An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.16, No.6, pp.2445-2457 (2026)
https://doi.org/10.36922/IJOCTA026330215

RESEARCH ARTICLE

Legendre spectral-collocation approximation for an optimal control problem governed by generalized Caputo integro-differential equations

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ARTICLE INFO

Article history:
Received: August 14, 2026
Revised: September 18, 2026
Accepted: September 22, 2026
Published Online: October 13, 2026

Keywords:
Generalized Caputo fractional derivative
Legendre polynomials
Fractional Integro-differential equation
Spectral methods
Optimal control

AMS Classification 2010:
26A33, 49K15, 65K10

ABSTRACT

This work investigates a class of optimal control problems governed by generalized Caputo fractional integro-differential equations. The numerical solution is based on a Legendre spectral-collocation method combined with the Gauss-Legendre quadrature rule. To reduce the problem to a finite-dimensional optimization problem, the Rayleigh-Ritz method is applied, and the resulting nonlinear system is solved using Newton's iterative procedure. We derive an approximation to the generalized Caputo fractional derivative. Additionally, we present an error bound for this approximation and for the proposed method. The accuracy and efficiency of the developed method are verified by numerical experiments.

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1. Introduction

Fractional differential equations have increasingly attracted attention due to their wide-ranging applications in science, mechanics, and engineering. In contrast to traditional integer-order models, fractional formulations provide a more accurate, flexible, and reliable framework for describing the dynamics of complex physical systems. Several definitions of fractional operators have been proposed in the literature, which are of considerable relevance to diverse applications.[1–6] Among them, Katugampola introduced a generalized form of the Caputo fractional derivative.[7,8] The behavior of the generalized Caputo derivative depends critically on both the fractional order ν and the parameter σ, a feature that is particularly useful for graphical simulations with real-world data.[9–13]

Optimal control problems (OCPs) arise in a variety of fields, including economics, engineering, and science. These problems involve designing a control strategy for a dynamical system, typically governed by differential or integral equations, to optimize an objective function that depends on both the system's state and the control variables. A fractional-order control problem (FOCP) involves determining control and state trajectories that minimize a given objective functional while satisfying dynamic constraints defined by fractional-order models. The nonlinear nature of FOCPs makes obtaining analytical solutions particularly challenging, especially when the systems are governed by integro-differential equations. As a result, many researchers have developed various numerical methods to approximate solutions with satisfactory accuracy.

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Nonlinear FOCPs subject to fractional integro-differential constraints have been investigated numerically using different definitions of fractional derivatives.[14–19] These studies demonstrate the effectiveness of numerical and spectral techniques for treating fractional optimal control problems and provide useful computational frameworks for obtaining approximate state and control functions. Moreover, numerical schemes for fractional-order differential equations have been developed using various computational approaches, with particular attention to the numerical treatment and stability of fractional models.[20] However, the existing numerical approaches mainly focus on particular fractional operators, and the numerical treatment of nonlinear optimal control problems governed by integro-differential equations involving the generalized Caputo derivative remains limited.

Recent studies have also investigated related fractional optimal control problems from theoretical perspectives, including fractional delay and integro-differential systems.[21,22] These works provide valuable analytical foundations for the study of fractional optimal control problems. Nevertheless, the problem considered in the present work has a different structure, involving a nonlinear Volterra-type integro-differential constraint together with the generalized Caputo fractional derivative. In particular, the additional parameter σ associated with the generalized Caputo operator introduces a further feature that is not present in the standard Caputo formulation and must be incorporated into the numerical approximation.

Motivated by this gap, we develop a numerical framework for nonlinear fractional optimal control problems governed by integro-differential equations involving the generalized Caputo derivative. The proposed method combines shifted-Legendre spectral collocation, Gauss–Legendre quadrature, and the Rayleigh–Ritz procedure in a unified computational framework. The state and control variables are approximated by shifted-Legendre polynomial expansions, while the Volterra integral term is evaluated using Gauss–Legendre quadrature. The Rayleigh–Ritz procedure then transforms the continuous optimal control problem into a finite-dimensional nonlinear algebraic system, which is solved by Newton's method. In addition to the numerical formulation, we derive convergence and error estimates for the generalized Caputo fractional derivative and the proposed approximation. Thus, the contribution of this work is the development and analysis of a spectral–quadrature framework specifically adapted to the combination of a generalized Caputo fractional operator, a nonlinear Volterra integral term, and an optimal control formulation. To the best of our knowledge, this combination has not previously been investigated numerically for the class of nonlinear fractional integro-differential optimal control problems considered here. Therefore, our investigation focuses on developing a numerical solution for the FOCP, formulated as follows:

$$\min J = \int_0^1 \mathcal{F}(\tau, \varkappa(\tau), \varsigma(\tau))\, d\tau, \tag{1}$$

governed by fractional integro-differential equation:

$$\begin{aligned} &{}^{C}\mathcal{D}_{a^+}^{\nu,\sigma}\varkappa(\tau) = \theta(\tau)\varkappa(\tau) + \vartheta(\tau)\varsigma(\tau) \\ &+ \zeta(\tau)\int_0^\tau \psi(\tau, t, \varkappa(t))\, dt,\ 0 < \nu \le 1, \\ &\sigma > 0,\ \varkappa(0) = \varkappa_0, \end{aligned} \tag{2}$$

where $\varkappa(\tau)$ and $\varsigma(\tau)$ represent the state and control variables, respectively. Here, τ is the current independent variable, while t is the integration variable in the Volterra-type integral, with $0 \le t \le \tau$. The operator ${}^{C}\mathcal{D}_{a^+}^{\nu,\sigma}\varkappa(\tau)$ denotes the generalized Caputo fractional derivative. The functions $\zeta(\tau)$ and $\vartheta(\tau)$ are prescribed coefficients, with $\zeta(\tau)$ multiplying the Volterra integral term and not representing an additional control variable. Since $\zeta(\tau)$ depends only on τ, it is taken outside the integral with respect to t. Finally, $\mathcal{F}$ and ψ are continuously differentiable functions, with ψ allowed to be linear or nonlinear.

The content of the paper proceeds as follows: Section 2 introduces the shifted-Legendre polynomials of the first kind and their properties, along with several useful definitions of fractional derivatives. In Section 3, we explain how to numerically approximate the function and the integral involved in the suggested problem, while Section 4 presents the complete numerical technique used to solve it. Section 5 is devoted to the error estimates for the generalized Caputo fractional derivative and the proposed method. In Section 6, numerical examples are presented to illustrate the efficiency of the method. Finally, Section 7 provides the conclusions.

2. Preliminary concepts and notations

This section provides a concise overview of the definitions of fractional integrals and derivatives in the literature, together with selected properties of Legendre polynomials of the first kind relevant to our study.

2.1. Fractional operators and their properties

In this subsection, we will cover several definitions of fractional integrals and derivatives that are important to our work.

