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RESEARCH ARTICLE

Functional analytic approach on time optimal control problem in frechet space using graph of linear transformation

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ABSTRACT

This paper investigates the minimum-time optimal control problem in a Frechet graph space G_T associated with a continuous linear transformation $T : F \times I \to X$, where F is a Frechet space, $I = [t_0, t_1]$ is the time interval, and X is the state space. The study is motivated by the need to extend classical time-optimal control theory from Banach and finite-dimensional settings to more general Frechet spaces. A continuous linear dynamical system $\sum = \{G_T, U_G^T, I, X, Y, \phi, \eta\}$ is formulated on the Frechet graph space G_T and the unit ball U_G^T is employed as a net to characterise the structure of G_T. A key contribution of this work is to determine controls in Frechet graph space through a unit ball framework U_G^T for the analysis of admissible controls and attainable states, aiming to reach a prescribed target state from an initial state while minimizing a time cost function J_t. Under suitable assumptions including reflexivity and rotundity of the graph space, sufficient conditions are established for the existence and uniqueness of minimum-time optimal controls. Furthermore, necessary and sufficient conditions are derived for both normal and proper systems. In the normal case, the optimal control is shown to be unique and is characterised by a sign condition involving the outward normal vector η to a supporting hyperplane M of the attainable set $A_G^T(t) \subseteq X$ whereas in the proper case optimal controls exist but need not be unique. The proposed graph space framework provides a novel approach to characterise optimal trajectories and derive optimal control laws in Frechet graph space. Finally, illustrative examples are presented to demonstrate and validate the theoretical results.

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1. Introduction

Optimal control theory plays a central role in engineering, physics, and applied mathematics, where the objective is to determine control strategies that optimise system performance. Among various formulations, the minimum-time control problem is of particular importance, as it aims to steer a system to a desired state in the shortest possible time. Such problems arise naturally in time-critical applications, including stabilisation, robotics, and distributed parameter systems.

The optimal control problem has been extensively studied in the context of linear topological spaces, and its formulation has been extended to products of locally uniformly convex linear topological spaces. Necessary and sufficient conditions for the existence and uniqueness of solutions have been established using tools from functional analysis.[1] In particular, reflexivity, rotundity, and smoothness in normed spaces play a crucial role in the analysis of linear control systems. Reflexive spaces ensure the existence of optimal controls

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via weak compactness arguments, while rotund and smooth spaces contribute to uniqueness and stability. These geometric properties are closely related to controllability and are often linked to algebraic conditions such as rank criteria and Kalman decomposition.[2]

Furthermore, linear optimal control problems in Banach spaces have been widely investigated, where existence and uniqueness of optimal controls are derived by exploiting reflexivity and rotundity, thereby extending classical Hilbert space results.[3] The geometry of reachable sets has also been shown to play a fundamental role in determining the structure and uniqueness of optimal controls. In particular, normal systems exhibit strictly expanding reachable sets, and each boundary point corresponds to an extremal trajectory. Under appropriate rank conditions, Pontryagin's maximum principle provides sufficient conditions for minimum-time optimality.[4]

In addition, several important results from convex analysis and functional analysis support the theoretical framework of optimal control. The supporting hyperplane theorem ensures that every boundary point of a convex set admits a supporting hyperplane, which is essential in duality theory and optimality conditions.[5] Support functions constitute a fundamental tool in convex geometry, providing geometric characterisations of convex sets in both Euclidean and non-Euclidean settings.[6] The strict convexity of the objective function guarantees uniqueness of minimisers, while general convexity ensures global optimality.[7] Moreover, the Riesz–Frechet representation theorem establishes that, in Hilbert spaces, every continuous linear functional can be represented via an inner product with a unique vector.[8]

While these results provide a solid foundation in normed spaces, their extension to more general topological vector spaces is not straightforward. In this direction, Frechet spaces arise as a natural generalisation, being complete, metrizable, and locally convex topological vector spaces that are not necessarily normable.[9] These spaces frequently appear in functional analysis, particularly in contexts where normability is too restrictive, such as spaces of smooth functions and solutions of partial differential equations. They are stable under the formation of subspaces, quotients by closed subspaces, and finite products.[10] Kolmogorov's normability criterion provides necessary and sufficient conditions for a topological vector space to be normable.[11]

However, the analysis of control problems in Frechet spaces presents significant challenges. The absence of a norm prevents the direct application of standard techniques based on weak compactness, duality, and variational methods. In addition, the structure of the dual space is more intricate, and classical geometric arguments used in Banach spaces are not readily applicable. Although certain geometric properties, such as rotundity and local uniform rotundity, can be introduced via equivalent norms under suitable conditions, the overall analytical framework remains substantially more complex.[12,13]

Several foundational results, such as the closed graph theorem and the structural analysis of linear operators via their graphs, extend to complete metrizable topological vector spaces and provide useful tools in this setting.[14,15] Moreover, linear transformations between topological vector spaces can be studied via their graphs, viewed as subsets of product spaces, which provides a natural structural framework for analysis.[16] The preservation of structural properties under product spaces also plays a significant role in the analytical framework adopted in this study.[17]

In addition, robustness considerations play an important role in control theory, particularly in the presence of system uncertainties. Classical results, such as Kharitonov-type stability criteria, provide powerful tools for analysing families of systems with parametric uncertainties and ensuring stability and positive real behaviour. Although these approaches are primarily developed in finite-dimensional settings, they highlight the importance of incorporating structural and stability considerations into optimal control frameworks.[18,19]

Minimum-time optimal control is a fundamental topic in optimal control theory, concerned with transferring a dynamical system from an initial state to a desired target state in the shortest possible time. Various minimum-time control problems have been studied under different assumptions and settings.[20–23]

The analysis of minimum-time optimal control problems in Frechet spaces is still not as extensively developed as the corresponding theory in Banach and Hilbert spaces. One of the main reasons lies in the non-normable nature of Frechet spaces, which limits the applicability of several standard methods based on compactness, duality principles, and variational arguments. Despite these analytical difficulties, Frechet spaces provide a natural setting for many infinite-dimensional problems arising in smooth function spaces, distribution theory, evolution equations, and partial differential equation models.

In this paper, a framework for the study of minimum-time optimal control problems in Frechet spaces is established from a functional-analytic viewpoint. The analysis relies on geometric properties of reachable sets together with tools from convex analysis in order to derive conditions ensuring the existence and uniqueness of optimal controls. In this manner, several classical results from Banach-space control theory are generalised to a wider class of non-normable spaces.

Moreover, Frechet space formulations appear naturally in control systems involving delay terms, memory effects, and fractional-order dynamics. Recent contributions concerning controllability of fractional integro-differential systems in Frechet spaces further indicate the importance of this framework in infinite-dimensional control analysis.[24] Optimal control theory has attracted considerable attention due to its wide range of applications in engineering, physics, and distributed parameter systems. Significant progress has been made in the study of evolution equations and optimal control of infinite-dimensional systems.[25–31] Time optimal and feedback control problems have also been extensively investigated,[32,33] while geometric control methods and controllability properties have been widely studied.[34–40] These developments provide part of the motivation for investigating minimum-time control problems in a Frechet space setting.

Comparison: In this study, we constructed a minimum-time optimal control problem in the Frechet graph space $G_T \subseteq (F \times I) \times X$ in Section 3.1. This problem was also investigated in.[2,4] In,[18,19] the authors investigated robust stability and robustly SPR conditions for a family of interval plants, using a novel approach based on the Kharitonov box within a certain frequency range. In,[1] the authors considered the unit ball as a polar set E_π in the product of locally uniformly convex linear topological spaces. In,[17] the author introduced the Cartesian product for Banach spaces, whereas we employ the unit ball U_G^T as a net in G_T. Using this unit ball U_G^T in Section 3.2, we establish the existence and uniqueness at the minimum time $t = t_1$ for our problem in Section 3.3. This proof is based on the reflexivity and rotundity of the Frechet space G_T, whereas in,[3,7] the authors investigated Banach spaces using the unit ball $U = \{u : \|u\| \leq 1, u \in U\}$ and in[10] the author introduced a norm for Frechet spaces. We have shown that every normable Frechet space is an L^p space. However, the Frechet space is more general than Banach and Hilbert spaces due to its non-normable structure , as discussed in.[9,11] In[8] the author characterised the Riesz-Frechet theorem in the dual of a Hilbert space through an inner product with a unique vector, whereas we apply the Riesz-Frechet theorem to the graph of the linear transformation G_T. The flatness of the unit ball in[12] yields the Mazur-Ulam property in non-smooth 2D spaces whereas we take $G_T = V^*$, where V^* is the dual of the Frechet space F. Then G_T is strictly convex because the smoothness of V removes flatness , where $V = F^*$ and F^* is the set of all continuous functionals defined on F. In,[6] the support function was defined as the hyperbolic distance to the support geodesic within the Klein model, while in[5] the authors proved the supporting hyperplane theorem, which states that every boundary point of a convex set has a hyperplane passing through it. This result is key in advancing optimisation theory, especially in duality and optimality. In our work, we use the support function on the attainable set $A_G^T(t)$ in Section 3.4 via the maximum inner product in the direction of the outward normal η to the hyperplane M at $A_G^T(t)$.

The authors in[14–16] proved that continuity and closed graphs are equivalent for general complete metrizable spaces, whereas we have shown that G_T as a Frechet space , is closed for the continuous linear transformation $T : F \times I \to X$ by applying the closed graph theorem in the Frechet space $(F \times I) \times X$ in Section 3.1.

