An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.x, No.x, pp.xx-xx (2026)
https://doi.org/10.36922/IJOCTA026120045

RESEARCH ARTICLE

Optimal control of financial inclusion for women entrepreneurs with bifurcation, statistical and stochastic analysis

Muhammad Farman[1,2,3*], Aqeel Ahmad[1,4], Usama Atta[4,5], Zubaria Parvez[6], Evren Hincal[1,7], Aseel Smerat[8,9], and Mohamad Hafez[10,11]

[1]Department of Mathematics, Mathematics Research Center, Near East University, Mersin-10, Turkey
[2]Faculty of Medicine, Department of Biostatistics and Medical Informatics, Karadeniz Technical University, Trabzon, Turkey
[3]Department of Software Engineering, Ankara Science University, Ankara, Turkey
[4]Department of Mathematics, Ghazi University, Dera Ghazi Khan, Pakistan
[5]International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore, Pakistan
[6]AI and Software Engineering Department, AI and IoT Research Center, AI and Informatics Faculty, Near East University, Mersin, Turkey
[7]Research Center of Applied Mathematics, Khazar University, Baku, Azerbaijan
[8]Faculty of Educational Sciences, Al-Ahliyya Amman University, Amman, Jordan
[9]Department of Biosciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai, India
[10]Faculty of Engineering and Quantity Surviving, INTI International University Colleges, Nilai, Malaysia
[11]Faculty of Management, Shinawatra University, Pathum Thani, Thailand
farmanlink@gmail.com, aqeelahmad.740@gmail.com, usamaatta1111@gmail.com, zubariaparvez394@gmail.com, evren.hincal@neu.edu.tr, smerat.2020@gmail.com, mohdahmed.hafez@newinti.edu.my

ARTICLE INFO

Article History:
Received: March 19, 2026
Revised: April 19, 2026
Accepted: April 27, 2026
Published Online: November 2, 2026

Keywords:
Stochastic model
Social protection
Economic resources
Statistical analysis
Flip bifurcation
Euler-Maruyama
NSFD
Optimal control

ABSTRACT

In this paper, we develop a comprehensive mathematical model incorporating market linkages, sociocultural interventions, social capital, and access to finance to examine the dynamics of women entrepreneurs. We extend the deterministic model by incorporating the random variations present in socioeconomic systems, in addition to constructiing and analyzing a stochastic nonstandard finite difference (NSFD) scheme. The methodology overcomes significant limitations of traditional numerical approaches by ensuring the preservation of positivity and boundedness for all step sizes. We compute key thresholds at which the system undergoes qualitative behavioral changes by using eigenvalue analysis, one-dimensional parameter sweeps, and two-dimensional stability surfaces. The absence of flip bifurcation verifies that the behaviour of the system is bounded and that the system does not exhibit period-doubling instabilities. The stochastic and deterministic simulations highlight the dynamic behavior of the suggested model. The impact of parameters, illustrated through radial, tornado, chord, and heat-map diagrams, shows that maintenance programs that lower the depreciation rate through regulatory change should be combined with initiatives that improve investment rates through financial literacy and digital infrastructure. Statistical validation shows the superiority of the nonstandard finite difference scheme and the effectiveness of control measures compared with the no-control case. By formulating a stochastic optimal control problem, we compare four policy cases with the no-control case, and the results show that the combination of all controls produces the highest access to finance. The NSFD scheme preserves positivity for all step sizes, whereas Euler-Maruyama generates economically non-feasible negative results for large step sizes. The findings indicate that, to prevent irregular losses in entrepreneurial capacity, effective interventions should prioritize stabilizing social networks and controlling financial access during turbulent periods, which underscores the importance of stable policy environments and consistent financial support mechanisms. In women's entrepreneurial development programs, our study offers a strong foundation for policy modeling, uncertainty quantification, and evidence-based decision-making.

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1. Introduction

The contribution of women in entrepreneurship has attracted major global attention, although research consistently highlights the persistence of systemic hurdles that limit their advancement. Studies such as[1] and[2] show that women entrepreneurs in both developed and underdeveloped economies face multiple issues, ranging from limited access to finance and limited market structures to deeply entrenched sociocultural norms. These constraints not only limit entry into entrepreneurial activities but also limit the scale and persistence of women-owned enterprises. Some scholars argue that globalization and digital transformation are creating new opportunities for women entrepreneurs by providing alternative platforms and informal markets.[3] However, such advantages are not equally distributed and often fail to address the structural inequalities that perpetuate women's marginalization. South Asia offers an important example of persistent gender disparity, where women remain significantly underrepresented in the labour force despite multiple policy reforms.[4] and[5] argue that patriarchal structures and institutional gaps continue to limit women's opportunities, leading to stagnation in participation even as regional economies grow. Donor-funded and NGO-led training has emerged as a critical parallel system in contexts where formal provision remains limited or ineffective. Traditionally, women have faced significant barriers to achieving equality, including limited access to education, healthcare, financial opportunities, and political participation.[6–8] These difficulties have often been compounded by social norms, traditional practices, and legal contexts that constrain women's self-sufficiency and opportunities for development.[9,10] Consequently, gender inequality persists globally, with women in many regions continuing to incur lower wages, higher rates of unemployment, constrained access to management roles, and poorer health outcomes.[11] Recognizing the importance of addressing these inequalities, governments around the world have implied a range of measures to empower women.[12] These measures are diverse, ranging from policies focused on improving educational prospects for girls, improving parental health, ensuring workplace equality, supporting women's participation in governance, and addressing gender-based violence.[13] Global organizations, such as the United Nations, have also played a major role by setting global frameworks, such as the Sustainable Development Goals, which highlight women's empowerment and gender equality.[14]

1.1. Motivation and foundation of stochastic optimal control

The mathematical modeling of dynamic systems under uncertainty began in the middle of the 20th century, when[15,16] produced separate but complementary contributions to optimal control theory. Bellman's dynamic programming provided sufficient conditions through the concept of optimality, while Pontryagin's maximum principle, initially introduced in the late 1950s, supplied necessary conditions for optimality in deterministic systems. These significant contributions created the mathematical framework that would subsequently be expanded to stochastic settings, where systems evolve under random perturbations requiring more comprehensive analytical approaches.[17] Optimal control theory was extensively expanded to stochastic contexts as a result of the understanding that real-world systems are subject to uncertainties that deterministic models are unable to sufficiently explain. In the early phases of stochastic optimal control, which focused on diffusion processes represented by Ito stochastic differential equations, scholars such as[18–20] made important contributions to the theory of controlled diffusions and the associated Hamilton-Jacobi-Bellman equations. The combination of probability theory and control theory allowed for new approaches to problems in operations research, aeronautical engineering, and ultimately economic modeling.[21]

The use of stochastic control methods for economic problems gained popularity after a historic meeting of economists and control engineers at Princeton University in 1972, where the idea of cross-disciplinary interchange was first carefully examined. The use of linear-quadratic Gaussian frameworks for economic policy optimization was studied by.[22] Initial efforts focused on applying engineering control techniques to macroeconomic stabilization programs. These developments recognized that, like physical systems, economic systems are subject to random disruptions and change dynamically over time, which policymakers must take into account when creating intervention strategies.[23]

In recent years, stochastic optimal control has become increasingly important in the study of gender-specific economic distortions and entrepreneurship dynamics.[24] Systemic distortions affect market participation and company size distributions for women entrepreneurs, highlighting the need for targeted policy interventions grounded in comprehensive dynamic modeling. The convergence of development economics and stochastic control theory enables an understanding of how

market infrastructure, social capital, and financial access interact under uncertainty to produce entrepreneurial outcomes. The analytical toolset available for studying complex socioeconomic systems is being expanded by contemporary methodological advancements, such as the development of bias-corrected Monte Carlo operators and the use of artificial neural networks for dynamic programming problems.[25]

1.2. Research gap and justification

To the best of our knowledge, only one study has been conducted to date on the dynamics of women's entrepreneurship.[26] proposed a non-integer-order model that captured the dynamics of women's empowerment and illustrated the impact of government measures. The whole population was divided into five classes representing the entire range from early difficulties to complete empowerment. In order to assess the impact of controls, an optimal control model was also constructed and compared with controlled and uncontrolled cases. The results showed that empowerment outcomes are accelerated when policy changes and government support are combined.

There are still significant gaps in the literature on mathematical modeling, despite the growing recognition of women's entrepreneurship as a driver of economic growth. Current models are mostly deterministic, assuming completely predictable growth, despite actual evidence of substantial stochastic shifts in market conditions, social dynamics, and financial access. When stochastic components are present, traditional numerical techniques such as Euler-Maruyama are frequently employed. However, they may yield negative, economically nonsensical results and exhibit numerical instability over larger time steps. A comprehensive mathematical analysis proving the positivity, boundedness, convergence, and stability of numerical methods developed specifically for systems of female entrepreneurs is still lacking. Furthermore, stiffness arising from significantly varying time scales across different capital types is not adequately handled by existing approaches. Finally, there is a large gap between the development of mathematical models and practical implementation guidelines that development professionals and policymakers can use to design programs.

1.3. Novel contributions

This work closes these gaps through several coordinated contributions. The deterministic women entrepreneurship framework is extended to a stochastic formulation that includes Brownian motion disturbances in order to represent real-world random fluctuations while preserving mathematical tractability. We also illustrate the impact of parameters through tornado, heat map, chord, and radial diagrams, in addition to the socioeconomic interpretation of the bimodal distribution. We provide a complete statistical analysis that presents measures of dispersion and measures of central tendency for the compartments of the model. We also provide equilibrium points, local and global stability, the reproduction number, and verification of the bounded behaviour of the model by ensuring the absence of flip bifurcation. A nonstandard finite difference technique is developed for this system that preserves positivity and boundedness, independent of step size. A detailed comparison with the conventional Euler-Maruyama approach is used to quantify the superior numerical stability and qualitative property retention of the NSFD technique. Four distinct policy scenarios, each representing a different combination of active controls, are compared with a no-control case. The denominator function technique ensures that discrete schemes preserve the basic structure of the continuous system with a multiplicative noise structure that represents the fact that larger capital variations experience proportionately larger absolute fluctuations.

1.4. Arrangement of study

The remaining part of this study is arranged as follows: Section 2 presents the women entrepreneurship model in a deterministic setting, incorporating four compartments. Section 3 presents the equilibrium points, reproduction number, and local and global stability. Section 4 verifies the absence of flip bifurcation at the empowerment-free equilibrium. Section 5 presents the impact of parameters on the compartments in addition to the socioeconomic interpretation of the bimodal distribution. Section 6 formulates the stochastic women's entrepreneurship model, while Section 7 proves positivity, boundedness, and the reproduction number in the stochastic sense. In Section 8, we construct the two numerical schemes for the stochastic system, namely the Euler-Maruyama and NSFD techniques. Section 9 verifies the positivity and boundedness of the NSFD scheme. Numerical simulations are presented in Section 10 and include time series and histogram plots for low and high noise terms, comparison plots between solutions obtained by the Euler-Maruyama and NSFD schemes, comparison of solutions between the deterministic and stochastic systems, and 2D and 3D phase plots showing the relationship among

the compartments of the stochastic model. Section 11 formulates the optimal control model by extending the suggested women entrepreneurship system using three control variables. Finally, Section 12 summarizes the key findings and provides recommendations based on the outcomes.

1.5. Basic concepts

This part presents some fundamental concepts to be used in the remaining part of the research from.[27]

Definition 1. *A probability space is a triplet (S, F, P), where the sample space S is the collection of all possible outcomes, and F denotes the event space, which is a collection of subsets of S. Probability P is a function from F to $\mathbb{R}$ that assigns probabilities to events in F, and satisfies the following properties:*

(1) $P(E) \geq 0, \forall E \in F$,
(2) $P(S) = 1$,
(3) For any disjoint events $E_1, E_2 \in F$, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Definition 2. *A Brownian motion process $(B_t)_{t\geq 0}$ is characterized by:*

(1) $B_0 = 0$.
(2) Sample paths $t \mapsto B_t$ *are continuous with probability 1.*
(3) For any $0 \leq t_1 < t_2 < \cdots < t_n$, *the increments* $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent.*
(4) For $0 \leq s \leq t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$ *(normally distributed with mean 0 and variance* $t - s$*).*

2. Deterministic model formulation

In this section, we present the mathematical framework for simulating the dynamics of women entrepreneurs. For this, we incorporate four interrelated socioeconomic compartments that define the entrepreneurial ecosystem for women in developing economies.