Definition 1. [4,5,23] *For $\varkappa : [0,\mathcal{T}] \to \mathbb{R}$ and $\nu > 0$, the Riemann–Liouville integral of order ν is*

$$ {}_0\mathcal{I}^{\nu}\varkappa(\tau) = \frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-t)^{\nu-1}\varkappa(t)dt, \quad 0<\tau<\mathcal{T}. $$

with $k = [\nu]+1$, the corresponding derivative is

$$ {}_0\mathcal{D}^{\nu}\varkappa(\tau) = \left(\frac{d}{d\tau}\right)^{k} {}_0\mathcal{I}^{k-\nu}\varkappa(\tau), \quad 0<\tau<\mathcal{T}. $$

Definition 2. [24] *Let $\varkappa : [0,\mathcal{T}] \to \mathbb{R}$ and $\nu > 0$. The Caputo derivative of $\varkappa(\tau)$, corresponding to the parameter ν, is*

$$ {}_0D^{\nu}\varkappa(\tau) = {}_0I^{m-\nu}\varkappa^{(m)}(\tau), \quad m = [\nu]+1. $$

Definition 3. *(see[7,10,25,26]) Let $0 < c < d < \infty$, $\nu \in (0,1)$, $\sigma > 0$, and let $\varkappa \in AC[c,d]$. The generalized Caputo fractional derivative of order ν is defined by*

$$ {}^{C}D_{c^+}^{\nu,\sigma}\varkappa(\tau) = \frac{\sigma^{\nu}}{\Gamma(1-\nu)}\tau^{1-\sigma}\frac{d}{d\tau}\int_c^{\tau}\frac{t^{\sigma-1}}{(\tau^{\sigma}-t^{\sigma})^{\nu}} [\varkappa(t)-\varkappa(c)]dt, \quad \tau\in(c,d]. \tag{3} $$

Theorem 1. [25] *If $\varkappa \in C^1[c,d]$, then the Caputo–Katugampola derivative admits the following integral form:*

$$ {}^{C}D_{c^+}^{\nu,\sigma}\varkappa(\tau) = \frac{\sigma^{\nu}}{\Gamma(1-\nu)}\int_c^{\tau}\frac{\varkappa'(t)}{(\tau^{\sigma}-t^{\sigma})^{\nu}}dt. \tag{4} $$

Remark 1. *The generalized Caputo fractional derivative unifies several well-known operators. In particular, when $\sigma = 1$ it reduces to the classical Caputo fractional derivative, while as $\sigma \to 0^+$ it coincides with the Caputo–Hadamard fractional derivative.[26,27]*

Definition 4. [7,28] *The generalized Caputo fractional integral is defined by*

$$ {}_0I^{\nu,\sigma}\varkappa(\tau) = \frac{1}{\Gamma(\nu)}\int_0^{\tau}\left(\frac{\tau^{\sigma}-t^{\sigma}}{\sigma}\right)^{\nu-1}\varkappa(t)\frac{dt}{t^{1-\sigma}}. $$

2.2. Legendre polynomials in shifted form

One can generate the Legendre polynomials on $[-1,1]$,-is via the recurrence relation given below: $\mathscr{L}_{\kappa+1}(x) = \frac{2\kappa+1}{\kappa+1}x\mathscr{L}_{\kappa}(x) - \frac{\kappa}{\kappa+1}\mathscr{L}_{\kappa-1}(x), \quad \kappa \geq 1,$ $\mathscr{L}_0(x) = 1, \quad \mathscr{L}_1(x) = x.$

It is worth noting that the Legendre polynomials, $\mathscr{L}_{\kappa}(x)$, can be represented analytically as follows:[29]

$$ \mathscr{L}_{\kappa}(x) = \sum_{r=0}^{[\kappa/2]}(-1)^r\frac{(2\kappa-2r)!}{2^{\kappa}\,r!\,(\kappa-r)!\,(\kappa-2r)!}\,x^{\kappa-2r}. \tag{5} $$

The orthogonality relation of Legendre polynomials on $[-1,1]$, with weight $\omega(x) = 1$, is:

$$ \int_{-1}^{1}\mathscr{L}_{\kappa}(x)\mathscr{L}_j(x)dx = \beta_{\kappa}\delta_{\kappa j}, \;\; \beta_{\kappa} = \frac{2}{2\kappa+1}, $$

$$ \int_{-1}^{1}(1-x^2)\mathscr{L}'_{\kappa}(x)\mathscr{L}'_j(x)dx = \kappa(\kappa+1)\beta_{\kappa}\delta_{\kappa j}, $$

where $\delta_{\kappa j}$ is the Kronecker delta function. Furthermore, according to,[30] the Legendre polynomials $\mathscr{L}_{\kappa}(x)$ and their derivative satisfy the following bounds:

$$ |\mathscr{L}_{\kappa}(x)| \leq 1, \;\text{ and }\; |\mathscr{L}'_{\kappa}(x)| \leq \frac{\kappa(\kappa+1)}{2}, \quad \forall x\in[-1,1], \;\; \forall\kappa \geq 0. \tag{6} $$

By introducing the variable $x = \frac{2\tau}{\mathcal{L}} - 1$, one can define the so-called shifted Legendre polynomials, which map the classical Legendre polynomials from $[-1,1]$ onto the interval $[0,\mathcal{L}]$. They are defined as:
$\mathscr{L}_{\kappa,\mathcal{L}}(\tau) = \mathscr{L}_{\kappa}\left(\frac{2\tau}{\mathcal{L}} - 1\right)$, where $\mathscr{L}_{0,\mathcal{L}}(\tau) = 1$ and $\mathscr{L}_{1,\mathcal{L}}(\tau) = \frac{2\tau}{\mathcal{L}} - 1$.

The following formula provides an analytical representation of $\mathscr{L}_{i,\mathcal{L}}(\tau)$:

$$ \mathscr{L}_{\kappa,\mathcal{L}}(\tau) = \sum_{r=0}^{\kappa}(-1)^{\kappa+r}\frac{(\kappa+r)!\,\tau^r}{\mathcal{L}^r\,(\kappa-r)!\,(r!)^2}. \tag{7} $$

It is evident from (7) that $\mathscr{L}_{\kappa,\mathcal{L}}(0) = (-1)^{\kappa}$, and $\mathscr{L}_{\kappa,\mathcal{L}}(\mathcal{L}) = 1$.

It is worth mentioning that the Legendre basis is selected for its orthogonality, favorable approximation properties, and convenient recurrence and differentiation formulas. It also provides a consistent polynomial representation with the LGL collocation and Gauss–Legendre quadrature employed in the present formulation, while facilitating the construction of the matrices arising in the Rayleigh–Ritz procedure. Although Chebyshev polynomials are also a viable alternative, the Legendre basis is adopted to maintain consistency with the numerical framework used throughout this work.

3. Numerical approximations

Let $\varphi \in C[0,\mathcal{L}]$ and $\mathcal{M} \in \mathbb{N}_+$. An approximation of $\varphi(\tau)$ using shifted Legendre polynomials, $\mathscr{L}_{\lambda,\mathcal{L}}$, can be expressed as:

$$ \varphi_{\mathcal{M}}(\tau) = \sum_{\lambda=0}^{\mathcal{M}}\tilde{a}_{\lambda}\mathscr{L}_{\lambda,\mathcal{L}}(\tau). \tag{8} $$

If $\varphi_{\mathcal{M}}(\tau)$ interpolates $\varphi(\tau)$ at the Legendre-Gauss-Lobatto (LGL) points $\{\tau_\lambda\}_{\lambda=0}^{\mathcal{M}}$, then the coefficients $\tilde{a}_\lambda$ can be computed via the associated LGL quadrature formula as follows:[31]

$$\tilde{a}_\lambda = \frac{1}{\gamma_\lambda}\sum_{\mu=0}^{\mathcal{M}} \varphi(\tau_\mu)\,\mathscr{L}_{\lambda,\mathcal{L}}(\tau_\mu)\omega_\mu, \tag{9}$$

where $\omega_\mu = \frac{\mathcal{L}}{\mathcal{M}(\mathcal{M}+1)}\frac{1}{[\mathscr{L}_{\mathcal{M},\mathcal{L}}(\tau_\mu)]^2}$, are the weights corresponding to the nodes $\{\tau_\mu\}_{\mu=0}^{\mathcal{M}}$, the zeros of $(1-(\frac{2}{\mathcal{L}}\tau-1)^2)\mathscr{L}'_{\mathcal{M},\mathcal{L}}(\tau)$ on the interval $[0,\mathcal{L}]$, and $\gamma_\lambda = \frac{\mathcal{L}}{2\lambda+1}$, for $0\le\lambda\le\mathcal{M}-1$, $\gamma_{\mathcal{M}} = \frac{\mathcal{L}}{\mathcal{M}}$.