Advantage: By constructing such a unit ball in any space, we can characterize the entire space. In the production industry, the region or shape of the ball can be expanded or contracted as needed. In medical science, any affected region can spread through an extreme point, where the outward normal plays a vital role because pathogens move rapidly through such points. Therefore, if that point can be rendered immovable through medication, vacination or other techniques, the disease cannot spread through either the extreme point or the boundary of the affected region. In addition,for normal systems, it is possible to obtain a unique time - optimal and robust solution for small perturbations in initial conditions using the Pontryagin principle, the Bang-Bang principle, and the Agrachev-Sachkov approach for the time optimal control problem , based on the unit ball framework developed in our manuscript. However, for proper systems, at least one time-optimal control exists, but uniqueness is not guaranteed due to flat faces.

Contribution: Despite the extensive development of optimal control theory in Banach and Hilbert spaces, the study of minimum-time optimal control problems in Frechet spaces remains

largely underdeveloped. The primary difficulty arises from the non-normable structure of these spaces, which limits the applicability of classical analytical tools based on compactness, duality, and variational principles. At the same time, the natural occurrence of Frechet spaces in infinite-dimensional models, particularly in spaces of smooth functions, makes this investigation both relevant and necessary. Here, we contribute to the existence and uniqueness of the minimum - time optimal control problem in G_T in Section 3.1 as a Frechet space, constructing the unit ball U_G^T as a net in Section 3.2. To solve this problem, we consider the single-state continuous linear dynamical system

$$\sum = \{G_T, U_G^T, I, X, Y, \phi, \eta\} \tag{1}$$

where G_T is reflexive and rotund Frechet graph space. Thus, we determine the unique time optimal control $((u,t), T(u,t))^*$ in U_G^T at the minimum time $t = t_1$ such that

(1) $((x,t_1), T(x,t_1)) = ((z,t_1), T(z,t_1))$
(2) $|((u,t), T(u,t))|$ minimizes the time cost function $J_t((u,t),(x,t))$ and

$$J_t((u,t),(x,t)) = min\{|((u,t), T(u,t))| : ((u,t), T(u,t)) \in U_G^T\}$$

where $(x,t) = T(u,t)$ is defined in our problem in the Section 3.3. Also, $J_t((u,t),(x,t))$ is transformed by a mapping $R : U_G^T \to X$ such that

$$R((u,t), T(u,t)) = G(u,t)$$

where $G : F \to X$ is a continuous linear functional for some time $t \in I = [t_0, t_1]$, as discussed in Remark 11.

In addition, we provide examples of the net U_G^T in G_T arising from particular cost functions. Next, we consider normal and proper systems. In the normal system, we determine the unique minimum time optimal control in the form $((u,t), T(u,t))^*$, and then, by,[4]

$$((u,t), T(u,t))^* = sgn[\eta' Y(t)] \tag{2}$$

where η' is the transpose of the outward normal η to the hyperplane M at $A_G^T(t)$ in Section 3.4 and $Y(t) = [X^{-1}(t)Bu(t)]$ by[4] related to the differential equation $\dot{x} = Ax(t) + Bu(t)$ in Section 4 where $X = R^1$ is the one-dimensional state space in our problem in Section 3.3. Thus $Y(t)$ is the output space based on the necessary and sufficient conditions for some theorems. However, in the proper system, we determine the minimum-time optimal control in the form of (2), which is not unique. Finally, we present two examples to illustrate normal and proper systems in which we obtain minimum time optimal control in the Frechet space G_T.

2. Definitions, properties and theorems

We categorise the various notations in the Nomenclature.

2.1. Frechet space

A Frechet space F is a topological vector space (over the Real field $\mathbb{R}$ or Complex field $\mathbb{C}$) that is locally convex, metrizable, and complete.[13]

2.2. Graph of a linear transformation

Let T be a function with domain D. The graph of T, denoted by G_T , is a subset of $F \times X$ consisting of all pairs (u, Tu) for u in D. When F and X are linear spaces over the same scalar field K, a subset of $F \times X$ is the graph G_T of a linear transformation T whose domain and range are linear subspaces of F and X respectively , if and only if G_T is a linear subspace of $F \times X$ such that $(0, x) \in G_T$ implies $x = 0$.[16]

2.3. Net

Let F be a Frechet Space, and let the unit ball

$$U = \{u : u_\alpha \to u \text{ and } |u| \leq 1, u_\alpha \in F \subset F^{**}, u \in F \text{ and } \alpha \in \Lambda\}$$

where F^{**} and F^* are the sets of all continuous linear functionals defined on F^* and F respectively, and Λ is a directed set. Then the unit ball U is a net in F.

Remark 1. According to the definition of the unit ball U as a net in the Frechet space F, the unit ball U is a convex neighborhood of zero, circled, balanced and barrelled. Thus, the unit ball U characterises the Frechet space F.

2.4. Theorem

A Frechet space F is reflexive if and only if it is barrelled and every closed, convex, balanced and bounded set of F is compact in the weak topology of F.[17]

2.5. Reflexive space

A Frechet Space F is said to be reflexive when $F = F^{**}$.

Some properties of reflexive frechet spaces are:

(1) The intersection of every decreasing sequence of closed, bounded and convex sets is nonempty.[2]
(2) If Frechet spaces are reflexive, then their product and direct sum also exhibit reflexivity.[16]

2.6. Rotund

A Frechet space is rotund if the boundary, $\partial U = \{u : u_\alpha \to u \text{ and } |u| = 1, u_\alpha \in F \subset F^{**}\}$ contains no line segments.

The following properties are equivalent:

(1) The Frechet space F is rotund.
(2) Every point in ∂U is an extreme point of U.
(3) For each non zero $f \in F^*$, there is at most one $u \in U$ with $f(u) = |-f|$.
(4) Each supporting hyperplane meets U at exactly one point.
(5) Every convex set G_T in F has at most one minimum element satisfying $|u| \leq |z| \; \forall \; z \in G_T, and \; u \in G_T$.
(6) The product of rotund spaces is again rotund.[2]

2.7. Strict convexity

A norm on a Frechet space F is called rotund (strictly convex) if , for all (u_1, Tu_1) and $(u_2, Tu_2) \in G_T$, the condition $|(u_1, Tu_1) + (u_2, Tu_2)| = 2$ implies $(u_1, Tu_1) = (u_2, Tu_2)$.[11]

2.8. Smooth

A Frechet space is said to be smooth if at each point $u \in \partial U$ there is exactly one supporting hyperplane passing through it.[2]

2.9. Riesz-frechet theorem

The Riesz-Frechet theorem states that every continuous linear functional on U can be expressed as an inner product with a fixed vector of U.[8]

2.10. Closed graph theorem

If F and X are Frechet spaces, $T : F \to X$ is linear , and $G_T = \{(u, Tu) : u \in F, Tu \in X\}$ is closed in $F \times X$, then T is continuous.[15]

Remark 2. If F is reflexive and rotund then G_T is reflexive and rotund.[3]

Remark 3. A topological vector space is classified as normable if and only if a single norm characterizes its topology. In other words, a topological vector space over the real number $\mathbb{R}$ or complex number $\mathbb{C}$ is normable if and only if it is Hausdorff and has a convex bounded neighbourhood of zero.[11]

Remark 4. A reflexive Frechet space F is strictly convex if and only if its dual Frechet space F^* is smooth.[12]

Remark 5. If F and X are Frechet spaces, then $F \times X$ is also a Frechet space.[17]

Remark 6. If F is a reflexive Frechet space and G_T is a closed subspace of F, then G_T is reflexive.[15]

Remark 7. By Definition 2.2 and Theorem 2.10, we can say that G_T is a closed linear subspace of $F \times X$.

Remark 8. A closed subspace of a Frechet space is also a Frechet space.[9] Then , by Remark 7, G_T is a Frechet space.

3. Definition of net and statement of problem

3.1. Definition of G_T related to the time interval

Let F be a Frechet space and let $I = [t_0, t_1]$ be the time interval , which is also a Frechet space. Let $X = R^1$ be the state space of a single state variable which is also a Frechet space. Then $(F \times I) \times X$ is a Frechet space by Remark 5. Following Definition 2.2, we define G_T in 2.2 related to the time interval $t \in I = [t_0, t_1]$ where $T : F \times I \to X$ is a continuous linear transformation and $G_T \subseteq (F \times I) \times X$. Clearly, G_T is a Frechet space related to the time interval I and G_T is closed in $(F \times I) \times X$ by the closed graph theorem in Section 2.10.

3.2. Definition of net in G_T

Let the unit ball in G_T be U_G^T related to the time $t \in I = [t_0, t_1]$. Following Definition 2.3, the unit ball U_G^T in $G_T \subseteq (F \times I) \times X$ can be written as:

$$U_G^T = \{((u,t), T(u,t)) : ((u_\alpha, t), T(u_\alpha, t)) \to ((u,t), T(u,t)) \text{ and } |((u,t), T(u,t))| \leq 1, (u_\alpha, t) \in F \subset F^{**}, T(u_\alpha, t) \in X, \; \alpha \in \Lambda \text{ and } t \in I\}$$

which is a net in G_T defined on a directed set Λ. The boundary of U_G^T is denoted by ∂U_G^T.