2.1. Description of compartments

We incorporate four compartments into the entrepreneurial environment. The market-linkage infrastructure $M(t)$ comprises the institutional and physical systems that link female entrepreneurs to markets and consumers. This includes market facilities, digital platforms, transportation networks, and institutional structures that facilitate access to markets. The main factor motivation for compartment is empirical evidence that market-access barriers disproportionately affect women entrepreneurs in emerging nations. Sociocultural interventions $S_i(t)$ are community-based programs and cultural changes that question gender norms and foster settings that encourage women to pursue entrepreneurship. This compartment includes community-mobilization initiatives, gendersensitization initiatives, and awareness campaigns. The inclusion of this component recognizes that the economic participation of women is significantly influenced by societal attitudes and that entrepreneurship does not occur in a cultural vacuum. Social capital $S_c(t)$ comprises networks, connections, and trust that provide women entrepreneurs with access to resources, knowledge, and business opportunities. Examples include formal business associations, informal peer networks, and trust-based relationships with suppliers, customers, and mentors. The motivation stems from numerous sociological studies that demonstrate how social capital mediates access to economic opportunities. Access to finance $A_f(t)$ represents the availability of financing, savings options, and financial literacy support that help women grow and sustain businesses. This includes conventional banking services, digital financial services, microfinance institutions, and financial education programs. It is included as a fundamental component of the system because women entrepreneurs continue to face financial exclusion.

2.2. Parameter interpretation

The following parameter interpretations reflect the underlying socioeconomic dynamics. The growth contribution rates ψ and ω represent the growth of market linkages and sociocultural interventions, respectively, reflecting baseline investment and policy support. The influence coefficient α captures the contribution of market-linkage infrastructure to social capital formation. The coefficient β captures the interaction between sociocultural interventions and social capital, which depletes both compartments. The coefficient γ captures the transformation of social capital into better financial access, reflecting how networks and trust promote financial inclusion, lessen information asymmetries, and facilitate credit acceptance. The decay parameters m_1, m_2, and m_3 represent different mechanisms of market-infrastructure depreciation capturing the significance of rural-urban linkages, the negative effects of restricted access, and the perception of improvements. Similarly, s_1, s_2, and s_3 capture the devaluation of sociocultural interventions through cultural resistance, awareness degradation, and community-initiative fatigue. The parameters c_1, c_2, and c_3 show how social capital is decreased by weakened relationships, decreased engagement, and network attrition. The

parameters a_1, a_2, and a_3 capture the depreciation of financial access due to service withdrawal, credit limitations, and declining financial literacy.

2.3. Governing equations

The evolution of market-linkage infrastructure is governed by the balance between baseline expansion, depletion through interaction with social capital, and natural decay. The term ψ represents baseline growth of market-linkage infrastructure, $\alpha M S_c$ represents its depletion through interaction with social capital, and ΛM represents natural decay. The resulting equation takes the form:

$$\frac{dM}{dt} = \psi - \alpha M S_c - (m_1 + m_2 + m_3)M.$$

The dynamics of sociocultural interventions, with growth ω, interaction depletion $\beta S_i S_c$, and cumulative decay, have a similar structure:

$$\frac{dS_i}{dt} = \omega - \beta S_i S_c - (s_1 + s_2 + s_3)S_i.$$

Social capital evolves through a positive contribution from linkages, $\alpha M S_c$; depletion through interaction with sociocultural interventions, $-\beta S_i S_c$; natural decay; and conversion to financial access through $-\gamma S_c A_f$. The conversion term $\gamma S_c A_f$, subtracted from S_c and added to A_f, represents the mechanism by which social capital is used to create better financial access:

$$\frac{dS_c}{dt} = \alpha M S_c - \beta S_i S_c - (c_1 + c_2 + c_3)S_c - \gamma S_c A_f.$$

Access-to-finance dynamics capture both natural decay through cumulative depreciation and positive reinforcement from social capital through $\gamma S_c A_f$:

$$\frac{dA_f}{dt} = \gamma S_c A_f - (a_1 + a_2 + a_3)A_f.$$

We define the aggregated decay rates for notational ease:

$$\Lambda = m_1 + m_2 + m_3, \quad \rho = s_1 + s_2 + s_3,$$
$$\Phi = c_1 + c_2 + c_3, \quad \xi = a_1 + a_2 + a_3.$$

Figure 1 llustrates the relationships among the compartments of the model.

Using the above description of parameters and compartments, we formulate the deterministic women entrepreneurship model in the following form:

$$\begin{aligned}
\frac{dM}{dt} &= \psi - \alpha M S_c - \Lambda M, \\
\frac{dS_i}{dt} &= \omega - \beta S_i S_c - \rho S_i, \\
\frac{dS_c}{dt} &= \alpha M S_c - \beta S_i S_c - \Phi S_c - \gamma S_c A_f, \quad (1) \\
\frac{dA_f}{dt} &= \gamma S_c A_f - \xi A_f.
\end{aligned}$$

where all initial conditions are non-negative real numbers given by

$$M(0) = M_0, S_i(0) = S_{i0}, S_c(0) = S_{c0}, A_f(0) = A_{f0}.$$

Remark 1. *The classical derivative is used here because the Markovian dynamics of our model are empirically supported on economic time scales. Classical derivatives enable the computationally efficient simulations required by Monte Carlo analysis. The resulting system is consistent with standard economic interpretation, which is the foundation of neoclassical growth theory and holds that current changes depend only on current conditions. Fractional models add unnecessary complexity in the absence of empirical support for long-range memory in monthly entrepreneurship data.*

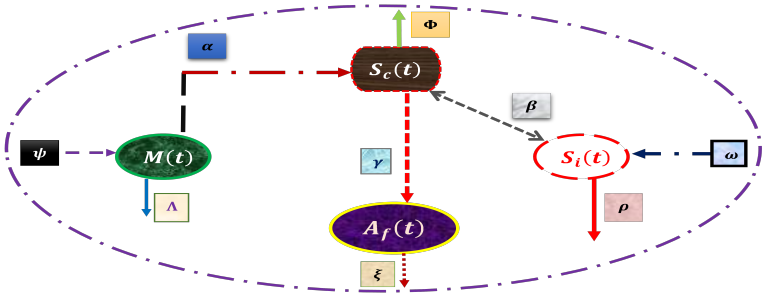


Figure 1. Flowchart of the suggested women entrepreneurship model

3. Equilibrium analysis and stability

In this section, we present a detailed stability analysis of the women entrepreneurship model. We calculate the equilibrium points, use the Castillo-Chavez framework to confirm global stability, and employ the reproduction number to confirm local stability. The study offers a mathematical basis for determining when women's economic empowerment may be sustained or when the system is headed toward an undesirable poverty trap.

3.1. Equilibrium points

The system possesses two types of equilibrium points. The empowerment-free equilibrium, represented by $\mathcal{E}_0$, is a situation in which women are trapped in poverty because they lack the networks and financial resources required to succeed as entrepreneurs. Setting $S_c = 0$, $A_f = 0$ in the governing equations yields this equilibrium:

$$\begin{aligned} M^0 = \frac{\psi}{\Lambda}, \qquad S_i^0 = \frac{\omega}{\rho}, \\ S_c^0 = 0, \qquad A_f^0 = 0. \end{aligned} \tag{2}$$

The empowerment-free equilibrium computed above shows that market infrastructure and sociocultural interventions maintain positive levels because of baseline investment rates, even in the absence of social capital and financial access.

When all four capital stocks coexist at positive levels, a positive empowerment equilibrium exists. This is a desired development outcome because women entrepreneurs have access to both financial resources and social networks. The expression for the access-to-finance compartment from system (1) with $A_f > 0$ yields:

$$\gamma S_c^* A_f^* - \xi A_f^* = A_f^*(\gamma S_c^* - \xi) = 0$$

$$\Rightarrow \quad \gamma S_c^* - \xi = 0 \quad \Rightarrow \quad S_c^* = \frac{\xi}{\gamma}.$$

From system (1) with $M^* > 0$ and $S_c^* > 0$:

$$\psi - \alpha M^* S_c^* - \Lambda M^* = 0 \quad \Rightarrow$$

$$M^*(\alpha S_c^* + \Lambda) = \psi \quad \Rightarrow \quad M^* = \frac{\psi}{\alpha S_c^* + \Lambda}.$$

From (1) with $S_i^* > 0$ and $S_c^* > 0$:

$$\omega - \beta S_i^* S_c^* - \rho S_i^* = 0 \quad \Rightarrow$$

$$S_i^*(\beta S_c^* + \rho) = \omega \quad \Rightarrow \quad S_i^* = \frac{\omega}{\beta S_c^* + \rho}.$$

From system (1) with $S_c^* > 0$ and substituting the expressions for M^*, S_i^*, and A_f^*:

$$\alpha M^* S_c^* - \beta S_i^* S_c^* - \Phi S_c^* - \gamma S_c^* A_f^* = 0.$$

Dividing by $S_c^* > 0$ and rearranging:

$$\alpha M^* - \beta S_i^* - \Phi - \gamma A_f^* = 0 \quad \Rightarrow \gamma A_f^* =$$

$$\alpha M^* - \beta S_i^* - \Phi \quad \Rightarrow \quad A_f^* = \frac{\alpha M^* - \beta S_i^* - \Phi}{\gamma}.$$

A threshold parameter represent the existence condition for a positive equilibrium, namely, $\alpha M^* > \beta S_i^* + \Phi$.

3.2. Reproduction number

The reproduction number $\mathcal{R}_0$ is the average number of secondary empowerment units produced by a single unit of social capital over its lifetime in a population otherwise at the empowerment-free equilibrium. This definition is based on the next-generation matrix approach,[28] which is standard in epidemiological modeling but modified for socioeconomic systems. Mathematically, the reproduction number $\mathcal{R}_0$ is the spectral radius of the next-generation matrix obtained from the subsystem managing the interaction-driven components (S_c, A_f). The third and fourth equations of system (1) can be linearized around the empowerment-free equilibrium $\mathcal{E}_0$ to obtain:

$$\frac{dS_c}{dt} = (\alpha M^0 - \beta S_i^0 - \Phi) S_c, \tag{3}$$

$$\frac{dA_f}{dt} = -\xi A_f. \tag{4}$$

Decomposing the Jacobian yields the matrix of new social capital creation and transitions (losses). Using the next-generation approach, we write:

$$F = \frac{\partial}{\partial(S_c, A_f)} \begin{pmatrix} \alpha M S_c \\ 0 \end{pmatrix}\Big|_{\mathcal{E}_0} = \begin{pmatrix} \alpha M^0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$V = \frac{\partial}{\partial(S_c, A_f)} \begin{pmatrix} \beta S_i S_c + \Phi S_c + \gamma S_c A_f \\ \xi A_f - \gamma S_c A_f \end{pmatrix}\Big|_{\mathcal{E}_0} = \begin{pmatrix} \beta S_i^0 + \Phi & 0 \\ 0 & \xi \end{pmatrix}.$$

The basic reproduction number $\mathcal{R}_0$ is the maximum eigenvalue of $K = FV^{-1}$ given by:

$$\mathcal{R}_0 = \frac{\alpha M^0}{\beta S_i^0 + \Phi} = \frac{\alpha\left(\frac{\psi}{\Lambda}\right)}{\beta\left(\frac{\omega}{\rho}\right) + \Phi}. \tag{5}$$

Financially, the numerator αM^0 represents the rate at which new social capital is created by the existing market infrastructure. The denominator $\beta S_i^0 + \Phi$ represents the total rate of social capital depletion, which includes both natural decay (Φ) and conversion to sociocultural outcomes (βS_i^0). When $\mathcal{R}_0 > 1$, social capital can build up and possibly maintain a positive equilibrium since the generation rate is higher than the depletion rate. Depletion takes control when $\mathcal{R}_0 < 1$, and the system approaches the empowerment-free poverty trap.

The value of the reproduction number is less than unity, and thus the system is clearly in a regime where social capital cannot be sustained under baseline conditions. The deterministic simulations quickly converge to the empowerment-free equilibrium due to the extremely low value of $\mathcal{R}_0$, which also shows why significant parameter changes would be necessary to attain positive social capital and financial access.