To approximate the integral in the proposed problem, we employ Gauss-Legendre quadrature.[32] On the standard interval $[-1,1]$, the quadrature formula is

$$\int_{-1}^{1}\varphi(x)\,dx \;\approx\; \sum_{\iota=1}^{\mathcal{N}}\tilde{w}_\iota\varphi(x_\iota), \tag{10}$$

where $\{x_\iota\}_{\iota=1}^{\mathcal{N}}$ are the roots of $\mathscr{L}_{\mathcal{N}}(x)$ with corresponding weights given by:

$$\begin{aligned}&\tilde{w}_\iota = \frac{-2}{(\mathcal{N}+1)\mathscr{L}'_{\mathcal{N}}(x_\iota)\mathscr{L}_{\mathcal{N}+1}(x_\iota)},\\ &\iota = 1,2,\ldots,\mathcal{N}.\end{aligned} \tag{11}$$

The associated quadrature error can be expressed as:

$$\begin{aligned}&E_{\mathcal{N}}(\varphi) = e_{\mathcal{N}}\frac{\varphi^{(2\mathcal{N})}(\eta)}{(2\mathcal{N})!},\\ &\text{for some } -1<\eta<1,\end{aligned} \tag{12}$$

where $e_{\mathcal{N}} = \dfrac{2^{2\mathcal{N}+1}(\mathcal{N}!)^4}{(2\mathcal{N}+1)[(2\mathcal{N})!]^2}$.

To solve the proposed problem, we require an approximation of ${}^{C}D_{0^+}^{\nu,\sigma}\varphi_{\mathcal{M}}(\tau)$. Accordingly, the next lemma provides such an approximation in terms of the Legendre polynomial.

Lemma 1. *Let $\nu\in(0,1)$ and $\sigma>0$. The generalized Caputo fractional derivatives of $\varphi_{\mathcal{M}}(\tau)$, as given in* (8)*, can be derived as follows:*

$$\begin{aligned}&{}^{C}D_{0^+}^{\nu,\sigma}\varphi_{\mathcal{M}}(\tau) = \sum_{\lambda=1}^{\mathcal{M}}\tilde{a}_\lambda\\ &\sum_{r=1}^{\lambda}\frac{(-1)^{\lambda+r}(\lambda+r)!}{((r)!)^2(\lambda-r)!\mathcal{L}^r}\frac{\sigma^\nu\Gamma(\frac{r}{\sigma}+1)}{\Gamma(\frac{r}{\sigma}+1-\nu)}\tau^{r-\nu\sigma}.\end{aligned} \tag{13}$$

Proof. According to Theorem 3 in,[13] one has

$${}^{C}D_{0^+}^{\nu,\sigma}\tau^r = \frac{r\sigma^{\nu-1}\Gamma(\frac{r}{\sigma})}{\Gamma(\frac{r}{\sigma}+1-\nu)}\tau^{r-\nu\sigma}.$$

By applying this relation to the expansions in (7) and (8), this directly leads to the desired result. □

4. Methodology for numerical solution

This section describes the methodology for constructing an approximate solution to the proposed optimal control problem. We begin by approximating the state function, $\varkappa(\tau)$, using LGL interpolation (8),(9) as follows:

$$\varkappa_{\mathcal{M}}(\tau) = \sum_{\lambda=0}^{\mathcal{M}}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)\,c_{\lambda,\mu}(\tau), \tag{14}$$

where

$$c_{\lambda,\mu}(\tau) = \frac{\omega_\mu}{\gamma_\lambda}\mathscr{L}_{\lambda,1}(\tau_\mu)\,\mathscr{L}_{\lambda,1}(\tau).$$

Then, using (13), the generalized Caputo fractional derivative of $\varkappa_{\mathcal{M}}(\tau)$ can be represented as:

$${}^{C}D_{0^+}^{\nu,\sigma}\varkappa_{\mathcal{M}}(\tau) = \sum_{\lambda=1}^{\mathcal{M}}\sum_{r=1}^{\lambda}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)\,d_{\mu,r,\lambda}^{\nu,\sigma}(\tau),$$

where
$d_{\mu,r,\lambda}^{\nu,\sigma}(\tau) = \frac{(-1)^{\lambda+r}\,(\lambda+r)!\,\omega_\mu\,\mathscr{L}_{\lambda,1}(\tau_\mu)}{((r)!)^2\,(\lambda-r)!\,\gamma_\lambda}$
$\frac{\sigma^\nu\Gamma(\frac{r}{\sigma}+1)}{\Gamma(\frac{r}{\sigma}+1-\nu)}\,\tau^{r-\nu\sigma}$.

To approximate the integral in (2), we first express $\varkappa(t)$ as given in (14) and apply the substitution

$$t = \frac{\tau}{2}(\eta+1),$$

to transform the integration interval $[0,\tau]$ to $[-1,1]$. Hence, the integral becomes
$\int_0^\tau \psi(\tau,t,\varkappa_{\mathcal{M}}(t))\,dt =$
$\frac{\tau}{2}\int_{-1}^{1}\psi(\tau,\frac{\tau}{2}(\eta+1),\sum_{\lambda=0}^{\mathcal{M}}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)$
$c_{\lambda,\mu}(\frac{\tau}{2}(\eta+1)))\,d\eta$.

Then, applying the Gauss-Legendre quadrature (10), the integral can be approximated as:

$$\begin{aligned}&\int_0^\tau\psi(\tau,t,\varkappa_{\mathcal{M}}(t))\,dt = \frac{\tau}{2}\sum_{\iota=0}^{\mathcal{N}}\tilde{\omega}_\iota\,\psi\\ &(\tau,\frac{\tau}{2}(\eta_\iota+1),\sum_{\lambda=0}^{\mathcal{M}}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)c_{\lambda,\mu}(\frac{\tau}{2}(\eta_\iota+1))).\end{aligned} \tag{15}$$

Now, by substituting (15) into (2), we obtain the corresponding approximation of the control function, $\varsigma_{\mathcal{M}}(\tau)$, as follows:

$$\begin{aligned}\vartheta(\tau)\,\varsigma_{\mathcal{M}}(\tau) &= \sum_{\lambda=1}^{\mathcal{M}}\sum_{r=1}^{\lambda}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)\,d_{\mu,r,\lambda}^{\nu,\sigma}(\tau)\\ &-\theta(\tau)\sum_{\lambda=0}^{\mathcal{M}}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)\,c_{\lambda,\mu}(\tau)\\ &-\zeta(\tau)\,\frac{\tau}{2}\sum_{\iota=0}^{\mathcal{N}}\tilde{\omega}_\iota\,\psi(\tau,\frac{\tau}{2}(\eta_\iota+1),\\ &\sum_{\lambda=0}^{\mathcal{M}}\sum_{\mu=0}^{\mathcal{M}}\varkappa(\tau_\mu)\,c_{\lambda,\mu}(\frac{\tau}{2}(\eta_\iota+1))).\end{aligned} \tag{16}$$