Remark 9. By Section 2.3 and Remark 1, the unit ball U_G^T is a convex neighbourhood of zero, circled, balanced , and barrelled. Thus, the unit ball U_G^T characterizes the Frechet space G_T.

Remark 10. If G_T is reflexive, then by Remark 9 and by the Theorem 2.4, we can say that the unit ball U_G^T is weakly compact in G_T.

Now, we can define a time optimal control problem in G_T related to the time $t \in I = [t_0, t_1]$ using the constructed unit ball U_G^T which is a novel technique in G_T in this paper.

3.3. Problem statement in G_T

Consider a dynamical system $\sum = \{G_T, U_G^T, I, X, Y, \phi, \eta\}$ where G_T is the Frechet space.

Let $G : F \to X$ be denoted by the mapping ϕ for which Gu can be expressed as a continuous linear functional $\langle u, v\rangle$, $u \in F, v \in V$; where $V = F^*$ or $F = V^*$ by the Riesz-Frechet theorem considering that F is a Frechet space and $X = R^1$.

Let $T : F \times I \to X$ be a continuous linear transformation. Then , for all $t \in I = [t_0, t_1]$,

$$T(u,t) = (x,t)$$

Now, we formulate a minimum time optimal control problem in the Frechet space G_T using the unit ball as a net U_G^T and we determine $((u,t), T(u,t))$ in U_G^T such that

(1) $((x,t_1), T(x,t_1)) = ((z,t_1), T(z,t_1))$
(2) $|((u,t), T(u,t))|$ is minimized

where $\{((u,t), T(u,t)) : t \in I\}$ is the graph of T, denoted by G_T as a Frechet space, and the time cost function is denoted by $J_t((u,t),(x,t))$ for G_T as

$$J_t((u,t),(x,t)) = min\{|((u,t), T(u,t))| : ((u,t), T(u,t)) \in G_T\}$$

Here,

$$t_1 = \inf\{t : ((x,t), T(x,t)) = ((z,t), T(z,t)), t \in [t_0, \infty)\}$$

Also, a mapping $R : U_G^T \to X$ is defined such that

$$R((u,t), T(u,t)) = G(u,t)$$

Remark 11. Here, the unit ball is $U_G^T \subseteq G_T$ defined in Section 3.2. U_G^T is introduced to represent the admissible control-state pair (u,t) and $(x,t) = T(u,t)$ as a single element $((u,t),(x,t)) \in G_T$, automatically enforcing the dynamical constraints, whereas the mapping $R : U_G^T \to X$ defined by $R((u,t), T(u,t)) = G(u,t)$ converts the original time cost function $J_t((u,t),(x,t)) = G(u,t)$ depending on both (u,t) and (x,t) for some time $t \in I = [t_0, t_1]$.

Theorem 1. *R is a continuous linear functional on U_G^T for some time $t \in I = [t_0, t_1]$.*

Proof. We must show that R is a continuous linear functional in $U_G^T \subseteq G_T$ for some time $t \in I = [t_0, t_1]$. First,

$$\begin{aligned} R[(u_1,t) + (u_2,t)), & \\ T((u_1,t) + (u_2,t))] &= G((u_1,t) + (u_2,t)) \\ &= G(u_1,t) + G(u_2,t) \\ &= R((u_1,t), T(u_2,t)) \\ &\quad + R((u_1,t), T(u_2,t)) \end{aligned}$$

by taking from the Section 3.3 .

Also,

$$\begin{aligned} R[\alpha((u,t), T(u,t))] &= R(\alpha(u,t), \alpha T(u,t)) \\ &= G(\alpha(u,t)) \\ &= \alpha G(u,t), \end{aligned}$$

since G is linear functional.

$$= \alpha R[((u,t), T(u,t))]$$

by taking from the Section 3.3.

Hence, R is linear. To prove continuity, assume $((u_\alpha,t), T(u_\alpha,t)) \to ((u,t), T(u,t))$. We show that $R((u_\alpha,t), T(u_\alpha,t)) \to R((u,t), T(u,t))$.

Now,

$$R((u_\alpha,t), T(u_\alpha,t)) = G(u_\alpha,t). \tag{3}$$

Since $G : F \to X$ is continuous linear, $(u_\alpha,t) \to (u,t) \implies G(u_\alpha,t) \to G(u,t)$ for $t \in I = [t_0, t_1]$. Hence by **Equation (3)**

$$R((u_\alpha,t), T(u_\alpha,t)) \to G(u,t) \tag{4}$$

From Section 3.3, $G(u,t) = R((u,t), T(u,t))$. Therefore, by **Equation (4)**

$$R((u_\alpha,t), T(u_\alpha,t)) \to R((u,t), T(u,t))$$

□

Thus R is continuous, completing the proof.

Next , to solve the problem, we define the attainable set $A_G^T(t)$.

3.4. Attainable set

Let $R : U_G^T \to X$ denote the mapping ϕ which is a continuous linear functional at time $t \in I = [t_0, t_1]$ with a fixed $x(t_0)$. The attainable set is defined as

$$A_G^T(t) = \{((x,t), T(x,t)) : ((x,t), T(x,t)) = R((u,t), T(u,t)), ((u,t), T(u,t)) \in U_G^T\}.$$

Clearly, $((z,t), T(z,t)) \in A_G^T(t)$ for some $t \in I$ since the system $\sum$ is controllable.

Remark 12. If the Frechet graph space G_T is rotund and reflexive, then the unit ball U_G^T in G_T is convex and compact by Remarks 9 and 10. Moreover, for each fixed $t \in I = [t_0, t_1]$ and fixed initial state $x(t_0)$, the mapping $R : U_G^T \to X$ associated with the problem in Section 3.3 is a continuous linear functional. Hence, the image $R(U_G^T)$ is convex and compact. Also $R(U_G^T) \subseteq A_G^T(t) \subseteq X$. Thus, the attainable set $A_G^T(t)$ in the Section 3.4 is convex and compact.

Remark 13. For our problem, $\sum$ is a controllable continuous time dynamical system in **Equation (1)** and for that system $\sum$, $R : U_G^T \to X$ is

onto by.[2] Again by Remark 12, $A_G^T(t)$ is convex and compact. Then by,[1,2] we obtain the existence of a time optimal control $((u,t),T(u,t)) \in U_G^T$ such that $((u,t),T(u,t)) = R^{-1}((x,t),T(x,t))$ using the attainable set $A_G^T(t)$ and satisfying conditions 1 and 2 for our problem in Section 3.3.

Theorem 2. *Our problem has a unique minimum time optimal control in the unit ball U_G^T where U_G^T is a net in G_T if*

(1) $G_T \subseteq (F \times I) \times X$ is reflexive and rotund,
(2) G_T is the dual of a smooth linear space $V(t)$.

Proof. 1. Let F,X and I be reflexive and rotund Frechet spaces.Then $(F \times I) \times X$ is a reflexive, rotund Frechet space by Remark 5, Definition 2.5 property 5 and by Definition 2.6 property 7. By Section 3.1, G_T is a Frechet space with $G_T \subseteq (F \times I) \times X$. Hence, G_T is reflexive and rotund by Remark 2. Therefore the unit ball U_G^T in G_T, is convex and compact by Remark 10.

Also, R is a continuous linear functional by Theorem 1 and is onto for the dynamical system $\sum$ by Remark 13. Then, by Remark 12, $A_G^T(t)$ is convex and compact and $G_T = \{R^{-1}((z,t),T(z,t))\}$, where $((z,t),T(z,t)) \in A_G^T(t) \subseteq X$, is convex. Based on the rotundity of $(F \times I) \times X$, each convex set G_T has at most one element satisfying

$$\begin{aligned} &((u_1,t),T(u_1,t)) \leq ((u_2,t),T(u_2,t)) \\ &\text{for all } ((u_1,t),T(u_1,t)) \in G_T, \\ &((u_2,t),T(u_2,t)) \in G_T, \text{ by (2.6 , property 6)} \end{aligned} \tag{5}$$

In addition, R is continuous and G_T is nonempty , so G_T is closed. Let $J_t((u,t),T(u,t))$ be the time cost function of the above problem. Thus, we define

$$\begin{aligned} &J_t((u,t),T(u,t)) = min\{((u,t),T(u,t)) : \\ &((u,t),T(u,t)) \in G_T\} \end{aligned} \tag{6}$$

and

$$\begin{aligned} &G_{T\alpha} = \{((u,t),T(u,t)) : \ ((u,t),T(u,t)) \in \\ &G_T, J_t((u,t),T(u,t)) \leq \lambda + 1/\alpha\}, \ \alpha \in \Lambda \end{aligned}$$

Then $\{G_{T\alpha}\}$ is a decreasing sequence of closed bounded sets in U_G^T , each of which $G_{T\alpha}$ is closed, bounded , and convex. By Definition 2.5 property 4, $G_{T\alpha}$ is compact , and by Definition 2.5 (property 2), the intersection $\bigcap_{n=1}^{\infty} G_{T\alpha}$ is nonempty. Hence, there exists at least one minimum element $((u,t),T(u,t))$ and by **Equation (5)**

$$\begin{aligned} &((u_1,t),T(u_1,t)) = ((u_2,t), \\ &T(u_2,t)) = ((u,t),T(u,t)) \end{aligned} \tag{7}$$

Now, by Remark 13 and the rotundity of $(F \times I) \times X$, we can write from **Equation (7)**:

$$\begin{aligned} &R^{-1}((x,t_1),T(x,t_1)) = ((u,t_1),T(u,t_1)) \\ &\text{and } R^{-1}((z,t_1),T(z,t_1)) = ((u,t_1),T(u,t_1)) \\ &\text{where } ((x,t_1),T(x,t_1)),((z,t_1),T(z,t_1)) \in A_G^T(t) \\ &\text{and } (u,t_1),T(u,t_1)) \in \partial U_G^T, \\ &\text{boundary of } U_G^T \text{ by 3.2} \end{aligned} \tag{8}$$

Then , from **Equation (8)** we can have $((x,t_1),T(x,t_1)) = ((z,t_1),T(z,t_1))$ which proves condition 1 of our problem for which

$$\begin{aligned} &t_1 = \inf\{t : ((x,t),T(x,t)) = ((z,t),T(z,t)), \\ &t \in [t_0,\infty)\} \end{aligned}$$

Also, $|((u,t),T(u,t))|$ is minimized , so the time cost function J_t in **Equation (6)** becomes $J_t((u,t),T(u,t)) = J_t((u,t),(x,t)) = min\{|((u,t),T(u,t))| : ((u,t),T(u,t)) \in G_T\}$, which proves condition 2 of our problem.