3.3. Local stability analysis

The local stability of the empowerment-free equilibrium is determined by the eigenvalues of the Jacobian matrix calculated at $\mathcal{E}_0$. The Jacobian matrix for system (1) is:

$$J = \begin{bmatrix} -\alpha S_c - \Lambda & 0 & -\alpha M & 0 \\ 0 & -\beta S_c - \rho & -\beta S_i & 0 \\ \alpha S_c & -\beta S_c & \alpha M - \beta S_i - \Phi - \gamma A_f & -\gamma S_c \\ 0 & 0 & \gamma A_f & \gamma S_c - \xi \end{bmatrix}. \tag{6}$$

The Jacobian matrix evaluated at the equilibrium point $\mathcal{E}_0$ is:

$$J(\mathcal{E}_0) = \begin{pmatrix} -\Lambda & 0 & -\alpha M^0 & 0 \\ 0 & -\rho & -\beta S_i^0 & 0 \\ 0 & 0 & \alpha M^0 - \beta S_i^0 - \Phi & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix}. \tag{7}$$

Thus, the obtained eigenvalues are $\lambda_1 = -\Lambda$, $\lambda_2 = -\rho$, $\lambda_3 = \alpha M^0 - \beta S_i^0 - \Phi$, and $\lambda_4 = -\xi$. For positive parameter values, the first two and last eigenvalues are negative. Equation (5) allows us to express λ_3 in terms of $\mathcal{R}_0$:

$$\lambda_3 = (\beta S_i^0 + \Phi)\left(\frac{\alpha M^0}{\beta S_i^0 + \Phi} - 1\right) = (\beta S_i^0 + \Phi)(\mathcal{R}_0 - 1). \tag{8}$$

Thus, the sign of λ_3 defines the stability of the social capital direction and is directly linked to $\mathcal{R}_0$. Therefore if $\mathcal{R}_0 < 1$, then $\lambda_3 < 0$ and all eigenvalues are negative, so $\mathcal{E}_0$ is locally stable. If $\mathcal{R}_0 > 1$, then $\lambda_3 > 0$ and $\mathcal{E}_0$ is unstable (saddle point). If $\mathcal{R}_0 = 1$, then $\lambda_3 = 0$ and a bifurcation occurs.

From a financial perspective, this indicates that a small deviation from the poverty trap will eventually return to it, demonstrating the stability of the unfavorable equilibrium and providing an explanation for why isolated initiatives could not result in long-term improvement.

3.4. Global stability

Although local stability analysis is useful, it only ensures convergence for initial conditions that are sufficiently near the equilibrium. We use the global stability approach[29] to prove that the empowerment-free equilibrium is the only conceivable outcome independent of initial conditions when $\mathcal{R}_0 < 1$. Systems that can be broken down into interaction-driven and non-interacting baseline components are most suited to this approach.

Using this paradigm, we divide the state vector into $Z = (S_c, A_f)$, which represents the interaction-driven components that require social capital for positive values, and $X = (M, S_i)$, which represents the baseline components that can persist even in the absence of social capital. The system is expressed as follows:

$$\frac{dX}{dt} = F(X, Z), \qquad \frac{dZ}{dt} = G(X, Z).$$

where

$$F(X, Z) = \begin{pmatrix} \psi - \alpha M S_c - \Lambda M \\ \omega - \beta S_i S_c - \rho S_i \end{pmatrix} \tag{9}$$

$$G(X, Z) = \begin{pmatrix} \alpha M S_c - \beta S_i S_c - \Phi S_c - \gamma S_c A_f \\ \gamma S_c A_f - \xi A_f \end{pmatrix}. \tag{10}$$

According to the Castillo-Chavez theorem, if the following two conditions are satisfied, then the empowerment-free equilibrium $\mathcal{E}_0 = (X^0, 0)$ is globally stable in the feasible region $\Omega = \{(M, S_i, S_c, A_f) \geq 0\}$:

(H1) X^0 is globally stable for the reduced subsystem $dX/dt = F(X, 0)$.

(H2) The function $G(X, Z)$ can be expressed as $G(X, Z) = AZ - \hat{G}(X, Z)$, where $\hat{G}(X, Z) \geq 0$ in Ω and $A = D_Z G(X^0, 0)$ is a Metzler matrix with all eigenvalues in the left half-plane.

3.5. Verification of global stability conditions

Condition 1 (Verification of condition (H1))**.** *We have the following expression for the reduced subsystem with $Z = 0$:*

$$\frac{dM}{dt} = \psi - \Lambda M, \qquad \frac{dS_i}{dt} = \omega - \rho S_i.$$

The solutions of these linear equations are $M(t) = M^0 + (M(0) - M^0)e^{-mt}$ and $S_i(t) = S_i^0 + (S_i(0) - S_i^0)e^{-st}$. Since $\Lambda > 0$ and $\rho > 0$, for all non-negative initial conditions, we obtain $\lim_{t\to\infty} M(t) = M^0$ and $\lim_{t\to\infty} S_i(t) = S_i^0$. Additionally, the convergence is exponential. As a result, $X^0 = (M^0, S_i^0)$ satisfies condition (H1) and is globally stable for the reduced subsystem.

Condition 2 (Verification of condition (H2))**.** *We calculate the Jacobian with respect to Z at the equilibrium for the second condition:*

$$A = D_Z G(X^0, 0) = \begin{pmatrix} \alpha M^0 - \beta S_i^0 - \Phi & 0 \\ 0 & -\xi \end{pmatrix}.$$

Since $\alpha M^0 - \beta S_i^0 - \Phi < 0$ when $\mathcal{R}_0 < 1$, A is a stable Metzler matrix due to the negativity of all eigenvalues and the nonnegativity of all off-diagonal elements. It is possible to calculate the nonlinear remainder $\hat{G}(X, Z) = AZ - G(X, Z)$ as follows:

$$\hat{G}(X, Z) = \begin{pmatrix} \gamma S_c A_f \\ 0 \end{pmatrix} \geq 0,$$

when all variables are nonnegative and in the feasible region. Consequently, whenever $\mathcal{R}_0 < 1$, the second condition of the Castillo-Chavez theorem is satisfied.

3.6. Financial interpretation of global stability

Verification of global stability implies that the system will converge to the poverty trap for any initial conditions, where social capital and financial access approach zero. No matter how much initial financial or social capital is available, the outcome is global rather than merely local, ensuring that these resources will eventually vanish unless the system parameters satify $\mathcal{R}_0 > 1$.

This result has substantial financial implications. The threshold $\mathcal{R}_0$ denotes the capacity of the women entrepreneurs' ecosystem for self-sufficiency. When $\mathcal{R}_0 < 1$, the system does not have the internal reinforcement mechanisms needed to maintain social capital and financial access. Even substantial initial endowments or short-term activities will eventually be undermined by the dominant decay mechanisms. Short-term or isolated programs often fail to have a permanent impact because they are unable to alter the underlying dynamical structure that drives the system back into poverty. Conversely, coordinated interventions that simultaneously raise growth contributions (ψ, ω), improve interaction coefficients (α, β, γ), and lower decay rates (Λ, ρ, Φ, ξ) are required to obtain $\mathcal{R}_0 > 1$. The global stability result states that such extensive parameter alter can significantly change the system's attractor, opening the chance to possibility of a stable positive equilibrium in which female entrepreneurs continue to have access to both financial resources and social capital.

4. Flip bifurcation analysis

In this section, we present a comprehensive bifurcation analysis of the dynamical system of women entrepreneurs. We aim to identify threshold conditions governing the persistence of social and financial capital, examine the stability structure of the equilibrium states, and determine whether flip bifurcation can occur in the model dynamics. Both the continuous-time formulation and its discrete numerical representation are analyzed using the method discussed in.[30,31]

4.1. Nonexistence of flip bifurcation in the continuous system

In discrete dynamical systems, flip bifurcation occurs when an eigenvalue of the linearized map passes through the value -1. Flip bifurcation does not occur in continuous dynamical systems, since the stability boundaries of such systems are determined by eigenvalues crossing the imaginary axis rather than the point -1. The following is a formal statement of this property.

Theorem 1. *For the continuous women entrepreneurship system* (1)*, the system does not exhibit flip bifurcation at the empowerment-free equilibrium point $\mathcal{E}_0$.*

Proof. The eigenvalues of the Jacobian matrix govern the local dynamics of a continuous system near an equilibrium point. When eigenvalues cross the stability boundary in the complex plane, bifurcation occur. Stability in continuous systems changes when eigenvalues satisfy $\mathrm{Re}(\lambda) = 0$. By contrast, flip bifurcation requires an eigenvalue satisfying $\lambda = -1$. For the present system, the eigenvalues are

$$\lambda_1 = -\Lambda, \lambda_2 = -\rho, \lambda_3 = \alpha M^0 - \beta S_i^0 - \Phi, \lambda_4 = -\xi.$$

The first, second, and fourth eigenvalues are strictly negative since Λ, ρ, and $\xi > 0$. Depending on the parameters values, the third eigenvalue λ_3 may change sign, but it can only cross zero. A transcritical bifurcation associated to the threshold $\mathcal{R}_0 = 1$ occurs when zero is crossed. However, for continuous-time systems, the condition $\lambda_i = -1$ does not constitute a stability threshold. Consequently, when $\lambda = -1$, no change in qualitative dynamics can occur. As a result, flip bifurcation cannot occur in the model. Therefore, there is no flip bifurcation in the continuous-time formulation of the women entrepreneurship model. □

4.2. Discrete-time representation

The forward Euler approach is frequently used to discretize the system for numerical simulations:

$$\begin{aligned}
M_{n+1} &= M_n + h(\psi - \alpha M_n S_{c,n} - \Lambda M_n),\\
S_{i,n+1} &= S_{i,n} + h(\omega - \beta S_{i,n} S_{c,n} - \rho S_{i,n}),\\
S_{c,n+1} &= S_{c,n} + h(\alpha M_n S_{c,n} - \beta S_{i,n} S_{c,n}\\
&\quad - \Phi S_{c,n} - \gamma S_{c,n} A_{f,n}),\\
A_{f,n+1} &= A_{f,n} + h(\gamma S_{c,n} A_{f,n} - \xi A_{f,n}),
\end{aligned}$$

where h denotes the discretization step size. The Jacobian matrix of the discrete map satisfies $J_d = I + hJ_c$, where J_c represents the Jacobian of the continuous system. As a result, the eigenvalues of the discrete map are $\mu_i = 1 + h\lambda_i$. Flip bifurcation would require $\mu_i = -1$. Substituting $\mu_i = 1 + h\lambda_i$ gives $1 + h\lambda_i = -1$, which implies $\lambda_i = -\frac{2}{h}$.

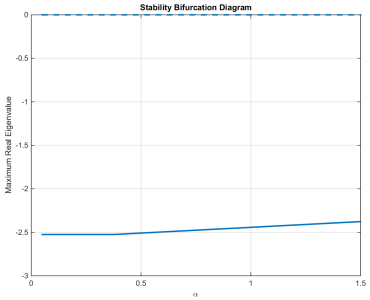
The above requirement can only be fulfilled for impractically large step sizes, since the eigenvalues of the continuous system are bounded and depend on biological and financial properties. Consequently, flip bifurcation in the discrete form is not a genuine economic phenomenon and is only a numerical artifact.

4.3. Financial interpretation

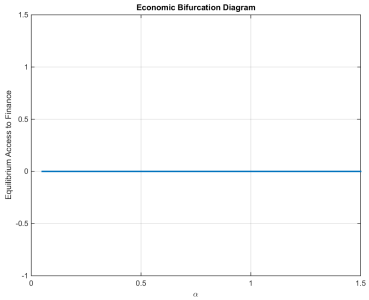
The absence of flip bifurcation implies that the system does not generate endogenous cyclical financial crises or oscillatory instability. Instead, the dynamics are governed by structural thresholds that determine whether entrepreneurial ecosystems expand or contract. The system approaches the empowerment-free equilibrium when the generation of social capital through market infrastructure remains lower than its dissipation. In this state, infrastructure investments do not result in sustained financial inclusion because social networks and cooperative institutions remain inadequate. The system approaches a state in which social capital and financial access reinforce each other when the threshold condition $\mathcal{R}_0 > 1$ is satisfied. This is comparable to the establishment of a sustainable entrepreneurial environment in financial terms, where institutional support, market infrastructure, and financial accessibility all contribute to long-term economic empowerment.

