i + 1].$$

It is clear that the solution is Ulam stable in J $\forall \in [0, \ell_1]$, and let $\epsilon < 1$ we get

$$E \left\| x(\ell) - \hat{y}(\ell) \right\|^p$$
$$\leq 4^{p-1} \Bigg\{ E \left\| \mathcal{L}(\ell, x(\ell - \rho_1(\ell)))\right.$$
$$\left. -\mathcal{L}(\ell, \hat{y}(\ell - \rho_1(\ell))) \right\|^p$$
$$+ E \left\| \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}})\right.$$
$$\times \mathcal{L}(\ell, x(\ell - \rho_1(\ell)))$$
$$\left. -A\mathcal{L}(\ell, \hat{y}(\ell - \rho_1(\ell))) \right\|^p$$

$$+E\left\|\int_0^t(\ell-\hat{\hat{\zeta}})^{q-1}T_q((\ell-\hat{\hat{\zeta}})\right.$$
$$\times[\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))$$
$$\left.-\hat{\mathfrak{f}}_1(\hat{\hat{\zeta}},\hat{y}(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))]d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_0^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})\right.$$
$$\times[\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3))$$
$$\left.\left.-\hat{\Lambda}(\hat{\hat{\zeta}},\hat{y}(\hat{\hat{\zeta}}-\rho_3))]dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\right\}$$

$$\leq 4^{p-1}\left\{\left[\hat{M}_T^p\left\|(A)^{-\alpha}\right\|^p\hat{C}_5\right.\right.$$
$$\times E\left\|x(\ell)-\hat{y}(\ell)\right\|^p+K(\alpha,q)$$
$$\times\ell_1^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}\hat{C}_5$$
$$\times E\left\|x(\ell)-\hat{y}(\ell)\right\|^p+\hat{M}_T^p\big(\frac{\ell^{pq}}{pq}\big)^p\epsilon+\tilde{K}$$
$$C_{\mathbb{H}}^{\frac{p}{2}}t^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p\ell_1^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}$$
$$\left.\left.\ell^{p\mathbb{H}-\frac{p}{2}}\hat{C}_3\hat{\theta}_3E\left\|x(\ell)-\hat{y}(\ell)\right\|^p\right]\right\}$$

$$E\left\|x(\ell)-\hat{y}(\ell)\right\|^p-4^{p-1}\left\{\left[\left\|(A)^{-\alpha}\right\|^p\right.\right.$$
$$\times\hat{C}_5E\left\|x(\ell)-\hat{y}(\ell)\right\|^p$$
$$+\ell_1^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}K(\alpha,\delta)$$
$$\times\hat{C}_5E\left\|x(\ell)-\hat{y}(\ell)\right\|^p$$
$$+\hat{M}_T^p\ell_1^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{C}_3\hat{\theta}_3$$
$$\left.\left.\times E\left\|x(\ell)-\hat{y}(\ell)\right\|^p\right]\right\}\leq 4^{p-1}\hat{M}_T^p\big(\frac{\ell^{pq}}{pq}\big)\epsilon.$$

$$E\left\|x(\ell)-\hat{y}(\ell)\right\|^p\left\{1-4^{p-1}\left[\left\|(A)^{-\alpha}\right\|^p\hat{C}_5\right.\right.$$
$$+\ell_1^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}k(\alpha,q)\hat{C}_5$$
$$\left.\left.+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p\ell_1^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3\right]\right\}$$
$$\leq 4^{p-1}\hat{M}_T^p\big(\frac{\ell^{pq}}{pq}\big)\epsilon.$$

Choosen$\hat{n}_1=1-4^{p-1}\Big[\left\|(A)^{-\alpha}\right\|^p\hat{C}_5$

$$+\ell_1^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}K(\alpha,q)\hat{C}_5$$
$$+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}t^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p\ell_1^{\frac{2pq-p-1}{2}}$$
$$\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3\Big]E\left\|x(\ell)-\hat{y}\right\|_{\mathcal{H}}^p$$
$$\leq\frac{4^{p-1}\hat{M}_T^p\big(\frac{\ell^{pq}}{pq}\big)\epsilon}{\hat{n}_1}=\mathcal{G}_0\epsilon,$$

where $\mathcal{G}_0=\frac{4^{p-1}\hat{M}_T^p\left(\frac{\ell^{pq}}{pq}\right)\epsilon}{\hat{n}_1}$.

Similarly for $\epsilon>0$, and a solution $\hat{y}(\ell)$ $\forall\ell\in(\ell_i,\hat{\hat{\zeta}}_i]$, $i=1,\cdots,k$, one can get

$$E\left\|x(\ell)-\hat{y}(\ell)\right\|_{\mathcal{H}}^p$$
$$\leq 5^{p-1}E\left\|\mathcal{N}_i(\ell,x(\ell-\rho_4(\ell))-\mathcal{N}_i(\ell,\hat{y}(\ell-\rho_4(\ell))\right\|^p+\epsilon$$
$$\leq\hat{C}_{\mathcal{N}_i}E\left\|\mathcal{N}_i(\ell,x(\ell-\rho_4(\ell))\right.$$
$$\left.-\mathcal{N}_i(\ell,\hat{y}(\ell-\rho_4(\ell))\right\|^p+\epsilon$$
$$\leq 5^{p-1}\max_{(i=1,\cdots,k)}\hat{C}_{\mathcal{N}_i}E\left\|x(\ell)-\hat{y}(\ell)\right\|^p+\epsilon$$

That is,

$$E\left\|x(\ell)-\hat{y}(\ell)\right\|^p-5^{p-1}\max_{(i=1,\cdots,k)}\hat{C}_{\mathcal{N}_i}E\left\|x(\ell)-\hat{y}(\ell)\right\|^p$$
$$\leq\epsilon$$
$$E\left\|x(\ell)-\hat{y}(\ell)\right\|^p\Big(1-5^{p-1}\max_{(i=1,\cdots,k)}\hat{C}_{\mathcal{N}_i}\Big)\leq\epsilon$$

where $\mathcal{G}_1=\frac{1}{\left(1-5^{p-1}\max_{(i=1,\cdots,k)}\hat{C}_{\mathcal{N}_i}\right)}$ is a constant.