Thus, there exists a unique minimum-time optimal control satisfying conditions 1 and 2 of our problem in Section 3.3. □

Proof. 2. Let $V(t)$ be a smooth linear space and let G_T be the dual of $V(t)$.Then $G_T = V^*(t)$ and each point $((v,t),T(v,t)) \in V(t)$ has at most one extremum $((u,t),T(u,t)) \in U_G^T$, $U_G^T \subseteq G_T$.[2] Let $((u_1,t),T(u_1,t))$ and $((u_2,t),T(u_2,t))$ be any two distinct elements in G_T. From Remark 4, the dual of a smooth Frechet space is strictly convex and hence rotund by Definition 2.7. Then we can write

$$\begin{aligned} &|((u_1,t),T(u_1,t))| = |((u_2,t),T(u_2,t))| = 1 \\ \implies &|((u_1,t),T(u_1,t)) + ((u_2,t),T(u_2,t))| = 2 \end{aligned}$$

then we have $((u_1,t),T(u_1,t)) = ((u_2,t),T(u_2,t))$

Thus, the time cost function $J_t((u,t),(x,t)) = |((u,t),T(u,t))|$ in **Equation (6)** is strictly convex on G_T. Hence, by Remark 13 and by strict convexity of $(F \times I) \times X$ Definition 2.7, the given problem has a unique time-optimal control . □

Theorem 3. *Let F, X, I be reflexive, rotund Frechet spaces. Then $(F \times I) \times X$ is a reflexive, rotund Frechet space for which a net can be defined as:*

$$\begin{aligned} &|(((u_1,t),T(u_1,t)),((u_2,t),T(u_2,t)))| = \\ &[|((u_1,t),T(u_1,t))|^p + |((u_2,t),T(u_2,t))|^p]^{\frac{1}{p}}, \\ &1 < p < \infty \end{aligned}$$

Proof. Let F, X, I be reflexive, rotund Frechet spaces. From the proof of Theorem 2, it is clear that $(F \times I) \times X$ is a reflexive, rotund Frechet space. Since a Hausdorff locally convex linear topological space is normable by Remark 3, and by the definition of a Frechet space, $(F \times I) \times X$ is

a complete, metrizable , Hausdorff locally convex linear topological vector space. Hence, $(F\times I)\times X$ is a normable Frechet space , and therefore it is an L^p space. Again, U_G^T is a net in G_T where $U_G^T \subseteq G_T \subseteq (F \times I) \times X$ and U_G^T characterize the Frechet space $(F \times I) \times X$, which is an L^p space. Hence, a net U_G^T in $G_T \subseteq (F \times I) \times X$ can be defined as

$$|(((u_1,t),T(u_1,t)),((u_2,t),T(u_2,t)))| =$$

$$[|((u_1,t),T(u_1,t))|^p + |((u_2,t),T(u_2,t))|^p]^{\frac{1}{p}},$$

$$1 < p < \infty$$

Since $(F \times I) \times X$ is reflexive and rotund , the Frechet space G_T is reflexive and rotund by Remark 2. Then by Section 2.6, property 2, every point in ∂U_G^T is an extreme point of U_G^T. Thus, we have

$$((u_1,t_1),T(u_1,t_1)) = ((u_2,t_1),T(u_2,t_1))$$

and $((u_1,t_1),T(u_1,t_1)),((u_2,t_1),T(u_2,t_1)) \in \partial U_G^T$ 3.2

Now by using Theorem 2 , we obtain the existence of a unique minimum time optimal control satisfying conditions 1 and 2 of our problem in Section 3.3

□

Theorem 4. *Let F, X, and I be reflexive, rotund Frechet spaces, and let $T : F \times I \to X$ be a linear continuous transformation. Then G_T is a closed subspace of $(F \times I) \times X$, and hence G_T is a rotund, reflexive Frechet space.*

Proof. Let F, X and I be reflexive, rotund Frechet spaces, and let $T : F \times I \to X$ be continuous and onto . We need to show that $G_T = \{((u,t),T(u,t)) : (u,t) \in F \times I$ and $T(u,t) \in X\}$ is a closed subspace of $(F \times I) \times X$. Consider a sequence $((u_\alpha,t),T(u_\alpha,t)) \in G_T$ that converging to $((u,t),(x,t))$. Since T is continuous

$$(u_\alpha,t) \to (u,t) \text{ and } T(u_\alpha,t) \to T(u,t)$$

Thus, $(x,t) = T(u,t)$, so $((u,t),(x,t)) \in G_T$ proving that G_T is closed. Therefore, by Remark 7, G_T is a closed subspace of $(F\times I)\times X$.It remains to show that G_T is a reflexive, rotund Frechet space. From the proof of Theorem 2, $(F \times I) \times X$ is reflexive and rotund. By Remarks 8 and 6, the closed subspace G_T of $(F \times I) \times X$ is reflexive. Moreover, in Theorem 2 we proved that G_T is the dual of a smooth linear space , and and by Remark 4, G_T is rotund. Therefore, G_T of $(F \times I) \times X$ is a rotund , reflexive Frechet space. □

Examples of the net U_G^T in G_T arising from particular cost indices.

Case 1: We show that a net of the form

$$(|u,t)|)^p + |T(u,t)|^p)^{\frac{1}{p}} =$$

$$\left(\int |(u,t)|^p + |(x,t)|^p dt\right)^{\frac{1}{p}}$$

can be expressed as a net on G_T for some time $t \in I = [t_0,t_1]$ such that

$$(|(u,t)|^p + |T(u,t)|^p)^{\frac{1}{p}} = |((u,t),T(u,t))| = |(y,t)|$$

where $(y,t) \in G_T$.

Proof. Consider the net in L^P space as

$$(|(u,t)|)^p + |T(u,t)|^p)^{\frac{1}{p}} =$$

$$\left(\int |(u,t)|^p + |(x,t)|^p dt\right)^{\frac{1}{p}} \tag{9}$$

For the given problem, $T : F \times I \to X$ is a continuous linear transformation. We show that the time cost function $(|(u,t)|^p + |T(u,t)|^p)^{\frac{1}{P}}$ at the time $t \in I = [t_0,t_1]$, defines a net in $G_T = \{((u,t),T(u,t)) : (u,t) \in F \times I,\ T(u,t) \in X\}$. Since $(F \times I) \times X$ is a normable Frechet space by Remark 3 and G_T is a closed subspace of $(F \times I) \times X$ by Theorem 4, G_T is a normable Frechet space. Every normable Frechet space is an L^p, so G_T is an L^P space. Hence, we define the time cost function on G_T at time $t \in I$ as

$$J((u,t),(x,t)) = (|(u,t)|)^p + |T(u,t)|^p), \tag{10}$$

Since U_G^T is a net in $G_T \subseteq (F \times I) \times X$ (Section 3.2), we have $((u,t),T(u,t)) \in G_T$. Using Theorem 3 we can write **Equation (10)** as

$$|((u_\alpha,t),T(u_\alpha,t))| = (|(u_\alpha,t)|^p + |T(u_\alpha,t)|^p)^{\frac{1}{p}}$$

$$\implies |((u,t),T(u,t))| = (|(u,t)|^p + |T(u,t)|^p)^{\frac{1}{p}} \tag{11}$$

$$\implies |(y,t)| = (|(u,t)|^p + |T(u,t)|^p)^{\frac{1}{p}} \tag{12}$$

where $(y,t) = ((u,t),T(u,t)) \in G_T$. Hence,this net arises from the time cost function on G_T. □

Case 2 : Consider a net of the form

$$J((u,t),(x,t)) = (|(u,t) - (\hat{u},t)|^p + |T(u,t) - (\hat{x},t)|^p)^{\frac{1}{p}}$$

Let $(\hat{w},t) = ((\hat{u},t),(\hat{x},t))$, and $(w,t) = ((u,t),T(u,t)) \in G_T$. Then the net can be expressed as

$$J((u,t),(x,t)) = |(w,t) - (\hat{w},t)|$$

Alternatively, if we define $(y,t) = (\hat{w},t)$, $(y,t) \in G_T$, then

$$J((u,t),(x,t)) = |(y,t)|$$

where we have assumed that $(\hat{w},t) \in G_T$.