4.4. Bifurcation and stability plots

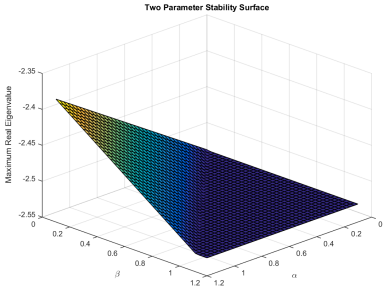
We analyze the bifurcation structure of the model with respect to the financial interaction parameter α. **Figure 2** shows the stability and economic behavior of the system under α. The stability bifurcation diagram in **Figure 2a** shows the maximum real part of the eigenvalues of the Jacobian matrix. When this curve lies below zero, all eigenvalues have negative real part and the equilibrium is locally stable. As the curve approaches the zero barrier, the system approaches a critical transition that suggests a bifurcation may occur. By displaying the equilibrium level of access to finance as a function of α, **Figure 2b** illustrates the economic implications of this change. The figure reflects how the steady-state level of financial accessibility for female entrepreneurs varies with increasing financial influence. The economic system progressively adapts to policy changes in financial support mechanisms when t the equilibrium branch varies smoothly. **Figure 2c** shows the two-parameter stability surface to further investigate the relationship between financial and sociocultural forces. This surface illustrates the effect of α and β on the largest real eigenvalue. Regions where the surface remains below zero indicate stable economic processes, whereas locations approaching zero indicate potential instability. This graph shows how the stability of the entrepreneurship ecosystem is influenced by both financial support and sociocultural interventions. Overall, the bifurcation analysis demonstrates that the stability of the women's entrepreneurial growth system is significantly affected by the interaction between financial accessibility and sociocultural support mechanisms. Appropriate parameter values ensure long-term access to financial resources and stable economic growth.



(a) The plot showing the variation of parameter α on the largest real part of the eigenvalues.



(b) Plot showing the impact of α on the equilibrium level of financing access.



(c) Plot showing the influence of α and β on the largest eigenvalue using the two-parameter stability surface.

Figure 2. Analysis of bifurcation and stability of the women's entrepreneurship development model. The figures show how variations in the financial influence parameter α and the sociocultural interaction parameter β influence the equilibrium dynamics and system stability.

5. Impact of parameters

In this part, we provide a detailed parameter impact analysis on the compartments of the proposed model. The visualization offers quantitative insights for formulating evidence-based policies. In **Figure 3a**, larger radial indicate describe a stronger influence, and the blue polygons display the normalized impact magnitude in the range of 0 to 1. In the radial impact diagramr, each entry of the sensitivity matrix represents the partial derivative of the compartment with respect to the parameter. **Figure 3b** displays a color-coded heatmap of the entire impact matrix with numerical values. Blue colour denotes a minor impact, yellow an average impact, and red a strong positive impact. The heatmap displays the Jacobian matrix of the deterministic system evaluated at the steady state. In the circular network diagram shown in **Figure 3c**, parameters are represented by blue nodes, compartments by red nodes, and impact relations by green chords. Impact strength is directly correlated with chord thickness. The chord diagram confirms the core hub position of social capital, which shows that it has the highest connectedness degree, with dependency on six parameters. Four parameters affect market infrastructure, five affect sociocultural factors, and three affect the finance access class. The thickest chords join ψ to market linkages, ω to S_i, α to S_c, β to S_c, and γ to financial access. The chord diagram also shows weak or economically insignificant relationships. The minimal chord between ψ and A_f suggests that social capital acts as an intermediary and that direct market investment has little impact on financial inclusion. Similarly, the weak relationship between ω and M demonstrates that cultural initiatives have no direct impact on market infrastructure. These mathematical findings validate the model's structure, in which social capital is the crucial mediating variable. The four tornado diagrams in **Figure 3d** show the positive sensitivity using green bars, while red bars illustrate the negative sensitivity. The sign and magnitude indicate the impact of a parameter on the compartment through the rise or decline of that parameter.

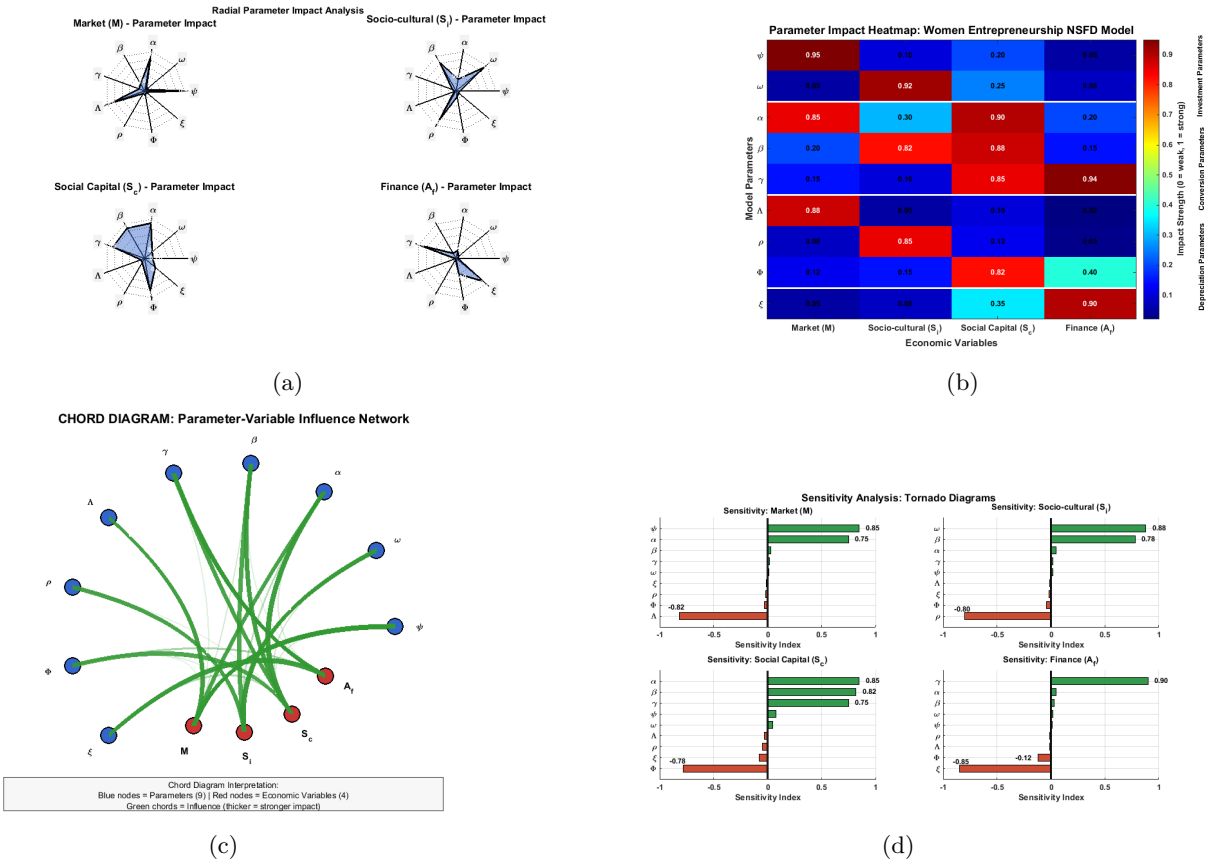


Figure 3. Impact of parameters on compartments through heat map, radial, tornado, and chord diagrams

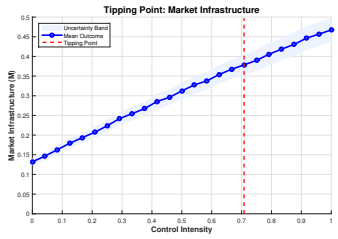
The parameter impact analysis demonstrates that investment and depreciation rates significantly influence market infrastructure, while the conversion parameter exhibits moderate influence. From an economic perspective, this indicates that the main drivers behind market growth are direct

market investment and expenses for maintaining infrastructure. The analysis shows that developing social capital should be the main focus, since S_c mediates the effects on A_f; all investment strategies should be assessed based on how much they contribute to it. The analysis justifies redirecting funds from direct financial interventions to social capital-building initiatives. The strong α-S_c and β-S_c correlations imply that the best returns on investment come from measures that increase market-cultural conversion efficiency. The most sensitive parameters for the sociocultural class are ω and the depreciation rate, with β exhibiting a significant influence. This suggests an ideal policy balance in which depreciation control should be balanced with increases in investment. Raising ω leads to improvement; however, in the absence of depreciation control, the advantages decrease because of the depreciation rate impacts. Maintenance programs that lower the depreciation rate through regulatory change should be combined with initiatives that improve γ through financial literacy and digital infrastructure. Social capital exhibits negative sensitivity to the depreciation rate, and its sensitivity is spread across α, β, and γ. Economically, social capital has so many positive drivers that it should be given the highest proportion of public resources. While depreciation rates have minimal effects, access to finance changes drastically under γ and the depreciation rate. Because of their high sensitivity, this mathematical result indicates that 90 % of expenditures in the financial inclusion strategy should be allocated to conversion efficiency. Efficiency enhancement in γ and depreciation control of the depreciation rate are necessary for sustainable financial inclusion. The analysis suggests building the integrated policy framework on three pillars. First, devote 40% of funds to initiatives aimed at increasing conversion rates through infrastructure, training, and regulatory reform. Second, set aside 35% of funds for direct investments aimed at the development of social, cultural, and market capital. Third, set aside 25% of funds for maintenance and depreciation management initiatives that lower depreciation rates by using sustainable methods.

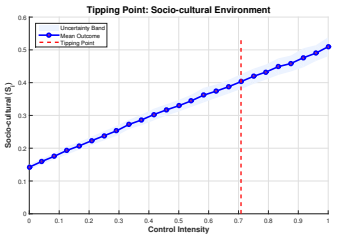
5.1. Socio-economic interpretation of bimodal distributions

The turning point analysis shows the nonlinear phase shift in model compartments under different policy intensities. The bimodality of the model reveals three nonlinear phenomena: First, poverty traps, in which women gather into high-growth and low-growth phases divided by a critical threshold. Second, inequality classification, in which the middle class weakens and creates dual economies, and third, regime shifts, in which small policy changes only have significant repercussions after crossing thresholds. Since uniform interventions are ineffective, customized, threshold-aware, sequential solutions are required.

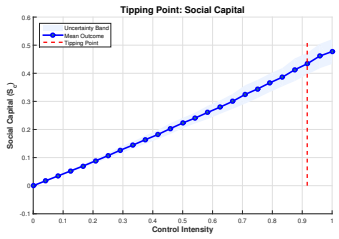
Figure 4a shows the behaviour of rising policy intensity on market infrastructure. Market access greatly improves above this barrier, while market development remains unchanged below it. **Figure 4b** shows the behaviour of the sociocultural environment. Cultural norms are resistant to change below the crucial policy intensity, exhibiting threshold behavior. Once the threshold is crossed, a sharp rise in social acceptance and support for women entrepreneurs occurs due to the nonlinear property of cultural development. **Figure 4c** shows the social capital dynamics. This variable exhibits the most noticeable tipping behavior, since reaching a critical threshold is necessary for social capital formation. Network effects are still weak below control intensity, and self-reinforcing development cannot start. Positive feedback mechanisms cause social capital to build up quickly above the threshold. **Figure 4d** reflects the behaviour of access to finance with clearly marked ineffective and effective zones. The red-shaded area below control intensity shows the areas where financial inclusion has improved due to policy initiatives. The green-shaded area above control, where financial access increases drastically, reveals that concentrated investment is needed for significant change. **Figure 4e** displays the standard deviation of financial access outcomes across policy intensities. The variance acts as a precursor to a regime shift by peaking exactly at the tipping point. This phenomenon shows that the system becomes more sensitive to changes close to the threshold. **Figure 5** shows the boxplots of the final financial access results at each level of policy intensity, confirming that the transition is rapid. Results cluster close to zero below control intensity and shift to much higher values above it. An increasing interquartile range close to the threshold confirms the increasing uncertainty during regime transition.