Hence, by substituting the approximate control function and applying the substitution

$$\tau = \frac{1}{2}(s+1),$$

to map the integration interval to $[-1,1]$, and then using the Gauss-Legendre quadrature (10), the cost functional J in (1) can be represented as:

$$\min J = \frac{1}{2}\sum_{\iota=0}^{\mathcal{N}} \tilde{\omega}_\iota \mathcal{F}(s_\iota, \varkappa_{\mathcal{M}}(s_\iota), \varsigma_{\mathcal{M}}(s_\iota)).$$

The initial condition is enforced strongly in the discrete formulation. Since the first LGL node satisfies $\tau_0 = 0$, we impose

$$\varkappa(\tau_0) = \varkappa(0) = \varkappa_0,$$

and exclude $\varkappa(\tau_0)$ from the Newton iteration. Thus, only the nodal values $\varkappa(\tau_\iota)$, $\iota = 1, \ldots, \mathcal{M}$, are treated as unknowns, and the initial condition is enforced by eliminating the corresponding degree of freedom without introducing a Lagrange multiplier.

Using the Rayleigh–Ritz technique, the stationary points of the reduced discrete cost functional are characterized by the first-order conditions

$$\frac{\partial J_{\mathcal{M},\mathcal{N}}}{\partial \varkappa(\tau_\iota)} = 0, \qquad \iota = 1, 2, \ldots, \mathcal{M}.$$

This yields a nonlinear algebraic system, which is solved iteratively using Newton's method to determine the unknown nodal values $\varkappa(\tau_\iota)$, $\iota = 1, \ldots, \mathcal{M}$.

The initial approximation for Newton's iteration is selected from the available spectral approximation and is consistent with the prescribed initial condition. For strongly nonlinear problems, the convergence of Newton's method may depend on the initial guess; in such cases, damping or continuation strategies can be employed, when necessary, to improve the robustness of the iteration.

Since the reduced discrete problem may be nonconvex, the first-order conditions identify stationary points but do not, by themselves, guarantee local minimality. A strict local minimum requires an additional second-order sufficient condition. In particular, let

$$\mathbf{x} = [\varkappa(\tau_1), \varkappa(\tau_2), \ldots, \varkappa(\tau_{\mathcal{M}})]^T.$$

Assuming that $J_{\mathcal{M},\mathcal{N}}$ is twice continuously differentiable in a neighborhood of a computed stationary point $\mathbf{x}^*$, a sufficient condition for $\mathbf{x}^*$ to be a strict local minimum is that the Hessian of the reduced discrete cost functional be positive definite at $\mathbf{x}^*$, namely,

$$\nabla^2_{\mathbf{x}} J_{\mathcal{M},\mathcal{N}}(\mathbf{x}^*) \succ 0.$$

5. Error estimate

In what follows, we provide an error bound for the approximations generated by the proposed method and derive a corresponding estimate for the approximation of the generalized fractional derivative operator.

To derive an error bound for the approximation of the generalized Caputo fractional derivative, sufficient regularity of the underlying function is required. In particular, the following result assumes that the function possesses absolutely continuous derivatives up to order r and that its derivative of order $r+1$ is bounded. These regularity assumptions allow us to apply the Legendre approximation estimate used in the proof.

Theorem 2. *Assume that $\varphi, \varphi', \ldots, \varphi^{(r)} \in AC[0,1]$ and that there exists a constant $\mathcal{Q}_r < \infty$ such that $|\varphi^{(r+1)}(\tau)| \le \mathcal{Q}_r$, $\forall \tau \in (0,1]$, with $r > 3$. Let $\varphi_{\mathcal{M}}$ be defined as (8). Then $\forall \tau \in (0,1]$, and $\mathcal{M} \ge r$,*

$$\begin{aligned} &\left| {}^{C}D^{\nu,\sigma}_{0^+}\varphi(\tau) - {}^{C}D^{\nu,\sigma}_{0^+}\varphi_{\mathcal{M}}(\tau) \right| \\ &\le \frac{\Omega_{r,\mathcal{M}}\Gamma(\frac{1}{\sigma}+1)\sigma^\nu}{\Gamma(\frac{1}{\sigma}+1-\nu)}\,\tau^{1-\nu\sigma}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Omega_{r,\mathcal{M}} = &\frac{\pi\,\mathcal{Q}_r\,\mathcal{M}(\mathcal{M}+1)}{2^{r+2}\,(\mathcal{M}^2-3\mathcal{M}+2)(r-3)(\mathcal{M}-3)(\mathcal{M}-4)\cdots(\mathcal{M}-r+1)} \\ &+2\mathcal{M}^2((1+\Lambda(\mathcal{M}))\frac{\mathcal{Q}_r}{2^r r\mathcal{M}(\mathcal{M}-1)(\mathcal{M}-2)\cdots(\mathcal{M}-r+1)} \\ &+\frac{\pi\,\mathcal{Q}_r}{2^{r+1}(r-1)(\mathcal{M}-1)(\mathcal{M}-2)\cdots(\mathcal{M}-r+1)}), \end{aligned}$$

and $\Lambda(\mathcal{M})$ denotes the Lebesgue constant, which satifies the estimate $\Lambda(\mathcal{M}) = O(\sqrt{\mathcal{M}})$ on $[-1,1]$.[33]

Proof. Applying the definition of the generalized Caputo derivative in (4), it follows that:

$$\begin{aligned} &\left| {}^{C}D^{\nu,\sigma}_{0^+}\varphi(\tau) - {}^{C}D^{\nu,\sigma}_{0^+}\varphi_{\mathcal{M}}(\tau) \right| \\ &= \left| \frac{\sigma^\nu}{\Gamma(1-\nu)} \int_0^\tau \left(\varphi'(t) - \varphi'_{\mathcal{M}}(t)\right)(\tau^\sigma - t^\sigma)^{-\nu}\,dt \right|, \end{aligned} \tag{18}$$

and according to,[34] we have

$$\begin{aligned} &\left|\varphi'(t) - \varphi'_{\mathcal{M}}(t)\right| \le \\ &\frac{\pi\,\mathcal{Q}_r\,\mathcal{M}(\mathcal{M}+1)}{2^{r+2}\,(\mathcal{M}^2-3\mathcal{M}+2)(r-3)(\mathcal{M}-3)(\mathcal{M}-4)...(\mathcal{M}-r+1)} \\ &+2\mathcal{M}^2((1+\Lambda(\mathcal{M}))\frac{\mathcal{Q}_r}{2^r r\mathcal{M}(\mathcal{M}-1)(\mathcal{M}-2)...(\mathcal{M}-r+1)} \\ &+\frac{\pi\,\mathcal{Q}_r}{2^{r+1}(r-1)(\mathcal{M}-1)(\mathcal{M}-2)...(\mathcal{M}-r+1)}). \end{aligned}$$

Then,

$$\begin{aligned} &\left| {}^{C}D^{\nu,\sigma}_{0^+}\varphi(\tau) - {}^{C}D^{\nu,\sigma}_{0^+}\varphi_{\mathcal{M}}(\tau) \right| \\ &\le \frac{\sigma^\nu}{\Gamma(1-\nu)}\,\Omega_{r,\mathcal{M}} \int_0^\tau (\tau^\sigma - t^\sigma)^{-\nu}\,dt \\ &\le \frac{\Omega_{r,\mathcal{M}}\Gamma(\frac{1}{\sigma}+1)\sigma^\nu}{\Gamma(\frac{1}{\sigma}+1-\nu)}\,\tau^{1-\nu\sigma}. \end{aligned}$$

□

Remark 2. *The estimate in Theorem 2 is a pointwise error estimate valid for every fixed $\tau \in (0,1]$. Moreover, for any fixed $\delta > 0$, the factor $\tau^{1-\nu\sigma}$ is bounded on $[\delta, 1]$. Consequently, the error estimate is uniform on every interval $[\delta, 1]$, $\delta > 0$, without requiring any restriction on $\nu\sigma$. If $\nu\sigma > 1$, then*

$$\lim_{\tau \to 0^+} \tau^{1-\nu\sigma} = +\infty;$$

hence, the present error bound does not provide uniform boundedness near $\tau = 0$ in this case. If $\nu\sigma \leq 1$, the factor $\tau^{1-\nu\sigma}$ remains bounded as $\tau \to 0^+$.