Thus $E\left\|x(\ell)-\hat{y}(\ell)\right\|_{\mathcal{H}}^p\leq\mathcal{G}_1\epsilon\Rightarrow(1)-(3)$ is Hyers- Ulam stable.

For every $\ell\in(\hat{\hat{\zeta}}_i,\ell_{i+1}]$, similarly, we obtain

$$E\left\|x(\ell)-\hat{y}(\ell)\right\|^p$$
$$\leq 5^{p-1}\left\{((\int_0^\infty\hat{\eta}_q(\hat{\theta})e^{-q((\ell-\hat{\hat{\zeta}}_i)^q\hat{\theta})})d\hat{\theta}\right.$$
$$E\left\|\big[\mathcal{N}_i(\hat{\hat{\zeta}}_i,x(\hat{\hat{\zeta}}_i-\rho_4(\hat{\hat{\zeta}}_i)))-\mathcal{N}_i(\hat{\hat{\zeta}}_i,\hat{y}(\hat{\hat{\zeta}}_i-\rho_4(\hat{\hat{\zeta}}_i)))\big]\right.$$
$$\left.-\big[\mathcal{L}(\hat{\hat{\zeta}}_i,x(\hat{\hat{\zeta}}_i-\rho_1(\hat{\hat{\zeta}}_i)))-\mathcal{L}(\hat{\hat{\zeta}}_i,\hat{y}(\hat{\hat{\zeta}}_i-\rho_1(\hat{\hat{\zeta}}_i)))\big]\right\|^p$$
$$+E\left\|\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))-\mathcal{L}(\ell,\hat{y}(\ell-\rho_1(\ell)))]\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})A\mathcal{L}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))\right.$$
$$\left.-A\mathcal{L}(\hat{\hat{\zeta}},\hat{y}(\hat{\hat{\zeta}}-\rho_1(\hat{\hat{\zeta}})))d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q((\ell-\hat{\hat{\zeta}})[\hat{\mathfrak{f}}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))\right.$$
$$\left.-\hat{\mathfrak{f}}_1(\hat{\hat{\zeta}},\hat{y}(\hat{\hat{\zeta}}-\rho_2(\hat{\hat{\zeta}})))]d\hat{\hat{\zeta}}\right\|^p$$
$$+E\left\|\int_{\hat{\hat{\zeta}}_i}^\ell(\ell-\hat{\hat{\zeta}})^{q-1}T_q(\ell-\hat{\hat{\zeta}})[\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3))\right.$$
$$\left.\left.-\hat{\Lambda}(\hat{\hat{\zeta}},\hat{y}(\hat{\hat{\zeta}}-\rho_3))]dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}})\right\|^p\right\}$$

$$\leq 5^{p-1}\left\{\hat{M}(\mathcal{G}_1\epsilon-\left\|(A)^{-\alpha}\right\|^p\hat{C}_5)\right.$$
$$\Big[+\left\|(A)^{-\alpha}\right\|^p\hat{C}_5+(\ell-\hat{\hat{\zeta}}_i)^{pq-pq\alpha-1}$$

$$\times\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}$$
$$\times K(\alpha,q)\mathcal{L}\,\|x(\ell-\rho_1)\|^p$$
$$+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p(\ell-\hat{\hat{\zeta}}_i)^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}$$
$$\hat{C}_3(\ell)\hat{\theta}_3(\|x(\ell-\rho_3(\ell))\|^p)\Big]$$
$$\times E\,\|x(\ell)-\hat{y}(\ell)\|_{\mathcal{H}}^p+\hat{M}_T^p\Big(\frac{(\ell-\hat{\hat{\zeta}}_i)^{pq}}{pq}\Big)\epsilon\Big\}$$
$$E\,\|x(\ell)-\hat{y}(\ell)\|^p$$
$$\le 5^{p-1}\Big[\,\|(A)^{-\alpha}\|^p\,\hat{C}_5$$
$$+(\ell-\hat{\hat{\zeta}}_i)^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}K(\alpha,q)$$
$$\times\mathcal{L}\,\|x(\ell-\rho_1)\|^p+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p\ell^{\frac{2pq-p-1}{2}}$$
$$\times\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3(t)\hat{\theta}_3(\|x(\ell-\rho_3(\ell))\|^p)\Big]$$
$$\times E\,\|x(t)-\hat{y}(\ell)\|^p$$
$$=E\,\|x(\ell)-\hat{y}(\ell)\|^p\Bigg[1-5^{p-1}\Big[\,\|(A)^{-\alpha}\|^p\,\hat{C}_5$$
$$+\|(A)^{-\alpha}\|^p\,\hat{C}_5+(\ell-\hat{\hat{\zeta}}_i)^{pq-pq\alpha-1}$$
$$+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}\hat{M}_T^p(\ell-\hat{\hat{\zeta}}_i)^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}$$
$$\times\hat{C}_3(\ell)\hat{\theta}_3(\|x(\ell-\rho_3(\ell))\|^p)\Big]\Bigg]$$
$$\le 5^{p-1}\Big\{\hat{M}(\mathcal{G}_1\epsilon-\|(A)^{-\alpha}\|^p\,\hat{C}_5)$$
$$+\hat{M}_T^p\Big(\frac{(\ell-\hat{\hat{\zeta}}_i)^{pq}}{pq}\Big)\epsilon\Big\}.\text{choose }\hat{n}_2(t)$$
$$=\Bigg[1-5^{p-1}\Big[\,\|(A)^{-\alpha}\|^p\,\hat{C}_5$$
$$+(t-\hat{\hat{\zeta}}_i)^{pq-pq\alpha-1}\Big(\frac{p-1}{pq-pq\alpha-1}\Big)^{p-1}$$
$$K(\alpha,q)\mathcal{L}\,\|x(\ell-\rho_1)\|^p+\tilde{K}C_{\mathbb{H}}^{\frac{p}{2}}\ell^{p\mathbb{H}-\frac{p}{2}}$$
$$\times\hat{M}_T^p(t-\hat{\hat{\zeta}}_i)^{\frac{2pq-p-1}{2}}\Big(\frac{p-2}{2pq-p-2}\Big)^{\frac{p-2}{2}}\hat{C}_3(\ell)$$
$$\hat{\theta}_3(\|x(\ell-\rho_3(\ell))\|^p)\Big]\Bigg]$$
$$E\,\|x(\ell)-\hat{y}(\ell)\|_{\mathcal{H}}^p$$
$$\le\frac{5^{p-1}\Big\{\hat{M}(\mathcal{G}_1\epsilon\,\|(A)^{-\alpha}\|^p\,\hat{C}_5)+\hat{M}_T^p\Big(\frac{(\ell-\hat{\hat{\zeta}}_i)^{pq}}{pq}\Big)\epsilon\Big\}}{\hat{n}_2}\le\mathcal{G}_2\epsilon,$$