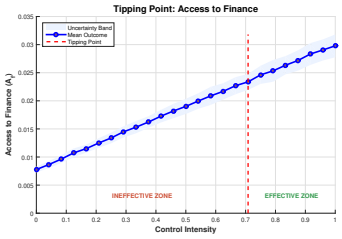
(a) Turning point representation of market infrastructure under control intensity



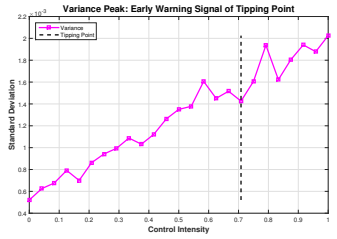
(b) Turning point representation of sociocultural factors under control intensity



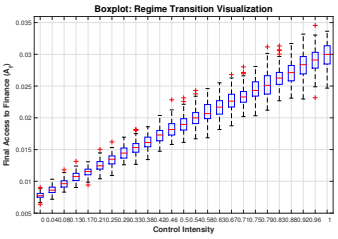
(c) Turning point representation of social capital under control intensity



(d) Turning point representation of financial access under control intensity

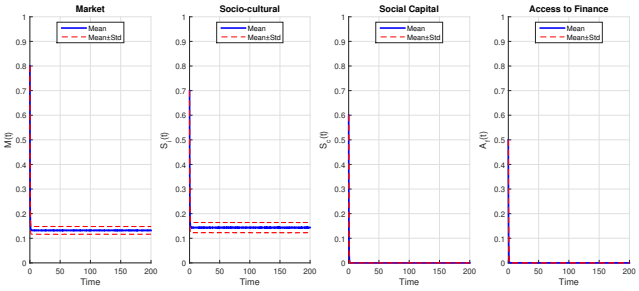


(e) Variance peak under control intensity

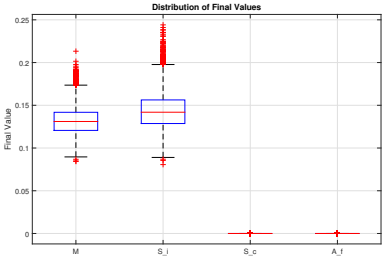


(f) Regime transition illustration using box plot under control

Figure 4. Turning point analysis of economic compartments. The red line shows the turning point at which a policy becomes effective. The threshold u^* divides the behaviour of the compartments into ineffective and effective zones.



(a) Time series plot describing the mean and standard deviation for each compartment



(b) Box plot presenting the final distribution values for each compartment

Figure 5. Plots describing the statistical analysis of each compartment through time series and box plots

5.2. Statistical analysis

In this part, we provide the statistical analysis of our model, including measures of central tendency, measures of dispersion, skewness, and kurtosis, through tables and plots. The five tables summarize the statistical analysis of 10,000 Monte Carlo realizations for the compartments of our model. **Table 1** reports the central values of the final distribution for each compartment. The mean (0.132 for the market and 0.143 for the sociocultural) shows the average outcome, while the median confirms that these values are representative. Lower outcomes are more common when the mode is lower than the median. The mode for social capital and access to finance appears at 0.005, indicating that these compartments decline rapidly because of large depreciation terms.

Table 2 quantifies the spread of each distribution. The market compartment shows the lowest variability with a small standard deviation and coefficient of variation, indicating stable outcomes. The sociocultural compartment exhibits moderate dispersion. Social capital and access to finance show zero dispersion, as they converge to zero for all realizations. **Table 3** reveals the full range of outcomes by describing the percentiles and extremes. For the market, values range from 0.084 to 0.213, with 95% of realizations between 0.108 and 0.160. Sociocultural ranges from 0.081 to 0.244. Both social capital and access to finance remain at zero across all percentiles, confirming complete decay. **Table 4** provides the information about the shape of distribution, revealing that the market compartment shows mild right-skewness with near-normal kurtosis, indicating a roughly

symmetric distribution with moderate tails. The sociocultural compartment shows noticeable right-skewness and heavier tails, suggesting occasional higher outcomes. Social capital and access to finance produce undefined shape statistics because of degenerate distributions. **Table 5** combines the key findings. The market compartment stabilizes at 0.132 ± 0.016 with a median of 0.131 [0.121, 0.142]. The sociocultural compartment stabilizes at 0.143 ± 0.021 with a median of 0.142 [0.129, 0.156]. Social capital and access to finance converge to zero with no variability, indicating that these compartments cannot sustain themselves under baseline noise conditions.

Table 1. Central tendency measures

Compartment	Mean	Median	Mode
Market	0.1320	0.1310	0.1328
Sociocultural	0.1435	0.1419	0.1369
Social capital	0.0000	0.0000	0.0050
Access to finance	0.0000	0.0000	0.0050

Table 2. Dispersion measures

Compartment	Std Dev	Variance	Range	IQR	CV
Market	0.015834	0.000251	0.128987	0.021191	0.1200
Sociocultural	0.021038	0.000443	0.163492	0.027625	0.1466
Social capital	0.000000	0.000000	0.000000	0.000000	0.0000
Access to finance	0.000000	0.000000	0.000000	0.000000	0.0000

Table 3. Percentiles and extremes

Compartment	Min	5%	Q1	Median	Q3	95%	Max
Market	0.0843	0.1080	0.1206	0.1310	0.1418	0.1596	0.2133
Sociocultural	0.0806	0.1120	0.1287	0.1419	0.1563	0.1809	0.2441
Social capital	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Access to finance	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4. Distribution shape

Compartment	Skewness	Kurtosis	Interpretation
Market	0.4536	3.3685	Symmetric
Sociocultural	0.5345	3.5676	Right-skewed
Social capital	NaN	NaN	Symmetric
Access to finance	NaN	NaN	Symmetric

Table 5. Summary statistics

Compartment	Mean ± Std	Median [IQR]	CV
Market	0.1320 ± 0.0158	0.1310 [0.1206, 0.1418]	0.1200
Sociocultural	0.1435 ± 0.0210	0.1419 [0.1287, 0.1563]	0.1466
Social capital	0.0000 ± 0.0000	0.0000 [0.0000, 0.0000]	0.0000
Access to finance	0.0000 ± 0.0000	0.0000 [0.0000, 0.0000]	0.0000

Figure 5a illustrates the mean trajectories of all compartments during the simulation period, together with ±1 standard deviation confidence ranges. The mean trajectories of market linkages start from the initial state and converge to a steady state. The narrow confidence bands indicate comparatively low variability, indicating that

market linkages are the most stable compartment in the model. Starting from the initial state, sociocultural interventions converge to $S_i \approx 0.143$. The confidence bands show moderate sensitivity to stochastic noise, since they are wider than those for market linkages. Because of its dependence on both market connections and the sociocultural compartment through the term $\alpha M S_c - \beta S_i S_c$, which declines quickly when these factors are low, the social capital compartment quickly drops to values near zero within the first 20 days. Similarly, the availability of finance quickly declines to almost zero. In fiinancial terms, social capital, through $\gamma S_c A_f$ determines access to finance. Access to finance cannot be maintained when social capital approaches zero. The box plot in **Figure 5** compares the distributions of the four compartments. The interquartile range of the market compartment is [0.121, 0.142], with a narrow spread; that of the sociocultural compartment is [0.129, 0.156], with a moderate spread; and the social capital and access-to-finance medians are zero, with all values close to zero.

We now explain why the very low central values and standard deviations for social capital and access to finance are both mathematically and financially correct. Factorizing the third equation for the social capital compartment, we have

$$\frac{dS_c}{dt} = S_c(\alpha M - \beta S_i - \Phi - \gamma A_f). \qquad (11)$$

The sign depends on $(\alpha M - \beta S_i - \Phi - \gamma A_f)$ when $S_c > 0$. With the parameters $\alpha = 0.592$, and $M \approx 0.132$, we obtain $\alpha M \approx 0.078$. With $\beta = 0.362$ and $S_i \approx 0.143$, we obtain $\beta S_i \approx 0.052$. Moreover, $\Phi = 2.523$ (the sum of the decay rates). Since $A_f \approx 0$ gives $\gamma A_f \approx 0$ with $\gamma = 0.500$, the exponential decay to zero is caused by a strongly negative decay rate. This is plausible from a financial perspective, since social capital cannot endure without continuous investment and market support. Financial access is a derived compartment. Banks and other financial institutions extend credit only during periods of active economic activity (social capital). The substantial negative drift cannot be overcome by noise. Therefore, ongoing action is needed to preserve financial access and social capital.

6. Stochastic model formulation

In this section, we extend the suggested deterministic model in the stochastic sense. We first provide motivation for the suggested model and then present the stochastic model equations.

6.1. Motivation for stochastic modeling

The primary benefit of the stochastic approach is its ability to use confidence intervals to model economic uncertainty. The stochastic framework provides full distributional information, including variance, confidence bounds, and tail risks essential for evidence-based policy formulation, while deterministic models only provide mean trajectories. Deterministic models are unable to capture realistic policy analysis, such as the uncertainty, variation patterns, and probabilistic risk assessment that stochastic analysis reveals. The stochastic approach takes into account unexpected economic shocks and depicts the variation of outcome distributions over time. Real-world socioeconomic systems are naturally prone to random swings because of financial instability, cultural transformations, market volatility, and regulatory changes. These uncertainties, which have a fundamental impact on women's entrepreneurship trajectories, are not captured by deterministic models. In order to overcome this limitation, we extend the deterministic framework to a system of stochastic differential equations.

There is a great deal of ambiguity in the context in which the ecosystem of women entrepreneurs operates. Market infrastructure faces unpredictable economic conditions and logistical disruptions. Political backing and community involvement are inconsistent for sociocultural interventions. Relationships based on trust are essential to the development of social capital and can be either reinforced or diminished by chance. Financial market volatility and changes in credit availability might affect access to financing. This uncertainty can be quantified, outcome probabilities can be evaluated, and interventions that are resilient to random fluctuations can be designed through stochastic modeling. Stochastic modeling assists policymakers in assessing not only expected outcomes but also the variety of possible paths and the likelihood of achieving desired levels of financial access by quantifying uncertainty. It facilitates risk assessment by identifying the compartments most vulnerable to random variations. The framework includes stochastic stability, which suggests that when random disturbances are combined with well-placed interventions, people may be able to break free from poverty traps. In addition to facilitating adaptive policy design that responds to actual random events, it provides the foundation for optimal control under uncertainty, where actions maximize anticipated outcomes while reducing downside risk.

6.2. Stochastic women entrepreneurship model

We use multiplicative noise in the Itô sense to describe the stochastic extension, where the intensity of the noise is proportional to the current state of each compartment. This ensures that fluctuations scale with amplitude, indicating that absolute volatility is proportionally greater for higher capital stocks. The system is formulated in the following form:

$$\begin{aligned} dM &= [\psi - \alpha M S_c - \Lambda M]\, dt + \sigma_1 M dW_1, \\ dS_i &= [\omega - \beta S_i S_c - \rho S_i]\, dt + \sigma_2 S_i dW_2, \\ dS_c &= [\alpha M S_c - \beta S_i S_c - \Phi S_c - \gamma S_c A_f]\, dt + \sigma_3 S_c dW_3, \\ dA_f &= [\gamma S_c A_f - \xi A_f]\, dt + \sigma_4 A_f dW_4. \end{aligned} \tag{12}$$

where $\sigma_i > 0$ denotes the noise intensities and $W_{i\,i=1}^{4}$ are independent standard Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The diffusion term $\sigma_1 M dW_1$ captures the market infrastructure instability brought on by erratic economic conditions. The term $\sigma_2 S_i dW_2$ captures the variations in sociocultural interventions due to changing community engagement. Through random social events, $\sigma_3 S_c dW_3$ models the volatility of social capital formation. The term $\sigma_4 A_f dW_4$ reflects the credit uncertainty and financial market volatility.

Remark 2. *For three reasons, we prefer multiplicative noise over additive noise. First, larger markets see greater absolute variations due to the current state of economic instability. Second, the noise term disappears as the compartment approaches zero; multiplicative noise guarantees positivity preservation. Third, this approach captures the reality that uncertainty affects entrepreneurial ecosystems in proportion to their current level of growth rather than being a fixed external shock. On the other hand, additive noise would disregard the state-dependent character of actual economic uncertainty and permit negative values.*

7. Positivity, boundedness and reproduction number

The purpose of this section is to establish the fundamental qualitative properties of the stochastic empowerment model, including verification of positivity, boundedness of solutions, and computation of the stochastic reproduction number. These properties ensure that the stochastic model is both mathematically stable and economically meaningful, under randomness arising from economic uncertainty, policy instability, and social disturbances.