Proof. From the problem statement in Section 3.3, $T : F \times I \to X$ is a continuous linear transformation. Also , by Definition 2.2

$$G_T = \{((u,t), T(u,t)) : (u,t) \in F \times I,\ T(u,t) \in X\}$$

and G_T is a closed subspace of $(F \times I) \times X$ by Theorem 4. Under the assumption in Case 2, the net has the form

$$\begin{aligned} &J((u,t),(x,t)) = (|(u,t) - (\hat{u},t)|^p \\ &+ |T(u,t) - (\hat{x},t)|^p)^{\frac{1}{p}} \\ &= |((u,t) - (\hat{u},t)), (T(u,t) - (\hat{x},t))| \\ &\text{by } \textbf{Equation (11)} \end{aligned} \tag{13}$$

Let $(\hat{w},t) = ((\hat{u},t),(\hat{x},t))$, and $(w,t) = ((u,t), T(u,t)) \in G_T$. Then, **Equation (13)** can be written as

$$\begin{aligned} &J((u,t),(x,t)) = |((u,t), T(u,t)) - \\ &((\hat{u},t), \hat{x}(t))| = |(w,t) - (\hat{w},t)| \end{aligned} \tag{14}$$

Since G_T is a closed subspace of $(F \times I) \times X$ by Theorem 4, it follows that $((w,t) - (\hat{w},t)) \in G_T$. From **Equation (14)**, we have

$$\begin{aligned} &J((u,t),(x,t)) = |(w,t) - (\hat{w},t)| \\ &= |(y,t)|, \text{ and hence } (y,t) \in G_T \end{aligned}$$

Thus, by Definition 3.2, we obtain a net U_G^T in G_T using the time cost function from **Equation (10)**. □

4. Determination of the unique optimal control in G_T by the necessary condition for the normal system

We consider the system

$$\begin{aligned} &\frac{d}{dt}((x,t), T(x,t)) = A(t)((x,t), \\ &T(x,t)) + B(t)((u,t), T(u,t)) \end{aligned} \tag{15}$$

where A and B are $n \times n$ and $n \times r$ matrices, respectively. Then, from Theorem 13.2 in,[4] if $((u,t), T(u,t))^* \in U_G^T$, with $U_G^T \subseteq G_T$ is an optimal control with minimum-time t_1, we can write the optimal control in the form of (2) as

$$((u,t), T(u,t))^* = sgn[\eta' Y(t)]$$

Thus, this form is the necessary condition for optimal control , and we can write

$$\begin{aligned} &((u_j,t), T(u_j,t))^* = sgn[\eta' y^j(t)] \\ &= sgn[\eta' X^{-1}(t) b^j(t)] \end{aligned} \tag{16}$$

on $[0, t_{1j}]$ for $j = 1, 2, ..., r$ where $y^j(t)$ and $b^j(t)$ are the jth column vectors of $Y(t)$ and $B(t)$ respectively.

4.1. Essentially normal

Let

$$\begin{aligned} &E_j(\eta) = \{t;\ \eta' y^j(t) = 0, b^j(t) \neq 0, \\ &t \in [0, t_1]\} \end{aligned} \tag{17}$$

Then the system

$$\begin{aligned} &\frac{d}{dt}((x,t), T(x,t)) = A(t)((x,t), \\ &T(x,t)) + B(t)((u,t), T(u,t)) \end{aligned}$$

is considered essentially normal in G_T if the set $E_j(\eta)$ in the **Equation (17)** has measure zero, where A and B are $n \times n$ and $n \times r$ matrices respectively. Hence, if the above system is essentially normal on $[0, t_1]$, then the time optimal control $((u,t), T(u,t))^* \in U_G^T$ where $U_G^T \subseteq G_T$, is essentially unique, exhibits the essentially bang bang property , and is essentially determined by the necessary condition $((u,t), T(u,t))^* = sgn[\eta' Y(t)]$ on $[0, t_1]$ by.[4]

Remark 14. The optimal control $((u,t), T(u,t))^*$ is essentially determined by $((u,t), T(u,t))^* = sgn[\eta' Y(t)]$ if and only if the set $E_j(\eta) = \{t : \eta' y^j(t) = 0, b^j(t) \neq 0, t \in [0, t_1]\}$ has measure zero, by.[4]

Theorem 5. *The control $((u,t), T(u,t))^* \in U_G^T$ to reach $((z,t), T(z,t))$ in time t_1 is essentially determined on $[0, t_1]$ by $((u,t), T(u,t))^* = sgn[\eta' Y(t)]$ where $Y(t) = X^{-1}(t)B(t)$ and for some $\eta \neq 0$ if and only if $((z,t), T(z,t)) = (x(t_1; u^*(t)), Tx(t_1; u^*(t))$ is the exposed point of $A_G^T(t)$.*

Proof. Assume that $((u,t), T(u,t))^*$ is essentially determined on $[0, t_1]$ by

$$((u,t), T(u,t))^* = sgn[\eta' Y(t)]$$

Let η be an arbitrary non zero vecto. Then the support function $\psi(\eta)$ of $A_G^T(t)$ is defined as

$$\psi(\eta) = \sup_{((x,t), T(x,t)) \in A_G^T(t)} \langle \eta, ((x,t), T(x,t)) \rangle$$

This represents the signed distance from the origin to the supporting hyperplane M of $A_G^T(t)$ that is perpendicular to η[6], with the supporting hyperplane given by

$$\begin{aligned} &M = \{((x,t), T(x,t)) : \langle \eta, ((x,t), T(x,t)) \rangle \\ &= \langle \eta, ((z,t), T(z,t)) \rangle = \psi(\eta)\} \end{aligned}$$

According to the supporting hyperplane theorem, let a point $((z,t), T(z,t)) \in \partial A_G^T(t)$ where $A_G^T(t)$ is convex by Remark 12. Then there exists $\eta \neq 0$ such that

$$\begin{aligned} &M = \{((x,t), T(x,t)) : \langle \eta, ((x,t), T(x,t)) \rangle \\ &= \langle \eta, ((z,t), T(z,t)) \rangle\} \end{aligned}$$

is a supporting hyperplane of $A_G^T(t)$ at $((z,t),T(z,t))$ and for all $((x,t),T(x,t)) \in A_G^T(t)$, by,[5] we have

$$\langle \eta, ((x,t),T(x,t))\rangle \leq \langle \eta, ((z,t),T(z,t))\rangle \quad (18)$$

However, from the theorem statement , we assume that there exists one optimal control $((u,t),T(u,t))^* \in G_T$ at time t_1 and

$$((u,t),T(u,t))^* = sgn[\eta' Y(t)] \quad (19)$$

where η' is the transpose of η.

From **Equations (18)** and **(19)**, we can say that the point $((z,t),T(z,t))$ on the supporting hyperplane M can be reached in time t_1 only by the control in the form of **Equation (19)** , and any other control of the same form is essentially equal to $((u,t),T(u,t))^*$. Since $((z,t),T(z,t))$ is the only point in $A_G^T(t)$, it is an exposed point of $A_G^T(t)$ on M (see **Figure 1**).

Now assume that $((z,t),T(z,t)) \in G_T$ is an exposed point of $A_G^T(t)$.By definition , there exists a closed supporting hyperplane

$$M = \{((x,t),T(x,t)); \langle \eta, ((x,t),T(x,t))\rangle = \langle \eta, ((z,t),T(z,t))\rangle\}$$

such that $M \cap A_G^T(t) = ((z,t),T(z,t))$, where η is the outward normal of $A_G^T(t)$ at $((z,t),T(z,t))$. Thus, $((z,t),T(z,t))$ is the unique point of $A_G^T(t)$ on M. Since $((z,t),T(z,t))$ is an exposed point, it is also an extreme point , and hence a boundary point of $A_G^T(t)$. Lemma 13.1 in[4] states that a boundary point $((z,t),T(z,t)) = (x(t_1,u^*(t)),Tx(t_1,u^*(t)))$ of $A_G^T(t)$ with η as the outward normal to a supporting hyperplane through $((z,t),T(z,t))$, is reached if and only if the optimal controls $((u,t),T(u,t))^*$ are of the form $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ for some $\eta \neq 0$. Hence, the control $((u,t),T(u,t))^*$ reaches to $((z,t),T(z,t))$ at the minimum time t_1. Moreover, since the exposed point $((z,t),T(z,t))$ is an extreme point, Theorem 14.2 in[4] implies the control $((u,t),T(u,t))^*$ reaching $((z,t),T(z,t))$ in minimum time such that $M \cap A_G^T(t) = ((z,t),T(z,t))$, where η is the outward normal of $A_G^T(t)$ at $((z,t),T(z,t))$. Thus, $((z,t),T(z,t))$ is the unique point of $A_G^T(t)$ on M. Since $((z,t),T(z,t))$ is an exposed point, it is also an extreme point , and hence a boundary point of $A_G^T(t)$. Lemma 13.1 in[4] states that a boundary point $((z,t),T(z,t)) = (x(t_1,u^*(t)),Tx(t_1,u^*(t)))$ of $A_G^T(t)$ with η as the outward normal to a supporting hyperplane through $((z,t),T(z,t))$, is reached if and only if the optimal controls $((u,t),T(u,t))^*$ are of the form $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ for some $\eta \neq 0$. Hence, the control $((u,t),T(u,t))^*$ reaches to $((z,t),T(z,t))$ at the minimum time t_1. Moreover, since the exposed point $((z,t),T(z,t))$ is an extreme point , Theorem 14.2 in[4] implies the control $((u,t),T(u,t))^*$ reaching $((z,t),T(z,t))$ in minimum time

$$t_1 = \inf\{t : ((x,t),T(x,t)) = ((z,t),T(z,t)), \; t \in [t_0 \; \infty)\}$$

is essentially unique. Therefore, $((u,t),T(u,t))^*$ is essentially determined by $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ on $[0,t_1]$ □