where $\mathcal{G}_2=\dfrac{5^{p-1}\Big\{\hat{M}(\mathcal{G}_1\epsilon\|(A)^{-\alpha}\|^p\hat{C}_5)+\hat{M}_T^p\Big(\frac{(\ell-\hat{\hat{\zeta}}_i)^{pq}}{pq}\Big)\epsilon\Big\}}{\hat{n}_2}$.

Hence $E\,\|x(\ell)-\hat{y}(\ell)\|^p\le\mathcal{G}\epsilon$, where $\mathcal{G}=max\{\mathcal{G}_0,\mathcal{G}_1,\mathcal{G}_2,\cdots\}$ is a constant. Thus, system $(1)-(3)$ is Ulam-Hyers stable.

We now illustrate the theoretical results with numerical examples.

$$\text{Illustration } {}^{C}D_{0^+}^{1.5}[x(\ell)-\frac{1}{e^{2\ell}}cos(x(\ell-\frac{\ell}{3}))]$$
$$=0.2x+\frac{1}{e^{\ell}}sinx(\ell-\ell/4)$$
$$+\frac{1}{e^{2\ell}}sinx(\ell-\ell/6)\frac{dS_Q^{\mathbb{H}}(\ell)}{d\ell},\ \ell\in(5,10]$$
$$x(\ell)=\frac{1}{e^{3\ell}}cos^2x(\ell-\ell/5),\ \ell\in(2,5]$$
$$x_0(\cdot)=\varphi\in([0,2],0).$$

From $(1)-(3)$, for $\ell\in[0,2]$, $\mathcal{L}^*(\ell)=0.490842$, and for $\ell\in(5,10]$, $\mathcal{L}^*(\ell)=0.0000226989$. $\hat{\mathfrak{f}}(t,x(t-\rho_2(t)))=\frac{1}{e^{\ell}}sin(x(\ell-\frac{\ell}{4}))$. and $\hat{\Lambda}(\hat{\hat{\zeta}},x(\hat{\hat{\zeta}}-\rho_3(\hat{\hat{\zeta}})))=\frac{1}{e^{2\ell}}sinx(\ell-\ell/6)$. By substituting these values, we get $\mathfrak{K}=0.70053\in(0,1)$. This means the obtained theory has a solution. $\mathcal{G}=max\{0.09517,1.00061,0.08387\},\mathcal{G}=1.00061>0$ is a constant. Hence the system $(1)-(3)$ is Hyers-Ulam stable.

Here $q=1.5$, $0.2x(\ell)+\frac{1}{e^{\ell}}sinx(\ell-\ell/4)$ is the drift coefficient, and ${}^{C}D^{\frac{1}{2}}x(\ell)$ denotes the state space rate of change, where $x(\ell)$ represents the newly connected stem cells and β cells.

The term $h(\ell,x_{\ell})=\frac{1}{e^{2\ell}}cos(x(\ell-\frac{\ell}{\ell/3})$ is used for self-balancing and also for lossless transmission between stem cells and β cells.

The term $f(\ell,x(\ell))=\frac{1}{e^{\ell}}sinx(\ell-\ell/4)$ represents the function that describes signal transformation between β cells.

The stochastic component of the sub-fBm process is taken as $\hat{\Lambda}(\ell,x(\ell))=\frac{1}{e^{2\ell}}sinx(\ell-\ell/6)$ which represents the death of β cells.

4.1. Disturbance rejection

The stochastic disturbance (death of β cells) is explained through a fractional differential analog circuit:

$$\begin{aligned}{}^{C}D_{0^+}^{q}&[x(\ell)-\mathcal{L}(\ell,x(\ell-\rho_1(\ell)))]\\&=\big[Ax(\ell)+\hat{\mathfrak{f}}(\ell,x(\ell-\rho_2(\ell)))\big]\\&-\Lambda(\ell,x(\ell-\rho_3(\ell)))\frac{dS_Q^{\mathbb{H}}(\ell)}{d\ell}\\&+\mathfrak{h}(\ell,x(\ell-\rho_5(\ell)))\ \ell\in[0,b]:=\hat{J}\\x_0(\cdot)&=\varphi\in\mathcal{B}_{\mathcal{F}_0}([\hat{m}(0),0],\mathcal{H})\end{aligned}\qquad(7)$$

Here $\hat{\mathfrak{f}}(\ell,x(\ell-\rho_2(\ell)))=K_p\frac{C(\ell-\rho(\ell))}{C(\ell-\rho(\ell))+K_d}$, $\mathfrak{h}(\ell,x(\ell-\rho_5(\ell)))=S_p\frac{C(\ell-\rho(\ell))}{C(\ell-\rho(\ell))+K_d}$ which represent the stem cell terms, and $\Lambda(\ell,x(\ell-\rho_3(\ell)))=K_{deg}AI1$.