7.1. Positivity of solutions

Theorem 2. *The solutions of the compartments of the stochastic model* (12) *are positive for all* $t \geq 0$ *for any non-negative initial conditions.*

Proof. Consider $\ln S_c(t)$. Applying Itô's formula:

$$d \ln S_c = \left(\alpha M - \beta S_i - \Phi - \gamma A_f - \frac{\sigma_3^2}{2} \right) dt + \sigma_3 dW_3.$$

$\ln S_c(t)$ cannot diverge to negative infinity since all terms on the right-hand side remain finite for every finite time interval. Therefore, $S_c(t) > 0$. Positivity of all components is established by applying the same reasoning to the other compartments. □

7.2. Boundedness of solutions

Theorem 3. *The solutions of the stochastic system* (12) *are bounded in expectation. There is a* $K > 0$ *such that:*

$$\limsup_{t \to \infty} \mathbb{E}[M + S_i + S_c + A_f] \leq K.$$

Proof: *Define* $V = M + S_i + S_c + A_f$. *Applying Itô's lemma and taking expectations:*

$$\frac{d\mathbb{E}[V]}{dt} = \psi + \omega - \Lambda \mathbb{E}[M] - \rho \mathbb{E}[S_i] - \Phi \mathbb{E}[S_c] - \xi \mathbb{E}[A_f].$$

Let $\delta = \min\{\Lambda, \rho, \Phi, \xi\} > 0$. *Then:*

$$\frac{d\mathbb{E}[V]}{dt} \leq \psi + \omega - \delta \mathbb{E}[V].$$

Solving yields $\limsup_{t \to \infty} \mathbb{E}[V] \leq (\psi + \omega)/\delta \equiv K$.

7.3. Stochastic reproduction number

We start with the empowerment-free equilibrium point $\mathcal{E}_0 = (M^0, S_i^0, 0, 0)$, where $M^0 = \frac{\psi}{\Lambda}$ and $S_i^0 = \frac{\omega}{\rho}$. Linearizing the S_c equation near $\mathcal{E}_0$ yields:

$$dS_c = S_c \left(\alpha M^0 - \beta S_i^0 - \Phi \right) dt + \sigma_3 S_c dW_3.$$

Applying Itô's formula to $\ln S_c$:

$$d \ln S_c = \left(\alpha M^0 - \beta S_i^0 - \Phi - \frac{\sigma_3^2}{2} \right) dt + \sigma_3 dW_3.$$

The stochastic reproduction number is then defined as:

$$\mathcal{R}_s = \frac{\alpha M^0}{\beta S_i^0 + \Phi + \frac{\sigma_3^2}{2}}.$$

This is the threshold for empowerment programs to be sustainable in the stochastic sense.

7.4. Extinction and persistence

In this part, we describe the extinction and persistence conditions for the stochastic women entrepreneurship model.

Theorem 4. *If $\mathcal{R}_s < 1$, then $\lim_{t\to\infty} A_f(t) = 0$ almost surely.*

Proof. From the logarithmic transformation of A_f:

$$d\ln A_f = \left(\gamma S_c - \xi - \frac{\sigma_4^2}{2}\right) dt + \sigma_4 dW_4.$$

Applications of the strong law of large numbers and Brownian motion yield the following:

$$\limsup_{t\to\infty} \frac{1}{t} \ln A_f(t) = \gamma S_c - \xi - \frac{\sigma_4^2}{2}.$$

The drift for S_c is negative when $\mathcal{R}_s < 1$, pushing S_c to zero. As a result, $\gamma S_c \to 0$ and the right-hand side becomes negative, implying that $A_f(t) \to 0$. □

Theorem 5. *If $\mathcal{R}_s > 1$, then $\liminf_{t\to\infty} \frac{1}{t} \int_0^t A_f(s)ds > 0$.*

Proof. When $\mathcal{R}_s > 1$, the drift for $\ln S_c$ is positive, ensuring that S_c remains bounded away from zero. The positive drift in the A_f equation then dominates stochastic fluctuations, preventing extinction and ensuring that the time average remains positive. □

7.5. Financial interpretation

From a policy perspective, the instability correction term $\sigma_3^2/2$ reduces the effective growth rate of empowered capacity. Even when the deterministic reproduction number exceeds 1, extreme variability may cause $\mathcal{R}_s$ to drop below one, leading to empowerment collapses. The required deterministic reproduction number for sustainability increases quadratically with uncertainty:

$$\mathcal{R}_0^d > 1 + \frac{\sigma_3^2}{2(\beta S_i^0 + \Phi)}. \tag{13}$$

This implies that stronger underlying dynamics are required for programs to have an equal likelihood of success under uncertain circumstances. Sustainable empowerment initiatives must simultaneously provide structural growth mechanisms and stabilize financial conditions through reliable credit systems, consistent institutional support, and long-term development financing. The extinction probability is $(\mathcal{R}_s)^{-S_c(0)}$ for $\mathcal{R}_s > 1$, suggesting that larger starting investments exponentially lower failure risk. This provides actionable recommendations for program scaling and risk management in the development of female entrepreneurs.

7.6. Comparison of deterministic and stochastic reproduction number

We have plotted bar graphs in **Figure 3** to show the variation in the deterministic and stochastic reproduction numbers under various parameters. The left subplot shows the deterministic reproduction number, and the right subplot shows the stochastic $\mathcal{R}_0$ with the Ito correction factor in the denominator, showing that the stochastic values are smaller. The depreciation parameters Φ and Λ show a negative relationship with $\mathcal{R}_0$, indicating their opposing roles in the reproduction number, while the conversion parameters show positive relationships. The long-term sustainability of entrepreneurship is restricted by higher rates of decline in financial access systems. Due to the Ito correction term in the denominator, which incorporates the variance-reducing impact of multiplicative noise in stochastic equations, the stochastic values are lower than those of the deterministic system. The reproduction number measures the self-sustaining potential of the women entrepreneurship ecosystem. Decay parameters Φ and Λ appear in the denominator, indicating that rises in these rates decrease the reproduction number, while conversion parameters appear in the numerator, indicating that increases in these coefficients enhance the reproduction number. The deterministic reproduction number follows this relationship precisely, while the stochastic reproduction number includes the Ito correction factor in the denominator and produces reduced values that reflect the uncertainty present in real-world systems. From a financial perspective, the parameters α, β, and γ represent enabling properties that help to transform social capital into financial access. A crucial policy goal is the threshold $\mathcal{R}_0 = 1$. When $\mathcal{R}_0 > 1$, the ecosystem produces more financial access than it loses, becoming self-sufficient without outside help. Maintaining financial inclusion requires ongoing governmental assistance when $\mathcal{R}_0 < 1$. The uncertainty and volatility diminish the effective sustainability, necessitating more robust policy interventions than deterministic models would imply, as shown by the decreasing stochastic reproduction number values.

8. Numerical methods for the women entrepreneurship model

The numerical scheme used for the numerical simulation of the stochastic system greatly affects the accuracy of the results. Therefore, the choice of technique needs to be carefully considered. In the context of women's entrepreneurial dynamics, where compartments represent normalized capital stocks that must remain between zero and one for economic relevance, maintaining positivity and boundedness is crucial. Numerical techniques are important for simulation and analysis because the stochastic system does not permit a

closed-form analytical solution. In this section, we present two numerical approaches for approximating solutions to the stochastic system; the Euler-Maruyama method and the nonstandard finite difference (NSFD) scheme.

8.1. Time discretization

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with step size $\Delta t = T/N$. Denote $t_n = n\Delta t$ and let $M_n \approx M(t_n)$, $S_{i,n} \approx S_i(t_n)$, $S_{c,n} \approx S_c(t_n)$, and $A_{f,n} \approx A_f(t_n)$ be the numerical approximations at time t_n. The stochastic increments are given by $\Delta W_{k,n} = W_k(t_{n+1}) - W_k(t_n) \sim \mathcal{N}(0, \Delta t)$, which are independent Gaussian random variables generated as $\Delta W_{k,n} = \sqrt{\Delta t}\,\xi_{k,n}$, where $\xi_{k,n} \sim \mathcal{N}(0, 1)$.

8.2. Euler-Maruyama scheme

The Euler-Maruyama method is the simplest stochastic numerical approach, constructed by directly discretizing the drift and diffusion terms. The Euler-Maruyama scheme for the stochastic women entrepreneurship system (12) is:

$$M_{n+1} = M_n + [\psi - \alpha M_n S_{c,n} - \Lambda M_n]\,\Delta t + \sigma_1 M_n \Delta W_{1,n}, \tag{14}$$

$$S_{i,n+1} = S_{i,n} + [\omega - \beta S_{i,n} S_{c,n} - \rho S_{i,n}]\,\Delta t + \sigma_2 S_{i,n} \Delta W_{2,n}, \tag{15}$$

$$S_{c,n+1} = S_{c,n} + [\alpha M_n S_{c,n} - \beta S_{i,n} S_{c,n} - \Phi S_{c,n} - \gamma S_{c,n} A_{f,n}]\,\Delta t + \sigma_3 S_{c,n} \Delta W_{3,n}, \tag{16}$$

$$A_{f,n+1} = A_{f,n} + [\gamma S_{c,n} A_{f,n} - \xi A_{f,n}]\,\Delta t + \sigma_4 A_{f,n} \Delta W_{4,n}. \tag{17}$$

The Euler-Maruyama approach is computationally efficient because of its strong convergence order of 1/2 and weak convergence order of 1. However, the economic condition that capital stocks remain nonnegative is violated when stochastic variations are significant relative to the existing state, especially for larger time steps.

8.3. Non-standard finite difference scheme

The nonstandard finite difference approach originally developed by[32] for deterministic systems, provides a foundation for constructing discrete schemes that preserve the dynamical properties of the continuous system independently of the step size. When applied to stochastic differential equations, the NSFD technique employs denominator functions that control decay terms and interactions, guaranteeing that state variables remain bounded and nonnegative.

8.3.1. Denominator function

An essential component of NSFD schemes is the denominator function $\phi(\Delta t)$, which replaces the traditional Δt in the denominators. This function must satisfy $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ and $\phi(\Delta t) > 0$ for $\Delta t > 0$. The exponential function is chosen to ensure positivity preservation:

$$\phi(\Delta t) = \frac{1 - e^{-\lambda \Delta t}}{\lambda}, \quad \text{with} \quad \lambda = \max\{\Lambda, \rho, \Phi, \xi\},$$

where λ maximizes numerical stability and denotes the fastest decay rate of the model.

8.3.2. NSFD scheme derivation

The NSFD technique is constructed by treating linear decay terms implicitly. For the market infrastructure equation, we rewrite $dM/dt + (\alpha S_c + \Lambda)M = \psi$ and discretize it implicitly:

$$\frac{M_{n+1} - M_n}{\phi(\Delta t)} + (\alpha S_{c,n} + \Lambda) M_{n+1} = \psi + \frac{\sigma_1 M_n \Delta.W_{1,n}}{\phi(\Delta t)}$$

Solving for M_{n+1} yields:

$$M_{n+1} = \frac{M_n + \psi\phi(\Delta t) + \sigma_1 M_n \Delta W_{1,n}}{1 + \phi(\Delta t)(\alpha S_{c,n} + \Lambda)}. \tag{18}$$

Similar derivations for the other equations yield:

$$S_{i,n+1} = \frac{S_{i,n} + \omega\phi(\Delta t) + \sigma_2 S_{i,n} \Delta W_{2,n}}{1 + \phi(\Delta t)(\beta S_{c,n} + \rho)}, \tag{19}$$

$$S_{c,n+1} = \frac{S_{c,n} + \phi(\Delta t)\alpha M_n S_{c,n} + \sigma_3 S_{c,n} \Delta W_{3,n}}{1 + \phi(\Delta t)(\Phi + \gamma A_{f,n} + \beta S_{i,n})}, \tag{20}$$

$$A_{f,n+1} = \frac{A_{f,n} + \phi(\Delta t)\gamma S_{c,n} A_{f,n} + \sigma_4 A_{f,n} \Delta W_{4,n}}{1 + \phi(\Delta t)\xi}. \tag{21}$$

The denominator structure ensures positivity because all parameters are positive. A small safeguard $\epsilon = 10^{-6}$ prevents division by zero in the numerical implementation.