Remark 3. *For every fixed $\tau \in (0,1]$, the right-hand side of the error estimate in Theorem 2 tends to zero as $\mathcal{M} \to \infty$, provided that $\Omega_{r,\mathcal{M}} \to 0$. Thus, Theorem 2 yields pointwise convergence of the generalized Caputo derivative approximation on $(0,1]$. Moreover, on every interval $[\delta, 1]$, $\delta > 0$, the convergence is uniform.*

To derive an error estimate for the approximation of the underlying function, sufficient regularity relative to the fractional order and approximation degree is required. In particular, the following result assumes that $\varphi \in AC^m[0,1]$, where $m = \lfloor (\mathcal{M}+1)\nu \rfloor + 1$, and that the corresponding generalized Caputo fractional derivative is continuous. These regularity assumptions allow us to apply the generalized fractional Taylor expansion used in the proof.

Theorem 3. *Let $\mathcal{M}$ be a non-negative integer, $m = \lfloor (\mathcal{M}+1)\nu \rfloor + 1$, and $\varphi(\tau) \in AC^m[0,1]$ satisfy the condition that ${}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma} \varphi \in C[0,1]$. Let $\varphi_{\mathcal{M}}(\tau) = \sum_{i=0}^{\mathcal{M}} a_i \mathscr{L}_{i,1}(\tau)$ be the best approximation of $\varphi(\tau)$ in the shifted Legendre polynomial space $\mathbb{P}_{\mathcal{M}}$. Then, the error bound is*

$$\|\varphi - \varphi_{\mathcal{M}}\|_{L^2(w)} \leq \frac{\mathscr{C}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)},$$

where

$$\mathscr{C} = \max_{\tau \in [0,1]} \left| {}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma} \varphi(\tau) \right|.$$

Proof. By applying the generalized Taylor expansion[35] with the initial point $a = 0$, the function $\varphi(\tau)$ can be expanded as follows:

$$\varphi(\tau) = \sum_{j=0}^{\mathcal{M}} \left(\frac{\tau^\sigma}{\sigma}\right)^{j\nu} \frac{{}^C D_{0^+}^{j\nu,\sigma}\varphi(0)}{\Gamma(j\nu+1)} + \left(\frac{\tau^\sigma}{\sigma}\right)^{(\mathcal{M}+1)\nu} \frac{{}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi(\Im)}{\Gamma((\mathcal{M}+1)\nu+1)},$$

where $\Im \in [0,\tau]$ and ${}^C D_{0^+}^{j\nu,\sigma}\varphi := {}^C D_{0^+}^{\nu,\sigma} \circ \cdots \circ {}^C D_{0^+}^{\nu,\sigma}\varphi$ (j-times).

Define

$$\widetilde{\varphi}_{\mathcal{M}}(\tau) = \sum_{j=0}^{\mathcal{M}} \left(\frac{\tau^\sigma}{\sigma}\right)^{j\nu} \frac{{}^C D_{0^+}^{j\nu,\sigma}\varphi(0)}{\Gamma(j\nu+1)}.$$

Then,

$$|\varphi(\tau) - \widetilde{\varphi}_{\mathcal{M}}(\tau)| = \left| \left(\frac{\tau^\sigma}{\sigma}\right)^{(\mathcal{M}+1)\nu} \frac{{}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi(\Im)}{\Gamma((\mathcal{M}+1)\nu+1)} \right|$$
$$= \frac{\tau^{(\mathcal{M}+1)\nu\sigma}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)} \left| {}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi(\Im) \right|.$$

Taking into account that $\left| {}^C D_{0^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi(\Im) \right| \leq \mathscr{C}$ and, for $0 \leq \tau \leq 1$ and $\nu, \sigma > 0$, $\tau^{(\mathcal{M}+1)\nu\sigma} \leq 1$, we obtain

$$|\varphi(\tau) - \widetilde{\varphi}_{\mathcal{M}}(\tau)| \leq \frac{\mathscr{C}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)}.$$

Since $\varphi_{\mathcal{M}}(\tau)$ is the best approximation of $\varphi(\tau)$ in $\mathbb{P}_{\mathcal{M}}$ and $\widetilde{\varphi}_{\mathcal{M}} \in \mathbb{P}_{\mathcal{M}}$, it follows that

$$\|\varphi - \varphi_{\mathcal{M}}\|_{L^2(w)} \leq \|\varphi - \widetilde{\varphi}_{\mathcal{M}}\|_{L^2(w)}.$$

Hence,

$$\|\varphi - \varphi_{\mathcal{M}}\|_{L^2(w)}^2 \leq \|\varphi - \widetilde{\varphi}_{\mathcal{M}}\|_{L^2(w)}^2$$
$$= \int_0^1 \omega(\tau) \left|\varphi(\tau) - \widetilde{\varphi}_{\mathcal{M}}(\tau)\right|^2 d\tau$$
$$\leq \int_0^1 \omega(\tau) \left[\frac{\mathscr{C}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)}\right]^2 d\tau.$$

Since the standard weight is $\omega(\tau) = 1$, we obtain

$$\|\varphi - \varphi_{\mathcal{M}}\|_{L^2(w)}^2 \leq \int_0^1 \left[\frac{\mathscr{C}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)}\right]^2 d\tau$$
$$= \left[\frac{\mathscr{C}}{\sigma^{(\mathcal{M}+1)\nu}\Gamma((\mathcal{M}+1)\nu+1)}\right]^2.$$

Hence, the theorem is proved upon taking the square root. □

Remark 4. *Since ${}^C D_{a^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi \in C[0,1]$, the constant $\mathscr{C} = \max_{\tau \in [0,1]} \left| {}^C D_{a^+}^{(\mathcal{M}+1)\nu,\sigma}\varphi(\tau) \right|$ is finite. Hence, the error bound in Theorem 3 is well defined.*

6. Analysis and interpretation of numerical results

This section demonstrates the application and validates the effectiveness of the numerical technique presented in Section 4 using two illustrative examples.