One can follow a similar method as in System (1) to obtain the existence and stability results for (14) as well. When $\mathfrak{h}(\ell, x(\ell-\rho_5(\ell))) = \frac{1}{e^{\ell}} sin(x(\ell - \frac{\ell}{5}))$ and using the values of $\hat{\mathfrak{f}}(\ell, x(\ell - \rho_2(\ell))), \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}})))$ as given in the previous numerical calculation.

In **Figure 1** shows the stability of system $(1)-(3)$ with non-instantaneous impulsive NFSDEs. Here, data 1 corresponds to $\ell \in [0, 2]$ and $\ell \in (2, 5]$ at the fractional power $q = 0.6$ and $q = 0.8$. One can observe the discontinuity between the impulses in the unstable and stable regions, indicating no transformation from dead β cells to other living cells.

In **Figure 2** illustrates the stability of system $(1)-(3)$ with non-instantaneous impulsive NFSDEs. Here, data 1 denotes $\ell \in (2, 5]$ and $\ell \in (5, 10]$ at the fractional power $q = 0.6$ and $q = 0.8$. The discontinuity between unstable and stable impulses is clearly visible, reflecting the absence of transformation from dead β cells to other living cells.

The approximate formula for q can be obtained from the complex frequency domain of the equivalent circuit in chain form, as shown in **Figure 3** for $q = 0.9$, according to circuit theory in the Laplace domain.

Figure 4 depicts the regular mechanism of the stem cell in detail: the diffusion part is shown in green the fractionally differentiated cell in black, and $Al1$ and the growth-affecting factor (GAF) in pink.

In **Figure 5**, the value of P_{SC} is attained through Kirchhoff's current law by subtracting Pss and P_{CC} from 1. Automatic regulation of P_{SS} and P_{CC} by the feedback loop occurs while the population of S remains stable.

An analog feedback circuit has been designed to improve performance while reducing instability. As shown in **Figure 6**, one can observe the precise homeostasis analog circuit controlling stem cells and β cells. The differentiated and stem-cell population numbers are represented by the C and S variables, respectively. S and C secrete AIC and AIS, which are diffusible molecules. AIC activates AC and RC, whereas AIS activates AS and RS, which in turn activate GAF, A_{CC}, and A_{SS} downstream signalling pathways, respectively. GAF is an effector molecule. A_{SS} governs SS division, A_{CC}regulates CC division, and Acc regulates overall growth. To govern differentiation, A_{CC} and A_{SS} enhance their respective conductances.P_{SS}, P_{CC}, and P_{SC} are probabilities indicated by current fluxes; the signal flows are represented by neutral terms. The FFL is implemented by RFFL (slow suppression), AFFL (fast activation), and AFFLOUT, which combine these to provide transient control signals at ASD^qC. The control of A_{SS} by ASF^qC promoting SS division whenever the risk of differentiated- cell death is high. The authors incorporated an incoherent feed-forward loop; its inclusion reduces steady-state error, which is equivalent to adding a fractional-order capacitor while significantly increasing loop gain. In analog circuit design, the feed-forward loop operates as a lead-zero compensator. P_{ss} is increased proportionally to $-D^qC$. The latter derivative approximated by the difference between two relatively fast derivatives and a filtered version of C.

From **Figure 7** shows the stability of signal transformation between healthy β cells and stem cells, without abrupt changes due to dead β cells. The advantage is that the unstable signal transformation of damaged cells is converted into stable signal transformation among cells through fractional power, which outperforms integer-order approaches.

In,[29] the authors describe that 70% recovery occurs in the feedback control shown by the differential circuit. The proposed model is enhanced into an NFSDEs analog circuit. On the other hand, the fractional differential equation approach creates a strong link between the classical affected model and a class of renewal models that can be used in more complex settings. We can test the capacity of the fractional-order circuit to reject stochastic disturbances, which supports significantly higher recovery. Generally, stem cells (S) divide into two stem cells (SS division), immediate two daughter cells (CC division), or stem cells and primary two daughter cells (asymmetrical division) (SC division). The mechanism of stem cell division, whether SS, CC, or SC, must be carefully controlled to provide optimal homeostatic behaviour when growth management is critical. The probability of each division mode P_{SS}, P_{CC}, and Psc sums to 1 in **Figure 5**, because a stem cell can only undergo one of the three types of differentiation. If the ideal ratio of stem cells to developed cells (C) is not optimal, the system adjusts the division probabilities. This division supports restoring the healthy level of stem cell population and restoring C through fractional differentiation. In **Figure 5** P_{cc} is proportional to $\frac{1}{S+C}$ and P_{CC} is proportional to $\frac{S}{C}$. There is no need to consider division when S and C are at the finest adequate population size. By applying the fractional-order circuit's ability to reject stochastic disturbances, recovery is greatly improved. The resulting closed-loop system can be obtained via Kirchhoff's law:

$$\begin{aligned}
&{}^{C}D^{q}_{0+}[x(\ell) - \mathcal{L}(\ell, x(\ell - \rho_1(\ell)))] \\
&+ \frac{\sigma^{1-q}}{RC_1} x(\ell) = \hat{\mathfrak{f}}(\ell, x(\ell - \rho_2(\ell))) \\
&- \Lambda(\ell, x(\ell - \rho_3(\ell))) \frac{dS^{\mathbb{H}}_{Q}(\ell)}{d\ell} \\
&+ \mathfrak{h}(\ell, x(\ell - \rho_5(\ell)))\ \ell \in [0, b] := \hat{J}, \ell \neq \ell_i \\
x_0(\cdot) &= \varphi \in \mathcal{B}_{\mathcal{F}_0}([\hat{m}(0), 0], \mathcal{H}) \qquad (8)
\end{aligned}$$