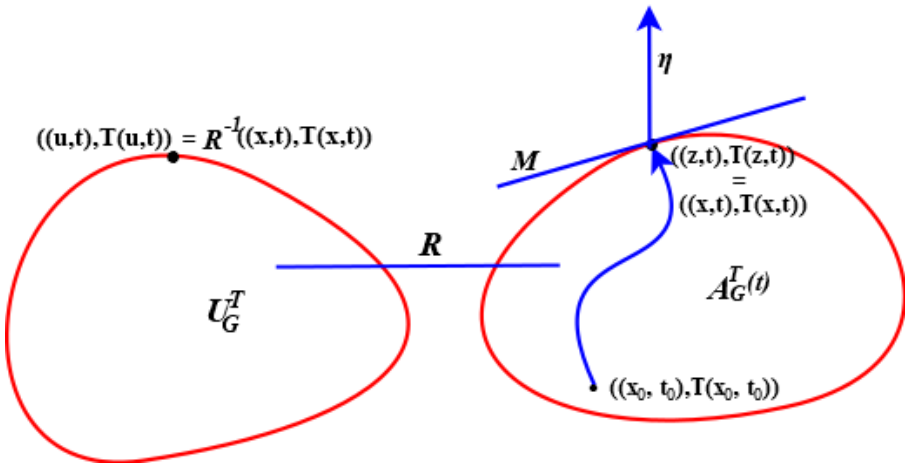


Figure 1. Geometric illustration of the exposed point property of the attainable set $A_G^T(t) \subseteq X$. The terminal point $\big((z,t),T(z,t)\big)$ is an exposed point of $A_G^T(t)$ determined by the supporting hyperplane M and its outward normal vector η and is associated with the corresponding unique time optimal control $((u,t),T(u,t))^* \in U_G^T$ in the Frechet graph space G_T for the normal system at the minimum time $t = t_1$, where $((u,t),T(u,t))^* = sgn(\eta' Y(t))$ with η' denoting the transpose of vector η , and $Y(t) = X^{-1}B(t)$ is the output space. In contrast, for the proper system, the time-optimal control need not be unique.

Theorem 6. *The system*

$$\frac{d}{dt}((x,t),T(x,t)) = A(t)((x,t),\\ T(x,t)) + B(t)((u,t),T(u,t))$$

is essentially normal on $[0,t_1]$ *if and only if* $A_G^T(t)$ *is strictly convex, where* A *and* B *are* $n\times n$ *and* $n\times r$ *matrices , respectively.*

Proof. Let $E_j(\eta) = \{t : \eta' y^j(t) = 0, b^j(t) \neq 0, t \in [0,t_1]\}$ where $y^j(t)$ and $b^j(t)$ are the $j-th$ columns of $Y(t)$ and $B(t)$ respectively. Assume the system is essentially normal on $[0,t_1]$. Then, by Section 4.1, the set $E_j(\eta)$ has measure zero for every nonzero vector η and for t on $[0,t_1]$, so the optimal control is essentially unique. By Theorem 5, the optimal control has the form $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ in minimum time t_1 on $[0,t_1]$, for some nonzero η , and it reaches the point $((z,t),T(z,t))$ which is an exposed point of $A_G^T(t)$ such that $((z,t),T(z,t)) = (x(t_1,u^*(t)),Tx(t_1,u^*(t)))$. Since an exposed point is an extreme point, Definitions 2.7 and 2.6 (property 3), imply that $A_G^T(t)$ is strictly convex.

Conversely, assume that $A_G^T(t)$ is strictly convex. Then $A_G^T(t)$ contains more than one boundary element, and each boundary element is an exposed point. By Theorem 5, the optimal control reaching each exposed point by $((z,t),T(z,t)) = (x(t_1,u^*(t)),Tx(t_1,u^*(t)))$ must be essentially of the form $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ for some $\eta \neq 0$. This implies that the set $E_j(\eta)$ has measure zero by Remark 14. Therefore, system (15) is essentially normal. □

5. Proper system

The system

$$\frac{d}{dt}((x,t),T(x,t)) = A(t)((x,t),\\ T(x,t)) + B(t)((u,t),T(u,t))$$

is proper over the interval $[t_0,t_1]$ if $\eta^/ X^{-1}(t)B = \eta^/ Y(t) = 0$ almost everywhere on $[t_0,t_1]$ implies $\eta = 0$ by.[4]

Remark 15. The attainable set $A_G^T(t)$ is symmetric if, for each x^0 belonging to $A_G^T(t)$, the vector $-x^0$ also belongs to $A_G^T(t)$ by.[4]

Remark 16. The attainable set $A_G^T(t_1)$ expands at time t_1 if , for every t with $0 \leq t < t_1$, the set $A_G^T(t)$ lies entirely in the interior of $A_G^T(t_1)$. If $A_G^T(t)$ expands for all $t_1 > 0$, then we say that $A_G^T(t)$ is expanding , by.[4]

Theorem 7. *If a control* $((u,t),T(u,t))^*$ *of the form* $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ *brings the system from* (x^0,Tx^0) *to the origin in time* t_1*, and if* $A_G^T(t)$ *expands at time* t_1*, then* $((u,t),T(u,t))^*$ *is an optimal control.*

Proof. Suppose that the control $((u,t),T(u,t))^*$ of the form $((u,t),T(u,t))^* = sgn[\eta' Y(t)]$ brings the system

$$\frac{d}{dt}((x,t),T(x,t)) = A(t)((x,t),\\ T(x,t)) + B(t)((u,t),T(u,t))$$

where A and B are $n\times n$ and $n\times r$ matrices respectively, from (x^0,Tx^0) to the origin at time t_1. By Remark 15, the attainable set $A_G^T(t_1)$ is symmetric , so $-(x^0,Tx^0)$ belong to $A_G^T(t_1)$. Hence , by Lemma (13.1) in[4], $-(x^0,Tx^0)$ lies on the boundary of $A_G^T(t_1)$ with η as an outward normal to a supporting the hyperplane of $A_G^T(t_1)$. Moreover, by Lemma (12.3) in[4], if $-(x^0,Tx^0)$ is an interior point of $A_G^T(t_1)$ for some $t_1 > 0$, then it is also an interior point of $A_G^T(t)$ for some $0 < t < t_1$. Thus, $A_G^T(t) \subset A_G^T(t_1)$, so $A_G^T(t)$ expands at t_1 by Remark 16. Now, by Theorem (13.2) in,[4] since $A_G^T(t)$ expands at t_1, we conclude that t_1 is the minimum time and $((u,t),T(u,t))^*$ is an optimal control. □

Theorem 8. *The autonomous system*

$$\frac{d}{dt}((x,t),T(x,t)) = A(t)((x,t),\\ T(x,t)) + B(t)((u,t),T(u,t))$$

where A *and* B *are constant matrices, is proper if and only if*

$$rank\,[B,AB,A^2B,....,A^{n-1}B] = n\ (controllable)$$

Proof. If the autonomous system is proper, then $\eta' Y(t) = 0 \implies \eta = 0$. We need to show that

$$rank\,[B,AB,A^2B,....,A^{n-1}B] = n$$

Assume to the contrary, that

$$rank\,[B,AB,A^2B,....,A^{n-1}B] < n$$

Then there exists a nonzero vector η $(\eta \neq 0)$ such that

$$\eta' Y(t) = 0, \quad \text{by the theorem 7.12 in}^2$$
$$\implies \eta' X^{-1}(t)B = 0, \text{by the Section 5}$$
$$\implies \eta' e^{-At}B = 0\ , \text{ for all } t \in [t_0,t_1],$$
$$X^{-1}(t) = e^{-At} \text{ by}^4$$

Since, e^{-At} is analytic, successive differentiation and evaluation at $t = 0$ give

$$\eta' B = 0,\ -\eta' AB = 0,$$
$$\eta' A^2B = 0, ..., (-1)^{n-1}\eta' A^{n-1}B = 0$$

Thus,

$$\eta' A^k B = 0, \text{ for all } k = 0, 1, 2, ..., n-1$$
$$\implies \eta' = 0$$
$$\implies \eta = 0$$

which is a contradiction of our assumption and hence

$$rank\,[B, AB, A^2B, ..., A^{n-1}B] = n$$

Conversely, suppose

$$rank\,[B, AB, A^2B, ..., A^{n-1}B] = n$$

We must show that the autonomous system is proper. If $\eta' Y(t) = 0$ with $\eta = 0$, there is nothing to prove. Assume instead that $\eta \neq 0$. Then the system is not controllable by Theorem 7.12 in[2], so

$$\eta' Y(t) = 0$$
$$\eta' X^{-1}(t) B = 0$$
$$\eta' e^{-At} B = 0 \text{ , for all } t \in [t_0, t_1]$$