Example 1. *Consider the following OCP:*

$$\begin{aligned} min\ J(\varkappa,\varsigma) &= \int_0^1 \Big[\Big(\varkappa(\tau)-\tau^{\nu\sigma+2}\Big)^2 \\ &+ \Big(\varsigma(\tau)-\tau^2\Big[\frac{\sigma^\nu\Gamma(\nu+\frac{2}{\sigma}+1)}{\Gamma(\frac{2}{\sigma}+1)} \\ &- \tau^{\nu\sigma}e^{-\tau^{2\nu\sigma+4}}\Big]\Big)^2\Big]d\tau, \end{aligned} \tag{19}$$

with the system dynamics given by:

$$\begin{aligned} &{}^C D_{0^+}^{\nu,\sigma}\varkappa(\tau) = \varkappa(\tau)+\varsigma(\tau)-\tau^{\nu\sigma+2} \\ &\int_0^\tau \Big((2\nu\sigma+4)t^{2\nu\sigma+3}e^{-\varkappa^2(t)}\Big)dt, \varkappa(0)=0, \end{aligned} \tag{20}$$

where, the exact solution has the form:

$$\begin{aligned} (\bar{\varkappa}(\tau),\bar{\varsigma}(\tau)) &= (\tau^{\nu\sigma+2},\tau^2\Big[\frac{\sigma^\nu\Gamma(\nu+\frac{2}{\sigma}+1)}{\Gamma(\frac{2}{\sigma}+1)} \\ &- \tau^{\nu\sigma}e^{-\tau^{2\nu\sigma+4}}\Big]). \end{aligned} \tag{21}$$

Table 1 exhibits the computed maximum absolute errors of the state $E_\varkappa$, control E_ς, together with the associated cost functional J, for various values of σ and ν at $\mathcal{M}=\mathcal{N}=8$. Moreover, **Table 2** reports the values of $E_\varkappa$, E_ς, and J for $\nu=0.8$ and $\sigma=1.2$ with several choices of $\mathcal{M}$ and $\mathcal{N}$. Here, $\mathcal{M}$ controls the LGL spectral approximation, whereas $\mathcal{N}$ controls the Gauss–Legendre quadrature resolution. These parameters may be selected independently; however, we set $\mathcal{M}=\mathcal{N}$ in this experiment to provide a balanced refinement of the two components of the numerical scheme. As shown in **Table 2**, the accuracy of the proposed approach improves significantly as $\mathcal{M}=\mathcal{N}$ increases. For instance, $E_\varkappa$ decreases from 4.2118×10^{-2} for $\mathcal{M}=\mathcal{N}=2$ to 1.6638×10^{-7} for $\mathcal{M}=\mathcal{N}=10$. This observed improvement is consistent with the theoretical error estimates in Theorems 2 and 3. In particular, Theorem 2 provides an error bound for the approximation of the generalized Caputo fractional derivative, while Theorem 3 gives an L^2-error estimate for the approximation of the underlying function. Although the errors reported in **Table 2** are maximum absolute errors, their systematic decrease under simultaneous refinement of $\mathcal{M}$ and $\mathcal{N}$ is consistent with the convergence behavior predicted by the theoretical analysis.

Figure 1 shows the absolute errors of $\varkappa(\tau)$ and $\varsigma(\tau)$ at $\nu=0.8$, $\sigma=1.2$, and $\mathcal{M}=\mathcal{N}=10$. Moreover, **Figures 2** and **3** compare the exact and approximate solutions with various values of σ and ν at $\mathcal{M}=\mathcal{N}=4$. It is clear that the approximate solutions closely match the exact ones, which validates the accuracy of the proposed approach.

Table 1. Computed errors $E_\varkappa$ and E_ς, with the corresponding J values, for various ν and σ at $\mathcal{N}=\mathcal{M}=8$

Example 1	$\boldsymbol{\sigma}$	$\boldsymbol{E_\varkappa}$	$\boldsymbol{E_\varsigma}$	$\boldsymbol{J}$
	0.5	1.9763e-5	1.1630e-4	6.4811e-11
$\nu=0.95$	1	8.7959e-7	8.0690e-5	1.6905e-13
	1.2	1.5180e-6	2.0620e-2	6.0703e-13
	0.5	1.9627e-5	1.2820e-4	6.4164e-11
$\nu=0.9$	1	1.7393e-6	1.2055e-4	7.9384e-13
	1.2	8.8855e-7	5.8532e-4	2.6822e-13
	0.5	1.9447e-5	1.2380e-4	6.4979e-11
$\nu=0.85$	1	2.5846e-6	1.3781e-4	2.1670e-12
	1.2	2.6566e-7	5.2609e-5	2.2443e-14

Table 2. Computed errors $E_\varkappa$ and E_ς, with the corresponding J values, at $\nu=0.8$, $\sigma=1.2$ and various values of $\mathcal{M}$ and $\mathcal{N}$

Example 1	$\boldsymbol{E_\varkappa}$	$\boldsymbol{E_\varsigma}$	$\boldsymbol{J}$
$\mathcal{M}=\mathcal{N}=2$	4.2118e-2	3.9573e-1	2.0657e-3
$\mathcal{M}=\mathcal{N}=4$	1.6996e-4	2.1762e-3	3.8907e-9
$\mathcal{M}=\mathcal{N}=6$	3.4517e-6	5.6407e-5	4.4608e-12
$\mathcal{M}=\mathcal{N}=10$	1.6638e-7	2.4337e-5	6.5020e-15

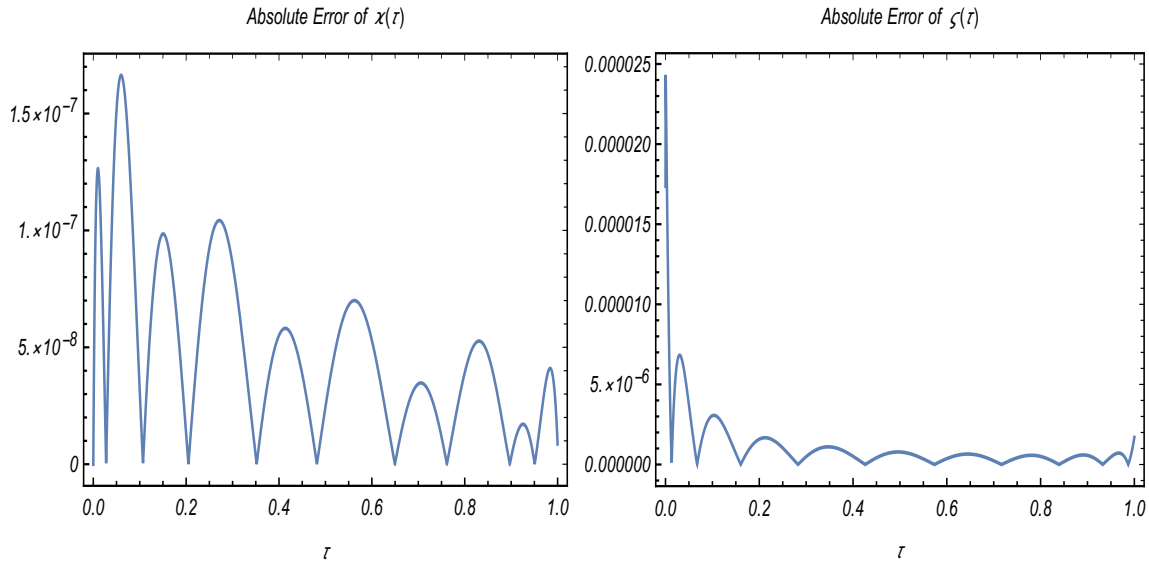


Figure 1. Illustration of the absolute errors for $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) for $\sigma = 1.2$, $\nu = 0.8$, and $\mathcal{M} = \mathcal{N} = 10$