In this case, $x(0) = x_0$ defines the initial condition. The expression $\hat{\mathfrak{f}}(\ell, x(\ell - \rho_2(\ell))) - \Lambda(\ell, x(\ell - \rho_3(\ell)))\frac{dS^{\mathbb{H}}_{Q}(\ell)}{d\ell} + \mathfrak{h}(\ell, x(\ell - \rho_5(\ell))) = \frac{\sigma^{1-q}}{RC_1} E(\ell)$ represents the exogenous input; the parameter σ is associated with the system's temporal components R, C_1 denote the resistor and capacitor, respectively; the source is the temporary parameter σ^{1-q} and $E(\ell)$ which eventually depend on time. The stability of signal transformation between stem cells and fractionally developed cells using the RC_1 circuit is demonstrated by the following theorem.

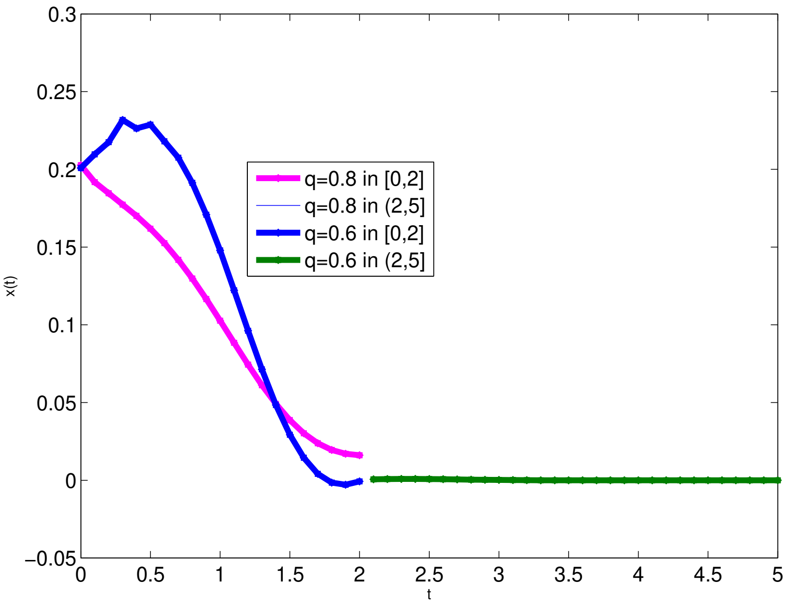

Figure 1. Stability trajectory for NFSDEs with non-instantaneous impulses on $\ell \in [0, 2]$, and $\ell \in (2, 5]$, showing fractional powers at $q = 0.6$ and $q = 0.8$

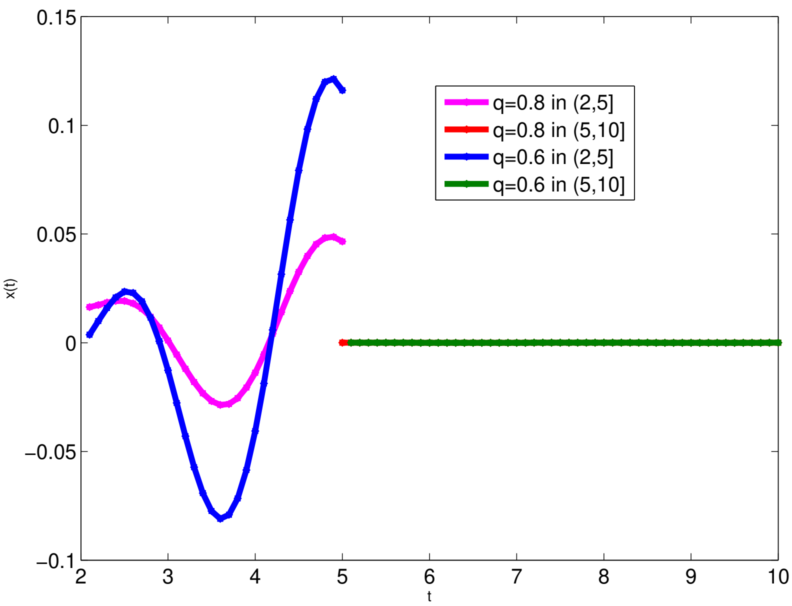

Figure 2. Stability trajectory for NFSDEs with non-instantaneous impulses on $\ell \in (2, 5]$ and $\ell \in (5, 10]$ showing fractional powers at $q = 0.6$ and $q = 0.8$

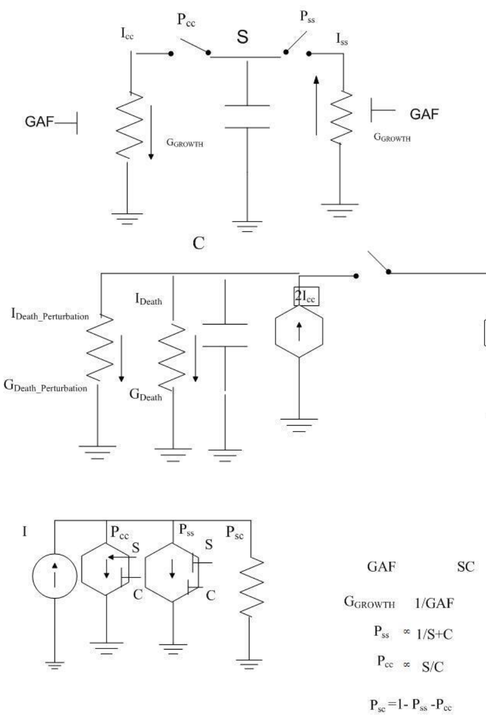

Figure 3. The feedback mechanism of the homeostasis circuit, which promotes G-growth via GAF