Again, since e^{-At} is analytic successive differentiation and evaluation at $t = 0$ give

$$\eta' B = 0, \ -\eta' AB = 0,$$
$$\eta' A^2 B = 0, ..., (-1)^{n-1} \eta' A^{n-1} B = 0$$

Hence,

$$\eta' A^k B = 0, \text{ for all } k = 0, 1, 2, ..., n-1$$
$$\implies \eta' = 0$$
$$\implies \eta = 0$$

which is a contradiction of our assumption

Also, by our assumption

$$rank\,[B, AB, A^2B, ..., A^{n-1}B] = n$$

Then, η must be zero, and the autonomous system in **Equation (8)** is proper. □

6. Example 1

Show that the following system is normal for the Frechet graph space:

$$\frac{d}{dt}(x_1, Tx_1) = (x_2, Tx_2)$$
$$\frac{d}{dt}(x_2, Tx_2) = -(x_1, Tx_1) + (u, Tu)$$

Solution: We express the given system in state–space (matrix) form as

$$\begin{bmatrix}(\dot{x}_1, T\dot{x}_1)\\(\dot{x}_2, T\dot{x}_2)\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}(x_1, Tx_1)\\(x_2, Tx_2)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}(u, Tu)$$

where

$$A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, B = \begin{bmatrix}0\\1\end{bmatrix}$$

We must show that this system is normal. For normality , the condition $\eta' Y(t) = 0$ where η' is the transpose of the outward normal η and $Y(t) = X^{-1}(t)B$ is the output space, must hold identically for all $t \in [0, t_1]$ only if $\eta = 0$. Now,

$$\eta' Y(t) = 0 \implies \eta' X^{-1}(t) B = 0$$
$$\implies \eta' e^{-At} B = 0$$

Hence,

$$e^{At} = \begin{bmatrix}cost & sint\\-sint & cost\end{bmatrix} \tag{20}$$

Now, from **Equation (20)**

$$e^{-At} = \begin{bmatrix}cost & -sint\\sint & cost\end{bmatrix} = X^{-1}(t)$$

Define,

$$Y(t) = X^{-1}(t)B = e^{-At}B = \begin{bmatrix}-sint\\cost\end{bmatrix}$$

Thus,

$$\eta' Y(t) = \begin{bmatrix}\eta_1 & \eta_2\end{bmatrix}\begin{bmatrix}-sint\\cost\end{bmatrix}$$

$$\implies -\eta_1 sint + \eta_2 cost = 0 \tag{21}$$

Case 1: When $\eta_1 \neq 0, \ \eta_2 = 0$. Then **Equation (21)** is

$$-\eta_1 sint = 0$$
$$\implies sint = 0$$
$$\implies sint = sin(n\pi)$$
$$\implies t = n\pi$$

Case 2: When $\eta_1 = 0, \ \eta_2 \neq 0$. Then **Equation (21)** is

$$\eta_2 cost = 0$$
$$\implies cost = 0$$
$$\implies cost = cos\left(\frac{\pi}{2} + n\pi\right)$$
$$\implies t = \frac{\pi}{2} + n\pi$$

Case 3: When $\eta_1 \neq 0, \ \eta_2 \neq 0$. Then **Equation (21)** is

$$-\eta_1 sint + \eta_2 cost = 0$$
$$\implies \eta_1 sint = \eta_2 cost$$
$$\implies tant = \left(\frac{\eta_2}{\eta_1}\right)$$
$$\implies t = tan^{-1}\left(\frac{\eta_2}{\eta_1}\right)$$

Case 4: When $\eta_1 = 0, \ \eta_2 = 0$. Then **Equation (21)** is

$$-\eta_1 sint + \eta_2 cost = 0 \tag{22}$$

In case 1, case 2 and case 3, it is clear that the given system is normal. But In case 4, $\eta = 0$,

which does not affect normality. Thus, by Section 4.1, the system is normal on $[0, t_1]$, and the unique time optimal control $((u,t), T(u,t))^* \in U_G^T$ at the minimum time $t = t_1$, is essentially unique , where U_G^T is the unit ball as a net in the Frechet graph space G_T. This control exhibits the essentially bang bang property and is essentially determined by the necessary condition $((u,t), T(u,t))^* = sgn[\eta' Y(t)]$ on $[0, t_1]$.

Overhead crane realisation of the abstract product-space system (an application of Example 1).

To provide a practical interpretation of the abstract product-space control formulation, we consider the time-optimal regulation of an industrial overhead gantry crane transporting a flexibly suspended payload. Overhead crane systems are widely studied in optimal control and industrial automation because of their coupled oscillatory dynamics and actuator constraints.[20–22]

Let $y(t) \in R$ denote the horizontal displacement of the trolley along the rail guide, and let $\theta(t) \in R$ represent the angular swing displacement of the suspended payload. Under the standard small-angle approximation about the lower vertical equilibrium configuration, the normalized crane dynamics are governed by

$$\ddot{y}(t) = m(t), \qquad \ddot{\theta}(t) = -\theta(t) + m(t). \tag{23}$$

Here, $m(t)$ is the bounded control force applied to the trolley mechanism.

Define the state vector

$$Y(t) = \begin{bmatrix} y(t) \\ \theta(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix}.$$

Then the system admits the state-space representation

$$\dot{Y}(t) = A_Y Y(t) + B_Y m(t), \tag{24}$$

where

$$A_Y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad B_Y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

To relate this physical model to the abstract product-space formulation, we consider the invertible transformation

$$X = MY, \tag{25}$$

where

$$X = \begin{bmatrix} x_1 \\ Tx_1 \\ x_2 \\ Tx_2 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The transformed coordinate relations are given by

$$\begin{cases} x_1(t) = y(t), \\ Tx_1(t) = y(t) + \theta(t), \\ x_2(t) = \dot{y}(t), \\ Tx_2(t) = \dot{y}(t) + \dot{\theta}(t). \end{cases} \tag{26}$$

Thus, the transformed variables combine trolley motion and payload oscillation into a coupled product-space representation.

Applying the transformation to the crane system yields

$$\dot{X} = A_X X + B_X U, \tag{27}$$

where

$$A_X = M A_Y M^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

and

$$B_X U = \begin{bmatrix} 0 \\ 0 \\ u \\ Tu \end{bmatrix}.$$

The transformed control variables are defined by

$$u(t) = m(t) + x_1(t), Tu(t) = 2u(t) - x_1(t). \tag{28}$$

We now verify that the transformed dynamics satisfy the abstract product-space structure.

Since

$$x_1(t) = y(t),$$

we obtain

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t).$$

Similarly,

$$Tx_1(t) = y(t) + \theta(t),$$

which implies

$$\frac{d}{dt}(Tx_1(t)) = \dot{y}(t) + \dot{\theta}(t) = Tx_2(t).$$

Hence,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ Tx_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ Tx_2 \end{pmatrix}. \tag{29}$$

Since,

$$x_2(t) = \dot{y}(t),$$

we have

$$\dot{x}_2(t) = \ddot{y}(t).$$

Using **Equation (23)**,

$$\dot{x}_2(t) = m(t).$$

From **Equation (28)**,

$$u(t) = m(t) + x_1(t),$$

and therefore

$$\dot{x}_2(t) = -x_1(t) + u(t).$$

Now,

$$Tx_2(t) = \dot{y}(t) + \dot{\theta}(t),$$

and hence

$$\frac{d}{dt}(Tx_2(t)) = \ddot{y}(t) + \ddot{\theta}(t).$$

Using **Equation (23)**,

$$m(t) + (-\theta(t) + m(t)) = 2m(t) - \theta(t).$$

From **Equation (26)**,

$$Tx_1(t) = x_1(t) + \theta(t),$$

which gives

$$\theta(t) = Tx_1(t) - x_1(t).$$

Hence,

$$\begin{aligned} 2m(t) - \theta(t) &= 2m(t) - \big(Tx_1(t) - x_1(t)\big) \\ &= x_1(t) - Tx_1(t) + 2m(t). \end{aligned}$$

Using **Equation (29)**,

$$Tu(t) = 2u(t) - x_1(t),$$

together with

$$u(t) = m(t) + x_1(t),$$

yields

$$Tu(t) = 2m(t) + x_1(t).$$

Therefore,

$$x_1(t) - Tx_1(t) + 2m(t) = -Tx_1(t) + Tu(t).$$

Consequently,

$$\frac{d}{dt}(Tx_2(t)) = -Tx_1(t) + Tu(t).$$

Combining the above relations yields

$$\frac{d}{dt}\begin{pmatrix} x_2 \\ Tx_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ Tx_1 \end{pmatrix} + \begin{pmatrix} u \\ Tu \end{pmatrix}. \tag{30}$$

From **Equations (29)** and **(30)**, the transformed crane dynamics realise the abstract product-space control system.

The controllability matrix corresponding to the transformed system is

$$C_X = [B_X, A_X B_X, A_X^2 B_X, A_X^3 B_X].$$

Because the transformation matrix M is invertible, controllability is preserved under the transformation. In particular,

$$C_X = MC_Y,$$

where

$$C_Y = [B_Y, A_Y B_Y, A_Y^2 B_Y, A_Y^3 B_Y]$$

denotes the controllability matrix of the original crane model. Therefore,

$$\operatorname{rank}(C_X) = \operatorname{rank}(C_Y).$$

It follows that the transformed system satisfies the condition

$$\eta' Y(t) = 0 \quad \forall\, t \in [0, t_1] \quad \Longrightarrow \quad \eta = 0.$$

Then the transformed system is proper and normal. If $\eta \neq 0$, we can accordingly, say that the associated optimal control is essentially unique and exhibits the bang-bang property.