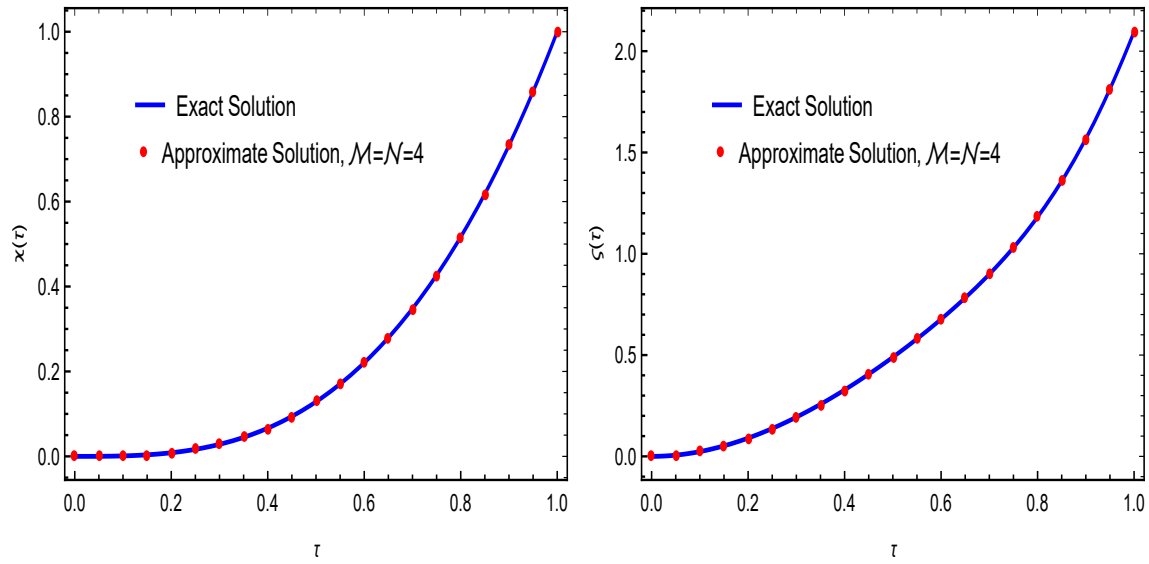


Figure 2. Illustration of analytical and numerical solutions corresponding to $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) for $\sigma = 1.2$, $\nu = 0.8$, and $\mathcal{M} = \mathcal{N} = 4$

Example 2. *Consider the following OCP:*

$$min\ J(\varkappa,\varsigma) = \frac{1}{2}\int_0^1 \left(\left(\varkappa(\tau) - e^{\tau}\right)^2 + \left(\varsigma(\tau) - e^{2\tau}\right)^2\right)d\tau, \tag{22}$$

with the system dynamics given by:

$$^{C}D_{0^+}^{\nu,\sigma}\varkappa(\tau) = \frac{3}{2}\varkappa(\tau) - \frac{1}{2}\varsigma(\tau) + \int_0^{\tau}\left(e^{(\tau-t)}(\varkappa(t))^3\right)dt, \quad \varkappa(0) = 1. \tag{23}$$

The exact solution for $\nu = \sigma = 1$ has the form:

$$(\bar{\varkappa}(\tau), \bar{\varsigma}(\tau)) = \left(e^{\tau}, e^{3\tau}\right). \tag{24}$$

Table 3 presents a comparison between our numerical results and those reported in the literature for the classical case. The results clearly show that our approach achieves higher accuracy than existing techniques for various values of $\mathcal{M}$. This behavior is also consistent with the theoretical error estimates in Theorems 2 and 3, since the exact solutions in this classical test case are analytic on $[0, 1]$ and therefore satisfy the regularity requirements underlying these estimates. Furthermore, **Table 4** demonstrates the computational efficiency of our technique by comparing its computational cost with the results reported in.[18] The CPU times reported in **Table 4** provide an indicative comparison with the results in.[18] Since

the computational environments and implementation details may differ, this comparison is not intended as a strictly controlled benchmark. Nevertheless, the results indicate competitive computational performance for the tested values of $\mathcal{M}$. As $\mathcal{M}$ increases, the number of unknowns increases, and the computational cost of evaluating the spectral approximation, fractional derivative, and Jacobian also increases, which is reflected in the CPU times reported in **Table 4**. **Figure 4** elucidates the absolute errors for $\varkappa(\tau)$ and $\varsigma(\tau)$ at $\nu = 1$, $\sigma = 1$, and $\mathcal{M} = \mathcal{N} = 10$. Moreover, **Figures 5** and **6** display the exact solution of the classical problem along with the approximate solution to $\varkappa(\tau)$ and $\varsigma(\tau)$ using our technique with $\mathcal{M} = \mathcal{N} = 8$. These figures indicate that, as $\nu \to 1$ and $\sigma \to 1$, the approximate solutions of our technique converge to the exact solution of the classical case, confirming that our technique provides an accurate approximation for the proposed problem. These results not only confirm the accuracy and convergence of the proposed method, but also indicate its stable numerical behavior as the approximation order increases.

Regarding numerical stability, the numerical experiments show a systematic reduction in the approximation errors as the approximation order $\mathcal{M}$ increases, without noticeable spurious oscillations or numerical instability. This behavior is consistent with the favorable approximation properties of the Legendre basis and the use of Gauss–Legendre quadrature. A rigorous stability analysis of the fully discrete nonlinear scheme is beyond the scope of the present work and is left for future investigation. Finally, all numerical experiments were performed using Mathematica on a computer with an Intel(R) Core(TM) i7-4790 processor (3.60 GHz) and 32 GB RAM.

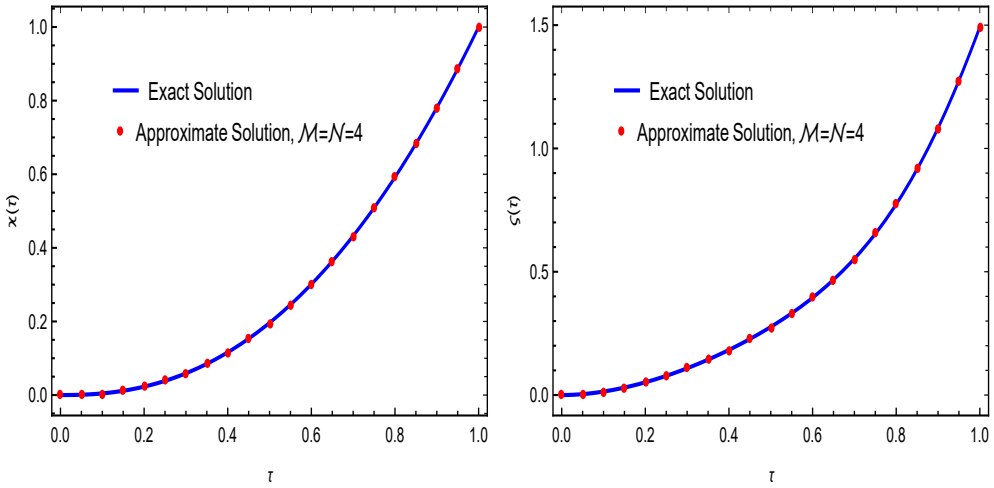


Figure 3. Illustration of analytical and numerical solutions corresponding to $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) for $\sigma = 0.5$, $\nu = 0.7$, and $\mathcal{M} = \mathcal{N} = 4$

Table 3. Numerical values of J for different $\mathcal{M}$

$\mathcal{M}$	Our Results	Singh *et al.*[18]	Mohammadi *et al.*[16]	Khanduzi *et al.*[36]	Maleknejad *et al.*[37]
2	1.9595e-4	4.5647e-5	7.5351e-4	9.0935e-5	1.0511e-2
4	6.4709e-10	1.4601e-9	3.8142e-9	1.3901e-7	1.9820e-7
6	1.4201e-15	1.9514e-15	1.0206e-14	8.0186e-9	1.1673e-11