Theorem 3. *The electrical RC_1 circuit defined by **Equation (8)** and characterized by the Caputo fractional derivative is stable.*

Proof. Let $\tilde{\mu} > 0$. Applying the Laplace transform to **Equation (14)**, we get

$$
\begin{aligned}
&S^q L[x(\tilde{\mu}) - \mathcal{L}(\tilde{\mu}, x(\tilde{\mu} - \rho_1(\tilde{\mu})))] - S^{q-1}x_0 \\
&+ \frac{\sigma^{1-q}}{RC_1} L[x(\tilde{\mu})] = L[\hat{\mathfrak{f}}(\tilde{\mu}, x(\tilde{\mu} - \rho_2(\tilde{\mu})))] \\
&- L[\Lambda(\tilde{\mu}, x(\tilde{\mu} - \rho_3(\tilde{\mu})))\frac{dS_Q^{\mathbb{H}}(\tilde{\mu})}{d\tilde{\mu}}] \\
&+ L[\mathfrak{h}(\tilde{\mu}, x(\tilde{\mu} - \rho_5(\tilde{\mu})))] \\
&+ S^{q-1} \sum_{i \in [1, \cdots, k]} I(\tilde{\mu})(x(\tilde{\mu} - \rho_4(\tilde{\mu})_i))
\end{aligned}
$$

$$
\begin{aligned}
&L[x(\tilde{\mu})\Big\{S^q + \frac{\sigma^{1-q}}{RC_1}\Big\} = S^{q-1} \\
&\times [x_0 - \mathcal{L}(o, x(0 - \rho_2(0)))] + S^q L[\mathcal{L}(\tilde{\mu}, x(\tilde{\mu} - \rho_1(\tilde{\mu})))] \\
&+ L[\hat{\mathfrak{f}}(\tilde{\mu}, x(\tilde{\mu} - \rho_2(\tilde{\mu})))] - L[\Lambda(\ell, x(\tilde{\mu} - \rho_3(\tilde{\mu})))\frac{dS_Q^{\mathbb{H}}(\tilde{\mu})}{d\tilde{\mu}}] \\
&+ L[\mathfrak{h}(\tilde{\mu}, x(\tilde{\mu} - \rho_5(\tilde{\mu})))]
\end{aligned}
$$

That is, $L[x(\tilde{\mu})]$

$$
\begin{aligned}
&= S^{q-1}\Big(S^q I + \frac{\sigma^{1-q}}{RC_1}\Big)^{-1}[x_0 - \mathcal{L}(o, x(0 - \rho_2(0)))] \\
&+ S^q\Big(S^q I + \frac{\sigma^{1-q}}{RC_1}\Big)^{-1} L[\mathcal{L}(\tilde{\mu}, x(\tilde{\mu} - \rho_1(\tilde{\mu})))] \\
&+ \Big(S^q I + \frac{\sigma^{1-q}}{RC_1}\Big)^{-1} L[\hat{\mathfrak{f}}(\tilde{\mu}, x(\tilde{\mu} - \rho_2(\tilde{\mu})))] \\
&- \Big(S^q I + \frac{\sigma^{1-q}}{RC_1}\Big)^{-1} L[\Lambda(\ell, x(\tilde{\mu} - \rho_3(\tilde{\mu})))\frac{dS_Q^{\mathbb{H}}(\tilde{\mu})}{d\tilde{\mu}}] \\
&+ \Big(S^q I + \frac{\sigma^{1-q}}{RC_1}\Big)^{-1} L[\mathfrak{h}(\tilde{\mu}, x(\tilde{\mu} - \rho_5(\tilde{\mu})))]
\end{aligned}
$$

Let $L[x(\tilde{\mu})] = V(\tilde{\mu}) = \int_0^\infty e^{-\tilde{\mu}\hat{\hat{\zeta}}} x(\hat{\hat{\zeta}}) d\hat{\hat{\zeta}}$, where L is the usual Laplace transform, and let $\frac{\sigma^{1-q}}{RC_1} = \hat{A}$, with I is the identity operator. Then we obtain

$$
\begin{aligned}
&V(\tilde{\mu}) = S^{q-1} \int_0^\infty e^{-\tilde{\mu}\hat{\hat{\zeta}}}[x_0 - \mathcal{L}(o, x(0 - \rho_2(0)))] d\hat{\hat{\zeta}} \\
&+ S^q \int_0^\infty T(\hat{\hat{\zeta}}) e^{-\tilde{\mu}\hat{\hat{\zeta}}} \mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\
&+ \int_0^\infty e^{-\tilde{\mu}\hat{\hat{\zeta}}} T(\hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\
&- \int_0^\infty e^{-\tilde{\mu}\hat{\hat{\zeta}}} T(\hat{\hat{\zeta}}) \Lambda(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}})) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}) \\
&+ \int_0^\infty e^{-\tilde{\mu}\hat{\hat{\zeta}}} \mathfrak{h}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_5(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \ \ell \in [0, b] := \hat{J}, \ell \neq \ell_i.
\end{aligned}
$$