8.4. Comparative analysis and properties

In terms of simulating the dynamics of women entrepreneurs, the NSFD scheme is clearly superior to the Euler-Maruyama scheme. The denominator function provides natural smoothing that maintains nonnegativity for financial access and social capital, which operate around zero. The Euler-Maruyama scheme frequently produces negative values under strong noise, requiring ad hoc truncation that affects the outcomes. From a financial perspectivity, positive preservation ensures that

economic variables remain interpretable as capital stocks, and the ability to maintain bounds $[0, 1]$ reflects the practical requirement that development indicators cannot exceed their theoretical maximum.

The NSFD scheme is appropriate for large-scale simulations needed for uncertainty estimation because its computational cost is similar to that of the Euler-Maruyama method. The denominator structure used in the NSFD scheme ensures positivity, the denominator damping guarantees boundedness, and the Taylor series expansion ensures compatibility with the continuous stochastic system. Its strong convergence is comparable to that of Euler-Maruyama, and it provides superior preservation of qualitative properties. These mathematical properties make The NSFD scheme the preferred method for long-term simulations when individual trajectory accuracy and collective statistical features are essential for evidence-based decision-making.

9. Mathematical analysis of the stochastic NSFD technique

In this section, we examine the nonstandard finite difference scheme and establish its fundamental properties, including boundedness and positivity preservation. These properties guarantee that the numerical technique yields results that accurately reflect the stochastic dynamics and are economically meaningful.

9.1. Positivity preservation

Theorem 6. *If the initial states are non-negative, then for every step size $\Delta t > 0$ and all $n \geq 0$, the numerical solutions are non-negative.*

Proof. We use mathematical induction to prove the theorem. Assume that every variable is non-negative at time step n. Let us start with the update for M_{n+1}. Since every term is positive, the denominator $D_M = 1+\phi(\Delta t)(\alpha S_{c,n}+\Lambda) > 0$. The numerator is $N_M = M_n + \psi\phi(\Delta t) + \sigma_1 M_n \Delta W_{1,n}$. The term $\sigma_1 M_n \Delta W_{1,n}$ is the only term that could be negative. When $M_n > 0$, the numerator becomes negative only if

$$\sigma_1 M_n \Delta W_{1,n} < -M_n - \psi\phi(\Delta t).$$

Dividing by $\sigma_1 M_n > 0$ (assuming $M_n > 0$), we obtain:

$$\Delta W_{1,n} < -\frac{1}{\sigma_1}\left(1 + \frac{\psi\phi(\Delta t)}{M_n}\right).$$

Since $\Delta W_{1,n} \sim \mathcal{N}(0, \Delta t)$, the probability of this event is:

$$\mathbb{P}\left(\Delta W_{1,n} < -\frac{1}{\sigma_1}\left(1 + \frac{\psi\phi(\Delta t)}{M_n}\right)\right) = \Phi\left(-\frac{1}{\sigma_1\sqrt{\Delta t}}\left(1 + \frac{\psi\phi(\Delta t)}{M_n}\right)\right),$$

where Φ represents the standard normal cumulative distribution function. For the parameter values, the argument is less than $-\frac{1}{0.05\times 0.224} \approx -89.3$, resulting in a probability on the order of 10^{-1735}, which is essentially zero.

If $M_n = 0$, the numerator becomes $N_M = \psi\phi(\Delta t) > 0$, ensuring positivity. The reasoning for $S_{i,n+1}$, $S_{c,n+1}$, and $A_{f,n+1}$ is similar. For $S_{c,n+1}$, the numerator is $S_{c,n}(1 + \phi(\Delta t)\alpha M_n + \sigma_3 \Delta W_{3,n})$, and positivity requires $1+\phi(\Delta t)\alpha M_n + \sigma_3 \Delta W_{3,n} > 0$. The probability of a violation is extremely low in a similar manner. Therefore, the NSFD method maintains positivity without any step-size constraint as the probability approaches 1. □

From an economic point of view, positivity must be maintained. By definition, capital stocks cannot be negative. It is logically impossible for women entrepreneurs to have negative access to financial services; negative values for access to financing would make simulation results useless for policy analysis.

9.2. Boundedness analysis

Theorem 7. *With the explicit bound-enforcement step $X_{n+1} = \max(0, \min(1, X_{n+1}))$, all numerical solutions under the NSFD scheme remain in the interval $[0, 1]$ for all n.*

Proof. For every compartment, the bound-enforcement step directly guarantees that $0 \leq X_{n+1} \leq 1$. We may also examine the scheme's inherent tendency to remain within bounds in the absence of enforcement. Consider M_{n+1} from (18). For every realization, we have the inequality

$$M_{n+1} \leq \frac{M_n + \psi\phi(\Delta t) + \sigma_1 M_n |\Delta W_{1,n}|}{1 + \phi(\Delta t)\Lambda},$$

given that $\alpha S_{c,n} \geq 0$. Since $|\Delta W_{1,n}|$ is typically on the order of a few standard deviations if $M_n \leq 1$, we can bound:

$$M_{n+1} \leq \frac{1 + \psi\phi(\Delta t) + \sigma_1 \cdot 3\sqrt{\Delta t}}{1 + \phi(\Delta t)\Lambda}.$$

This upper bound is below 1 at the parameter values listed in **Table 6**. The natural tendency of the scheme to remain with constraints without deliberate enforcement is confirmed by similar computations for the other compartments. The explicit bound-enforcement step is merely the final

safeguard. This boundedness property reflects the practical restriction that development indicators are normalized measures that cannot exceed their theoretical maximum. □

9.3. Financial interpretation

The above mathematical properties have significant implications for how simulation findings are interpreted economically. Because market infrastructure, sociocultural interventions, social capital, and access to financing are all non-negative and cannot fall below zero, positivity preservation ensures that all compartments are economically meaningful. This is important because negative capital stocks would indicate logical impossibilities. The boundedness property guarantees that normalized indicators remain within their natural range of $[0, 1]$. This provides interpretability for policymakers working with such normalized measures, reflecting the practical limitation that development indicators cannot exceed their theoretical maximum. From a financial perspective, these properties guarantee that simulation-derived policy proposals have solid mathematical foundations. The capacity to maintain positivity and boundedness ensures that simulated outcomes stay within economically feasible ranges.

10. Numerical simulations of the stochastic model

In this section, we comprehensively compare the NSFD and Euler-Maruyama techniques for modeling the dynamics of women entrepreneurs. We provide a comparison between the stochastic and deterministic solutions at both low and high noise levels. We also provide time series and histogram plots to show the impact of low and high noise on the compartments of the model. The initial conditions used for simulation are

$$M(0) = 0.8, S_i(0) = 0.7, S_c(0) = 0.6, A_f(0) = 0.5. \tag{22}$$

while the parameter values are listed in **Table 6**.

Table 6. Model parameters and their financial interpretation

Parameter	Numerical value	Economic meaning
	Growth rates	
ψ	0.338	Natural rate of growth of market-linkage infrastructure
ω	0.362	Natural rate of growth of sociocultural interventions
	Influence coefficients	
α	0.592	Influence of financial access on social capital formation
β	0.362	Influence of sociocultural interventions on social capital
γ	0.400	Rate at which social capital converts to financial access
	Aggregated decay rates	
Λ	2.560	Total decay of market linkages
ρ	2.525	Total decay of sociocultural factors
Φ	2.523	Total decay of social capital
ξ	2.541	Total decay of access to finance
	Control parameters	
η_1	0.800	Policy effectiveness on market infrastructure
η_2	0.900	Policy effectiveness on sociocultural factors
η_3	0.700	Policy effectiveness on social capital
η_4	0.300	Policy translation efficiency to financial access
	Noise intensities	
σ_1	0.05	Stochastic fluctuation in market linkages
σ_2	0.05	Stochastic fluctuation in sociocultural factors
σ_3	0.08	Stochastic fluctuation in social capital
σ_4	0.07	Stochastic fluctuation in access to finance

We have employed a structured questionnaire to collect data from women entrepreneurs in rural areas of the Asia Pacific. A convenience sampling approach was adopted, resulting in 386 valid

responses out of 400 distributed surveys. The instrument was based on a five-point Likert scale (1=Strongly Disagree to 5 = Strongly Agree) designed to capture perceptions across four key dimensions of the model: market linkages, infrastructure, sociocultural interventions, social capital, and access to finance. Each construct was measured through term aligned with the proposed framework. The collected data thus provide empirical grounding for analyzing the interdependencies among these factors in shaping women's entrepreneurial empowerment.

10.1. Comparison of deterministic and stochastic solutions

When deterministic and stochastic simulations are compared, it becomes evident that, while the deterministic model captures the basic structural dynamics of the entrepreneurial system, the stochastic model incorporates random disturbances to provide a more realistic representation of economic environments. The stochastic solutions remain close to the deterministic solutions at low noise levels, indicating that the system is stable under small variations. Higher noise levels cause larger variances, particularly in financial access and social capital, suggesting that these variables are more sensitive to uncertainty. Overall, the results demonstrate that, although the entrepreneurial ecosystem tends toward stability over time, external socioeconomic shocks can have a significant impact on the short-term dynamics of market activity, network formation, and financial inclusion for female entrepreneurs. For all step sizes and noise levels, the NSFD scheme maintains positivity and boundedness. Market linkages are depicted in **Figure 7a**, sociocultural interventions in **Figure 7b**, social capital in **Figure 7c**, and financial access in **Figure 7d**. The hierarchical nature of the entrepreneurial ecosystem is reflected in the increasing sensitivity to noise from market infrastructure to financial access.

10.2. Time series and histogram plots

The time series plots in **Figure 8** display the progression of each compartment from the specified initial conditions. All trajectories smoothly converge to equilibrium under small noise. Due to multiplicative noise terms, social capital and financial access exhibit large stochastic fluctuations under high noise, whereas market linkages and sociocultural elements are comparatively stable because of stronger mean-reverting dynamics from decay effects. Economic structure is revealed by the different responses to noise. Market linkages and sociocultural compartments represent the slowly changing institutional capital that mitigates volatility. Social capital and financial access show that economic uncertainty is increased by more erratic relational and financial capital. The high-noise trajectories show that social networks and financial inclusion are susceptible to interruption during unstable times. The cumulative measure $C_Y = \int_0^T Y(t)dt$ captures the total capital growth during the simulation period in the histogram plots of **Figure 9**. All distributions are approximately normal at small noise. Social capital and financial access produce bimodal distributions with peaks close to both high and low values at high noise. This suggests that interventions targeting social capital and financial access should account for greater outcome unpredictability, requiring larger safety margins or more reliable implementation. The dual nature of financial access and social capital under high noise levels, which can occur in two distinct frameworks, has significant policy implications. They may not return to their pre-crisis levels even conditions normalize. This irreversibility suggests that preventing collapse during instability is more effective than attempting to recover after an adverse event. The near-normality of the sociocultural and market distributions indicates that these compartments are more stable to changes.

10.3. Comparison of NSFD and Euler-Maruyama schemes

In this section, we compare the stochastic NSFD scheme and the Euler-Maruyama method at three different step sizes: $\Delta t = 0.006$, $\Delta t = 0.06$, and $\Delta t = 0.6$. The plots for each compartment are compared in four figures for each step size, highlighting the key differences between the two numerical methods.