More precisely,

$$u^*(t) = u_{\max} \operatorname{sgn}(\eta' Y(t)),$$

while the transformed control component satisfies

$$Tu^*(t) = 2u^*(t) - x_1(t).$$

Therefore, the actuator alternates between extreme admissible force levels according to the sign of the switching function $\eta' Y(t)$. From a physical perspective, this corresponds to alternating acceleration and braking actions in order to achieve minimum-time payload transportation, together with oscillation reduction.

Thus, the abstract product-space formulation admits a meaningful interpretation related to Example 6 within the framework of overhead crane stabilisation and constrained motion control , with $m(t)$ as the bounded control force applied to the trolley mechanism , represented by the time-optimal control $((u,t), T(u,t))^* \in U_G^T$ at the minimum time $t = t_1$ in the Frechet graph space G_T.

Remark 17. Through the invertible transformation $X = MY$, the overhead crane model attains the structure of the abstract product-space control system. In the transformed coordinates, the dynamics reflect the interaction between trolley motion and payload swing, and the associated optimal control possesses the normality and bang-bang features characteristic of constrained minimum-time control in the overhead crane system.

Example 2

Show that the following system is proper but not normal for the Frechet graph space:

$$\frac{d}{dt}(x_1, Tx_1) = -(x_1, Tx_1) + (u_1, Tu_1)$$
$$\frac{d}{dt}(x_2, Tx_2) = -2(x_2, Tx_2) + (u_1, Tu_1) + (u_2, Tu_2)$$

Solution: We express the given system in the state space (matrix) form as

$$\begin{bmatrix}(\dot{x}_1, T\dot{x}_1)\\(\dot{x}_2, T\dot{x}_2)\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}(x_1, Tx_1)\\(x_2, Tx_2)\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}(u_1, Tu_1)\\(u_2, Tu_2)\end{bmatrix}$$

where,

$$A = \begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}, B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

We must show that this system is proper. According to Theorem 8, an autonomous system is proper if and only if rank $[B, AB, A^2B, .., A^{n-1}B] = n$.

Now,

$$B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \quad AB = \begin{bmatrix}-1 & 0\\-2 & -2\end{bmatrix}$$

Hence, rank $[B, AB]$

$$= rank \begin{bmatrix}1 & 0 & -1 & 0\\1 & 1 & -2 & -2\end{bmatrix}$$

Using row and column operations

$$= rank \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

Thus, the rank $[B, AB] = 2 = n$ and the system is proper. Then, from Remark 15 and Theorem 7 in Section 5, there exists a time - optimal control $((u,t), T(u,t))^* \in U_G^T$ at the minimum time $t = t_1$ in the Frechet graph space G_T.

Now we must show that the system is not normal. According to Theorem (16.1) in[11] , the system is normal if and only if , for each $j = 1, 2, ..., r$ the vectors $b^j, Ab^j, ..., A^{n-1}b^j$ are linearly independent.

Case 1: For column b^1

$$b^1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad Ab^1 = \begin{bmatrix}1\\-2\end{bmatrix}$$

Now,

$$\{b^1, Ab^1\} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$$
$$\implies \begin{bmatrix}1 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = 0$$
$$\implies c_1 + c_2 = 0$$
$$\implies c_1 - 2c_2 = 0 \tag{31}$$

Thus, from **Equation (31)**

$$c_1 = 0 \text{ and } c_2 = 0.$$

Since all coefficients are zero, the system is linearly independent for column b^1 of B.

Case 2: For column b^2

$$b^2 = \begin{bmatrix}0\\1\end{bmatrix}, \quad Ab^2 = \begin{bmatrix}0\\-2\end{bmatrix}$$

Now,

$$\{b^1, Ab^1\} = \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\-2\end{bmatrix}\right\}$$
$$\implies \begin{bmatrix}0 & 0\\1 & -2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = 0$$
$$\implies c_1 - 2c_2 = 0$$
$$\implies c_1 = 2c_2$$

Choosing $c_2 = 1$, , we get $c_1 = 2$. Hence, the system is linearly dependent for column b^2 of B, and therefore the overall system is not normal.

Practical Interpretation. The above system may be viewed as a temperature control model used in smart building HVAC (Heating, Ventilation, and Air Conditioning) systems.[23] Here, x_1 and x_2 represent the temperatures of two connected rooms or zones. The control input u_1 acts like a central heater supplying heat to both zones, while u_2 acts as a local heater affecting only the second zone. The matrix

$$B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

indicates that the first control input influences both zones, whereas the second input has only a local effect. Because one control input does not affect all parts of the system independently, the system is not normal. However, the combined effect of both control inputs is sufficient to control the overall system, so the system remains proper. Similar distributed temperature-control

structures are commonly used in modern HVAC and thermal management systems.

Application to HVAC thermal control related to Example 2.

The above system can be interpreted as a thermal control model for a two-zone smart building. Let

$$X(t) = \begin{bmatrix}(x_1, Tx_1)\\(x_2, Tx_2)\end{bmatrix}$$

denote the thermal energy states of two connected rooms, and let

$$U(t) = \begin{bmatrix}(u_1, Tu_1)\\(u_2, Tu_2)\end{bmatrix}$$

represent the heating inputs. Then the system can be expressed in the form

$$\dot{X}(t) = AX(t) + BU(t),$$

where

$$A = \begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}, \qquad B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}.$$

In this model, the control input u_1 acts as a common heating source supplying energy to both rooms, whereas u_2 represents an additional local heater connected only to the second room. The negative entries of the matrix A describe the natural heat loss occurring in each zone.

From the structure of the control matrix B, the second actuator affects only the second subsystem. Hence,

$$b^2 = \begin{bmatrix}0\\1\end{bmatrix}, \qquad Ab^2 = \begin{bmatrix}0\\-2\end{bmatrix},$$

are linearly dependent, which shows that the system is not normal. However,

$$\text{rank}[B, AB] = 2,$$

and therefore the overall system remains proper. Such distributed heating configurations , represented by the time-optimal control $((u,t), T(u,t))^* \in U_G^T$ at the minimum time $t = t_1$ in the Frechet graph space G_T, frequently occur in multi-zone HVAC systems and smart-building thermal management applications.

7. Conclusion

In this paper, we establish the existence and uniqueness of the minimum-time optimal control problem in the Frechet graph space G_T in Section 3.1. To facilitate the analysis, the unit ball U_G^T in Section 3.2 is constructed as a net in G_T. We consider a single-state continuous linear dynamical system $\sum$ in 1 on a reflexive and rotund graph space G_T and determine the time-optimal control that minimises the associated time-cost function J_t defined in Section 3.3. In addition, we provide examples of nets generated from a particular cost function. Next, we investigate minimum time optimal control for both normal and proper systems through several theorems. For normal systems, a unique minimum time-optimal control is obtained, whereas for proper systems, the optimal control need not be unique. Finally, illustrative examples are presented to demonstrate and validate the theoretical results in the Frechet graph space G_T. Future research will focus on extending the existence and uniqueness results for the minimum-time optimal control problem to practical engineering applications. In particular, the theoretical framework developed in this work, together with the unit-ball construction in the Frechet graph space G_T, will be further explored using the Pontryagin Maximum Principle, the Bang–Bang Principle, and the geometric control-theoretic methods of Agrachev-Sachkov.

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Validation: Zeeshana Zahoor
Visualization: Zeeshana Zahoor
Writing – original draft: Zeeshana Zahoor
Writing – review & editing: All authors

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Appendix

Nomenclature

α	An index, $\alpha = 1, 2, 3, \ldots$
$\theta(t)$	Angular swing displacement of the payload
$A_G^T(t)$	Attainable set for G_T in X
∂U_G^T	Boundary of the unit ball U_G^T
$\mathbb{C}$	Complex numbers
T	Continuous linear transformation
B	Control-input matrix of order $n \times r$
$((u,t), T(u,t))^*$	Control variable in the admissible control set U_G^T
Λ	Directed set
D	Domain of T
V	Dual space of F
$((u,t), T(u,t))$	Element of G_T
$((x,t), T(x,t))$	Element of $A_G^T(t)$
F	Fréchet space
L^p	Lebesgue space of p-integrable functions
G	Linear mapping representing the mapping ϕ
R	Mapping defined as $R : U_G^T \to X$
e^{At}	Matrix exponential (state-transition matrix)
t_1	Minimum time
λ	Non-negative real scalar
Y	Output space
η	Outward normal to the boundary of the attainable set $A_G^T(t)$
$m(t)$	Physical actuator input
$\mathbb{R}$	Real numbers
F^*	Set of all continuous linear functionals on F
F^{**}	Set of all continuous linear functionals on F^*
$\sum$	Single-state linear dynamical system
X	Single-variable state space
A	State matrix of order $n \times n$
(x,t)	State variable at time t in X
(z,t)	State variable at time t in X
ψ	Support function
M	Supporting hyperplane to the attainable set $A_G^T(t)$
G_T	Graph of T
I	Time interval $I = [t_0, t]$
ϕ	Transition mapping represented by the linear mapping
U	Unit ball in the Fréchet space F
U_G^T	Unit ball in G_T as a net
$E_j(\eta)$	Zero set of the jth switching function

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