□

From,[18,26] we have

$$
\begin{aligned}
x(\ell) &= \hat{R}_q(\ell)[\phi(0) - \mathcal{L}(0, \varphi(\ell - \rho_1(0)))] \\
&+ \mathcal{L}(\ell, x(\ell - \rho_1(\ell))) + \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} A \hat{T}_q(\ell - \hat{\hat{\zeta}}) \\
&\mathcal{L}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_1(\hat{\hat{\zeta}}))) d\hat{\hat{\zeta}} \\
&+ \int_0^\ell (\ell - \hat{\hat{\zeta}})^{q-1} T_q(\ell - \hat{\hat{\zeta}}) \hat{\mathfrak{f}}(\hat{\hat{\zeta}} - \rho_2(\hat{\hat{\zeta}})) d\hat{\hat{\zeta}} \\
&+ \int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} \hat{T}_q(\ell - \hat{\hat{\zeta}}) \hat{\Lambda}(\hat{\hat{\zeta}}, x(\hat{\hat{\zeta}} - \rho_3(\hat{\hat{\zeta}}))) dS_Q^{\mathbb{H}}(\hat{\hat{\zeta}}) \\
&+ \int_0^t (\ell - \hat{\hat{\zeta}})^{q-1} \hat{T}_q(\ell - \hat{\hat{\zeta}}) \mathfrak{h}(\ell, x(\ell - \rho_5(\ell))) d\hat{\hat{\zeta}}, \qquad (9)
\end{aligned}
$$

where $\hat{R}_q(\ell)x = \int_0^\infty e^{-RC_1} \hat{\eta}(\hat{\theta}) R(\ell^q \hat{\theta}) x d\hat{\theta}$, $\hat{T}_q(\ell)x = q \int_0^\infty e^{-RC_1} \hat{\theta} \hat{\eta}(\hat{\theta}) R(\ell^q \hat{\theta}) x d\hat{\theta}$.