Table 4. Comparison of CPU time (in seconds) for various $\mathcal{M}$ at $\nu = \sigma = 1$ with the results reported in[18]

$\mathcal{M}$	Our Results	Singh *et al.*[18]
2	1.063	4.40
4	1.376	13.56
6	2.875	29.68

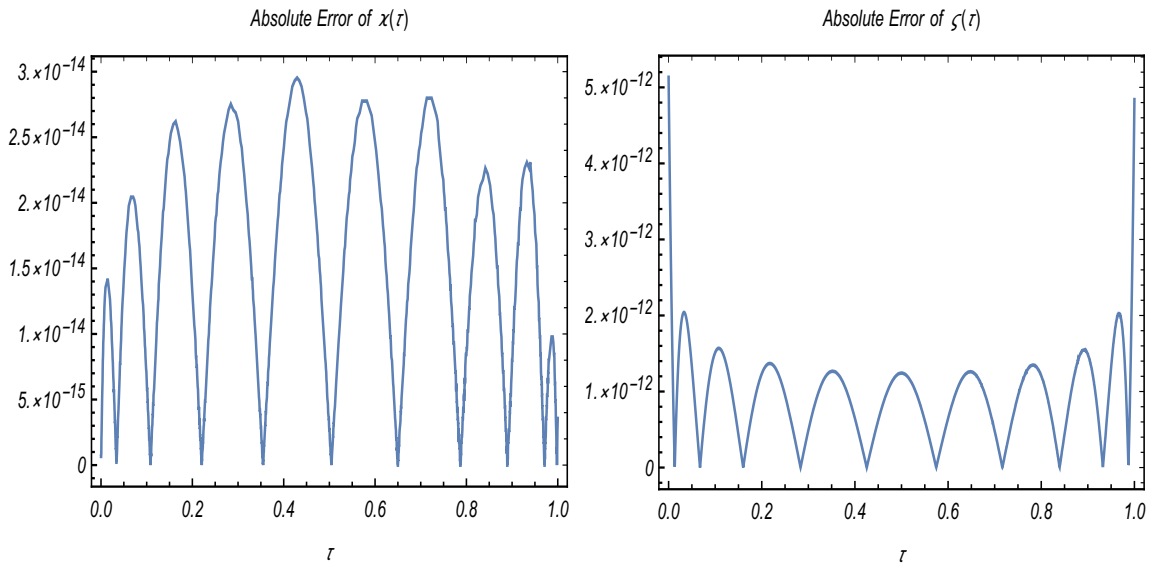


Figure 4. Illustration of the absolute errors for $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) for the classical case, $\sigma = \nu = 1$, at $\mathcal{M} = \mathcal{N} = 10$

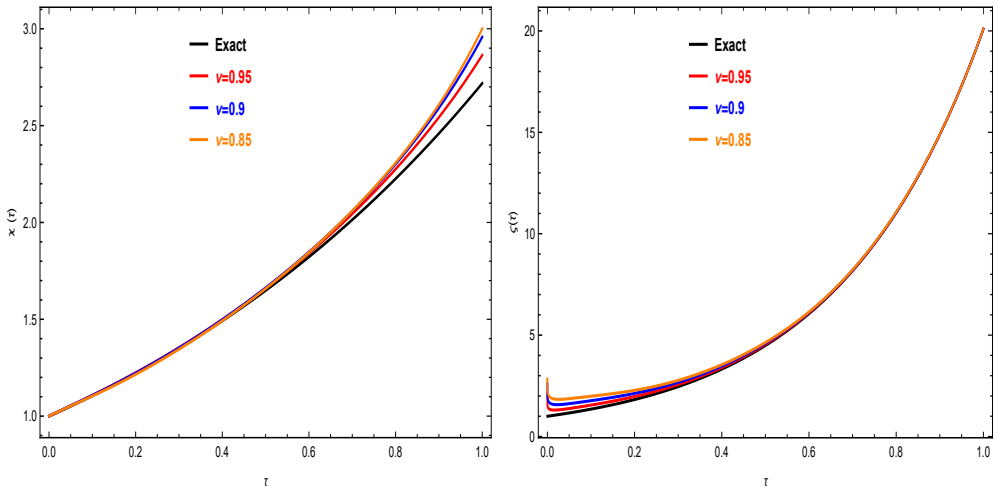


Figure 5. Illustration of analytical and numerical solutions corresponding to $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) at $\sigma = 1$, various values of ν, and $\mathcal{M} = \mathcal{N} = 8$

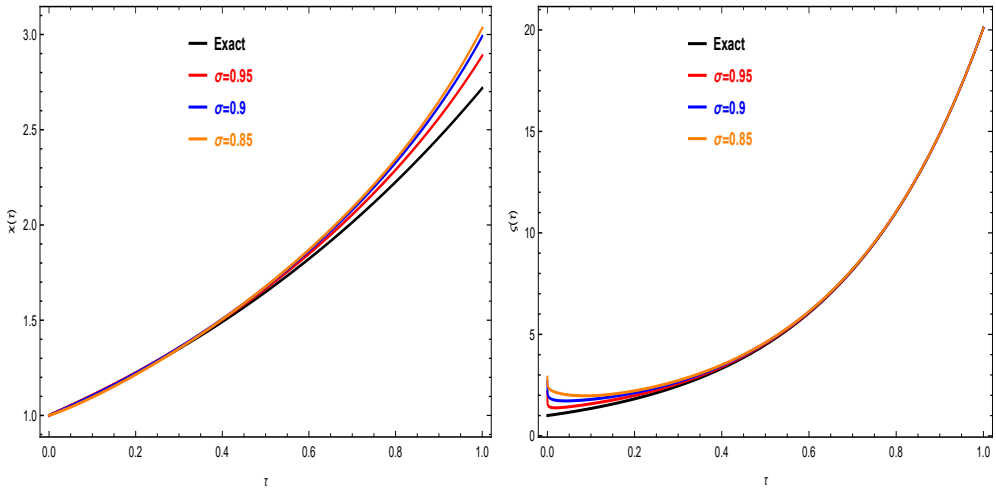


Figure 6. Illustration of analytical and numerical solutions corresponding to $\varkappa(\tau)$ (left panel) and $\varsigma(\tau)$ (right panel) at $\nu = 1$, various values of σ, and $\mathcal{M} = \mathcal{N} = 8$

7. Conclusions

The present work investigates a class of optimal control problems whose dynamics are described by generalized Caputo fractional integro-differential equations. The Gauss-Legendre quadrature technique and the Legendre spectral-collocation approach are employed to discretize the proposed problem. By applying the Rayleigh-Ritz method, the fractional optimal control problem is transformed into a finite-dimensional nonlinear algebraic system, which can be solved numerically using Newton's method. The generalized Caputo derivative is approximated by Legendre polynomials. An error bound for the approximate values produced by the proposed method is established. Moreover, an error bound for the approximation of the generalized fractional derivative operator is derived. Illustrative examples demonstrate the efficiency of our technique and confirm that its accuracy improves with higher degrees of freedom.

Acknowledgments

None.

Funding

None.

Conflict of interest

The authors confirm they have no conflicts of interest.

Author contributions

Conceptualization: All authors
Formal analysis: All authors
Investigation: All authors
Methodology: All authors
Writing–original draft: A. M. Nagy
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Availability of data

Not applicable.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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