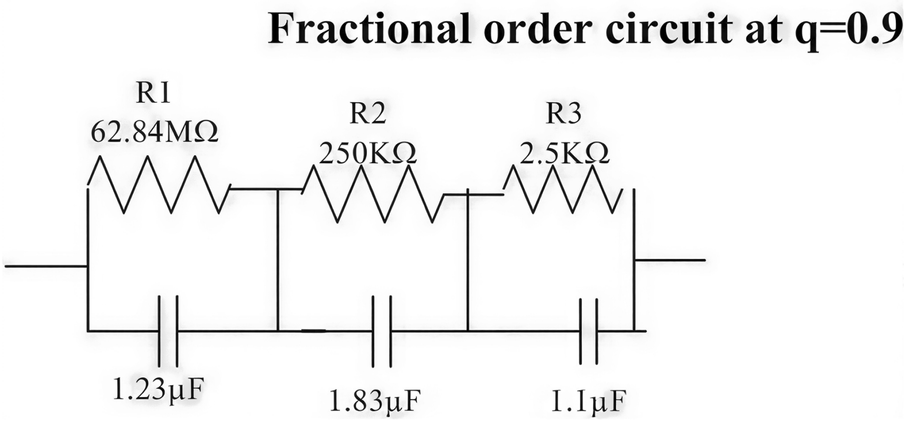

Figure 4. Fractional-order circuit

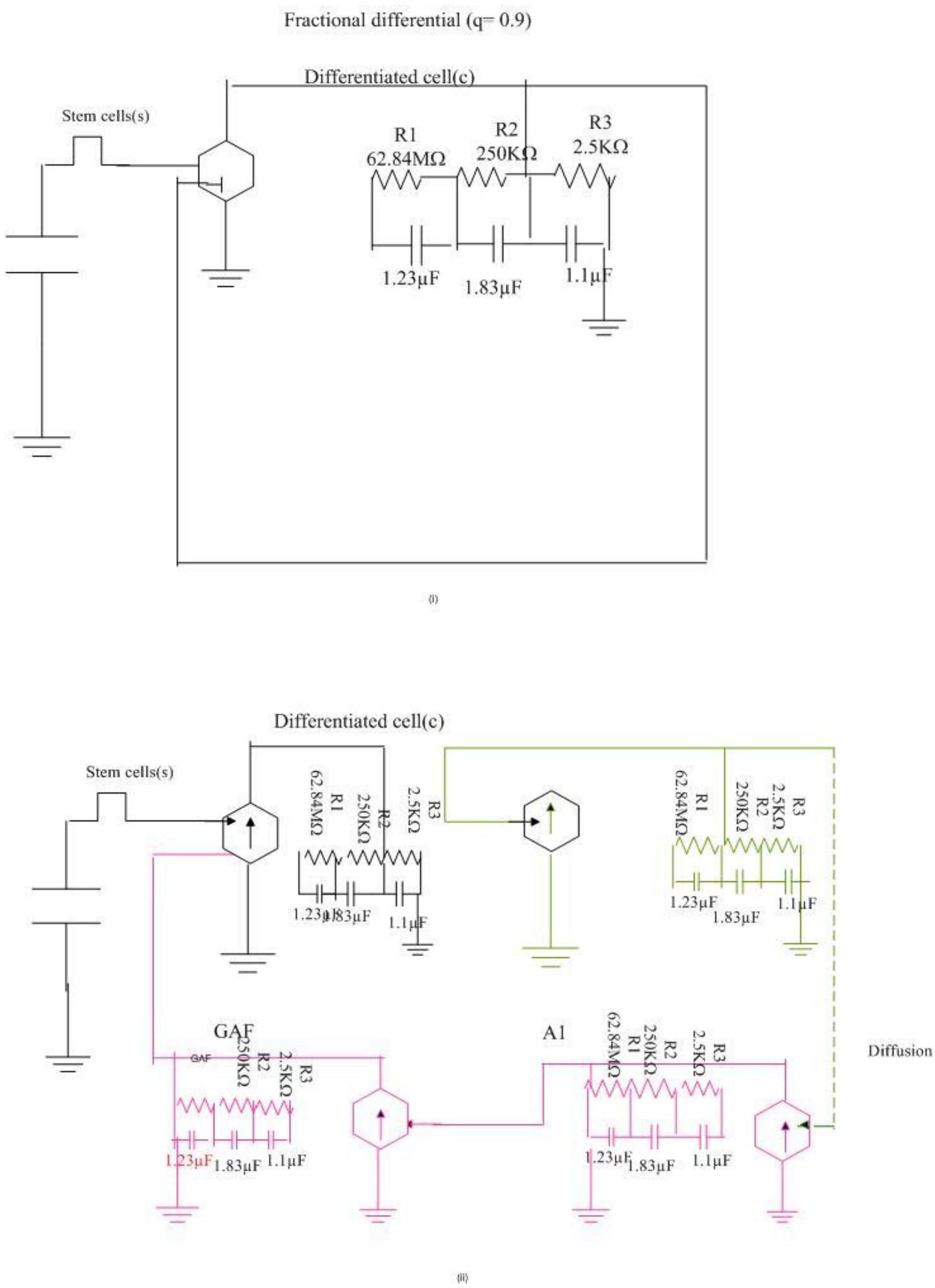

Figure 5. Homeostasis is circuit

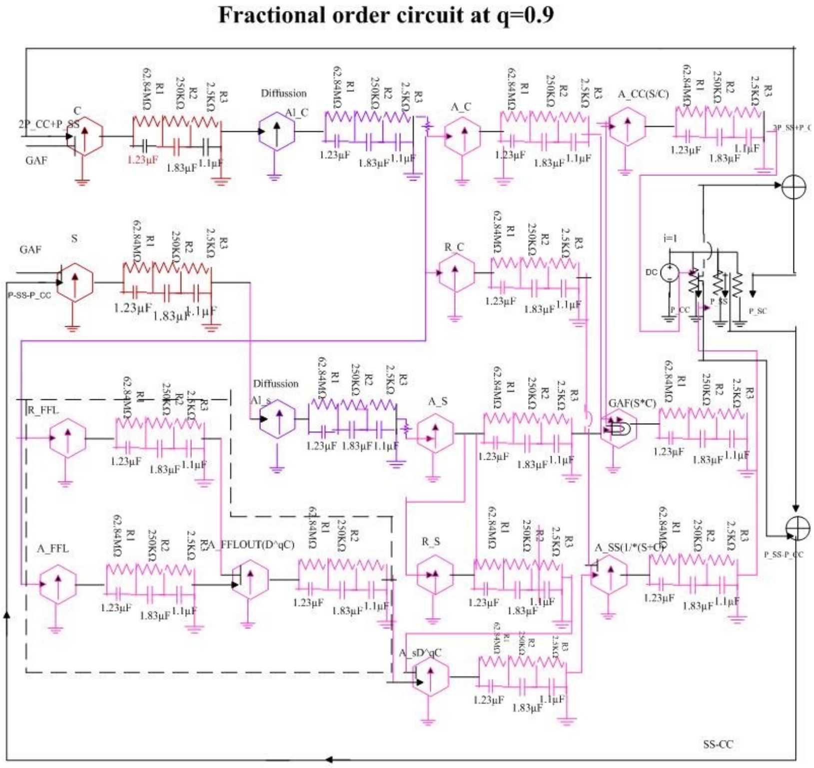

Figure 6. The homeostasis control circuit as a purely fractional-order analog circuit. Each circuit element can be viewed as a state variable defined by a circuit modification involving a current generator variable, a fractional-order resistor, and a capacitor.

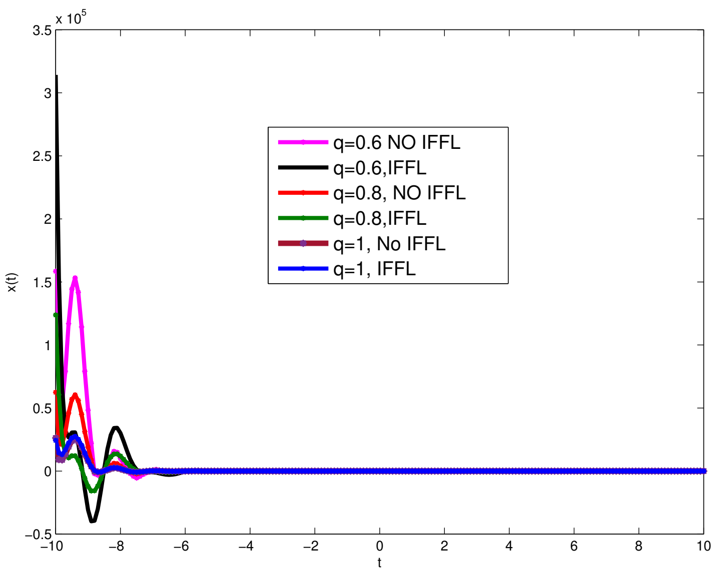

Figure 7. Stability trajectory of signal transformation between stem cells and fractionally differentiated cells, represented by NFSDEs

5. Conclusion

Sufficient conditions are obtained for the mild solutions of the Hyers-Ulam stability of NFSDEs driven by sfBm. Numerical simulations are applied to validate the main results, and a real-life model representing the artificial tissue homeostasis fractional differential system analog circuit is provided. In future work, the results will be extended to controllability problems for other abrupt changes, such as earthquakes and sudden fires in electrical wiring.

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Conflict of interest

The authors declare that they have no conflict of interest.

Author contributions

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Availability of data

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

AI tools statement

No artificial intelligence (AI) tools were used in the preparation, analysis, writing, or revision of this manuscript.

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