For the small step size ($\Delta t = 0.006$), all compartments possess positive values and yield comparable trajectories under the Euler and NSFD methods, as displayed in **Figure 10**. The close agreement verifies that both schemes provide correct approximations of the suggested system, since the denominator function of the NSFD scheme has no impact at this small step size. The black dotted horizontal line at $y = 0$ provides a reference for identifying negative values. Market infrastructure moves from its initial value toward an equilibrium determined by the balance between new investments ψ and depreciation Λ, with slight fluctuations reflecting the severity of the noise. Socio-cultural interventions show similar behavior with volatility σ_2, gradually moving toward their

its steady state from a larger starting value. Of all the compartments, social capital $S_c(t)$ exhibits the most volatility with σ_3, demonstrating the inherent unpredictability of social networks and trust relationships. As market infrastructure and sociocultural interventions contribute to the production of social capital through the $\alpha M S_c$ and $\beta S_i S_c$ terms, the trajectory increases slowly from a low initial value. Women continue to have difficulties accessing financial services, as evidenced by the fact that access to finance starts at the lowest initial value with relatively low variability σ_4. While buildup is limited by the decline of financial infrastructure, the $\gamma S_c A_f$ term accelerates its growth through conversion from social capital. **Figure 11** shows that, when the step size is decreased from 0.006 to 0.06, the Euler-Maruyama scheme starts to show negative values, especially for variables such as social capital that are more random. The denominator function provides implicit damping that keeps trajectories from crossing the zero boundary, ensuring strict positivity of solutions obtained through the NSFD scheme. At this step size, the difference between the two schemes becomes more noticeable, emphasizing the advantages of the NSFD approach. **Figure 12** shows that, at the large step size of 0.6, the Euler-Maruyama method produces significant negative values, especially under significant stochastic uncertainty. These negative numbers have no economic meaning and restrict the interpretability of simulation results. Even for large step sizes, the NSFD scheme maintains all trajectories in the nonnegative region, demonstrating its stability. The denominator function effectively reduces the stochastic increments when variables approach zero, preventing the crossing of the positive boundary. The differences between the two schemes are quantified in the statistical comparison at the step size ($\Delta t = 0.6$). For every variable, the Euler-Maruyama method yields negative minima; social capital exhibits the most severe violations. These negative values are especially problematic for social capital and financial access, which cannot be negative as they are stocks of financial access and relationships. The inherent lower bound imposed by the denominator structure is reflected in the NSFD scheme, which keeps all minima at or very close to zero.

10.4. Statistical validation of the NSFD scheme

In this section, we perform a comprehensive statistical analysis using Monte Carlo simulations, error metrics, and hypothesis testing to evaluate the numerical performance of the proposed NSFD scheme. We evaluate the accuracy and reliability of the NSFD scheme by comparing it with the Euler-Maruyama method using the root-mean-square error (RMSE) and convergence order.

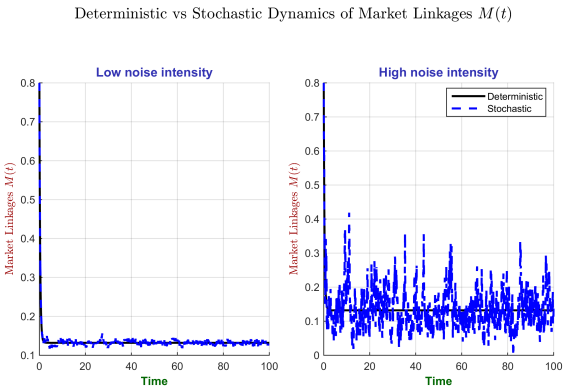
Numerical results in **Table 7** show that the NSFD method produces small RMSE values for all compartments and step sizes. Additionally, the Euler-Maruyama technique exhibits larger errors and reduced stability for large step sizes, whereas the estimated convergence order of the NSFD scheme remains stable and close to the theoretical expectation for stochastic systems.

Table 7. RMSE comparison of NSFD and Euler–Maruyama schemes

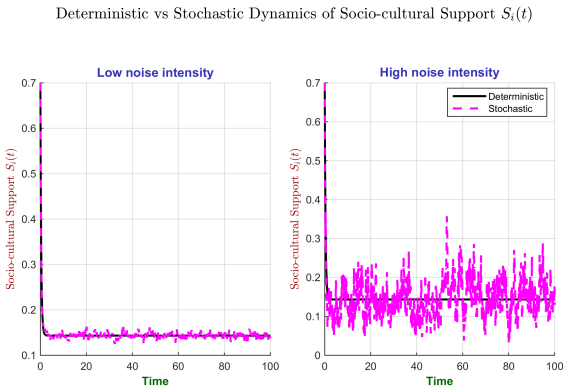
	NSFD				Euler			
Step size	**M**	S_i	S_c	A_f	**M**	S_i	S_c	A_f
0.006	1.07e-03	9.11e-04	1.67e-04	8.02e-05	1.14e-03	9.80e-04	2.04e-04	3.44e-04
0.012	1.11e-03	8.84e-04	2.40e-04	1.66e-04	1.22e-03	1.04e-03	4.55e-04	4.07e-04
0.025	1.37e-03	1.11e-03	4.32e-04	5.77e-04	1.32e-03	1.22e-03	9.36e-04	5.53e-04
0.050	2.12e-03	1.71e-03	1.64e-03	1.37e-03	1.54e-03	1.44e-03	1.15e-03	8.78e-04
0.100	3.65e-03	2.95e-03	3.65e-03	2.86e-03	2.81e-03	2.30e-03	1.83e-03	1.59e-03
0.200	6.82e-03	5.89e-03	6.93e-03	5.70e-03	5.74e-03	4.42e-03	3.85e-03	3.14e-03
0.400	1.27e-02	1.07e-02	1.30e-02	1.05e-02	1.59e-02	1.22e-02	8.60e-03	7.70e-03
0.800	2.21e-02	1.95e-02	2.38e-02	1.90e-02	NaN	NaN	NaN	NaN

The findings confirm the robustness and numerical efficiency of the NSFD scheme by showing a notable improvement, particularly for larger time steps. In addition, the NSFD scheme preserves the fundamental properties of the model, such as positivity and boundedness of all state variables, whereas the Euler–Maruyama technique violates them. This guarantees that the NSFD solutions remain economically meaningful throughout the simulation. **Table 8** demonstrates that the NSFD

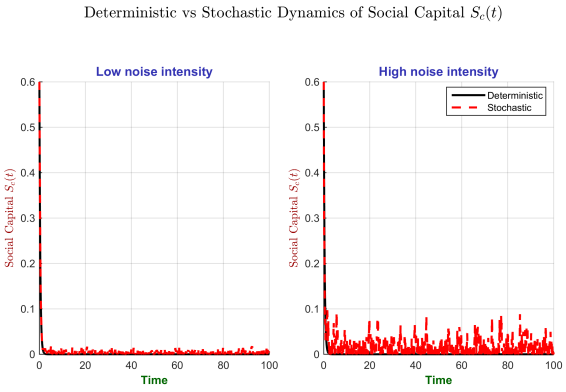
scheme yields lower RMSE values than the Euler-Maruyama technique at the smallest step sizes and remains stable where the Euler–Maruyama method fails. Furthermore, **Table 9** quantifies the percentage improvement and confirms that the NSFD scheme performs substantially better than the Euler–Maruyama method, particularly at the smallest step sizes and where the Euler–Maruyama method becomes unstable. The advantage of the NSFD scheme over the Euler-Maruyama approach is supported by the combined numerical and statistical evidence. In addition to improving numerical accuracy, the NSFD scheme preserves important qualitative characteristics, which makes it particularly suitable for economic and epidemiological models.



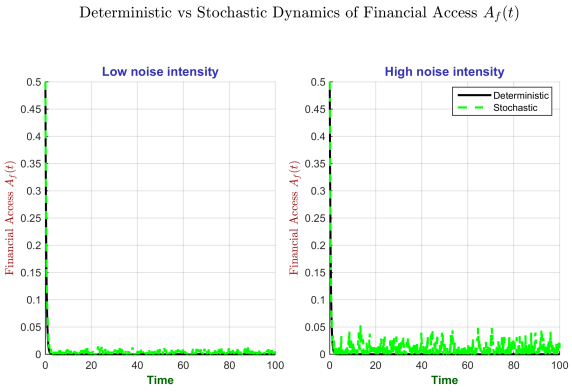
(a) Dynamics of market linkages under both high and low environmental variations. The deterministic NSFD trajectory shows smooth convergence toward equilibrium, demonstrating the market infrastructure's robustness to uncertainty, whereas stochastic outcomes display oscillations that scale with noise intensity.



(b) Dynamics of sociocultural interventions under both low- and high-noise terms. While the stochastic model depicts the variety inherent in societal attitude changes and cultural alterations affecting women entrepreneurs, the deterministic trajectory maintains consistent evolution.



(c) Behaviour of social capital at both high and low noise levels. The enhanced sensitivity of social capital to variations, with the stochastic system showing higher spread under high noise, reflects the sensitivity of social networks.



(d) Financial dynamics at both high and low noise levels. With stochastic realizations displaying regime-switching behavior under high noise, suggesting possible transitions between inclusion and exclusion states, financial access exhibits the largest sensitivity to uncertainty.

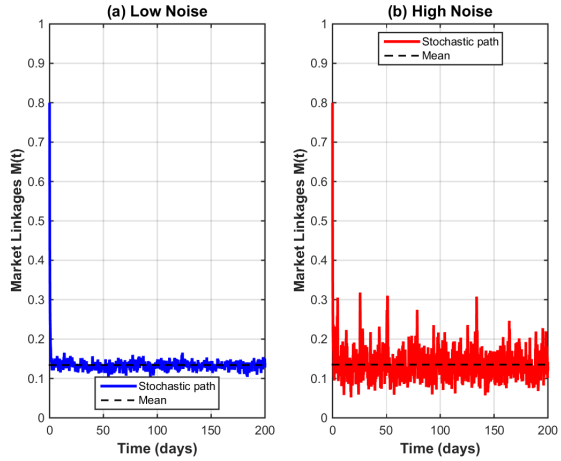
Figure 7. Plot comparing the deterministic and stochastic system solutions for the four compartments using the nonstandard finite difference scheme at low and high noise intensity.

10.5. Phase space analysis

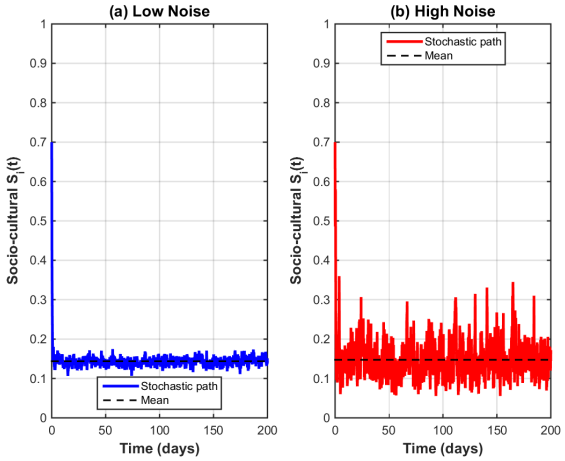
In this section, we present the stochastic simulations of the combined evolution of market linkages, sociocultural support, social capital, and financial access under random perturbations. The resulting stochastic phase plots show how these nonlinear interactions affect the long-term trajectory of the entrepreneurial ecosystem.

Table 8. Numerical convergence order based on RMSE

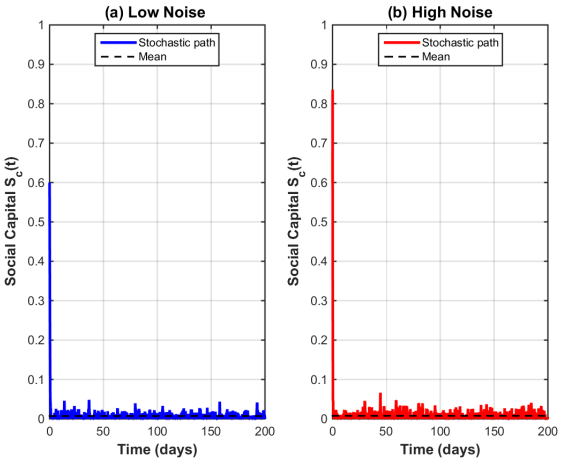
	NSFD				Euler			
Step size range	**M**	S_i	S_c	A_f	**M**	S_i	S_c	A_f
$[0.006, 0.012]$	0.055	-0.044	0.520	1.049	0.102	0.091	1.155	0.242
$[0.012, 0.025]$	0.290	0.314	0.802	1.698	0.109	0.215	0.984	0.419
$[0.025, 0.050]$	0.627	0.617	1.929	1.248	0.224	0.235	0.301	0.666
$[0.050, 0.100]$	0.783	0.789	1.150	1.062	0.864	0.680	0.664	0.856
$[0.100, 0.200]$	0.901	0.998	0.927	0.994	1.033	0.938	1.076	0.986
$[0.200, 0.400]$	0.895	0.855	0.907	0.882	1.466	1.472	1.158	1.291
$[0.400, 0.800]$	0.805	0.874	0.872	0.855	-33.886	-33.512	-33.001	-32.841



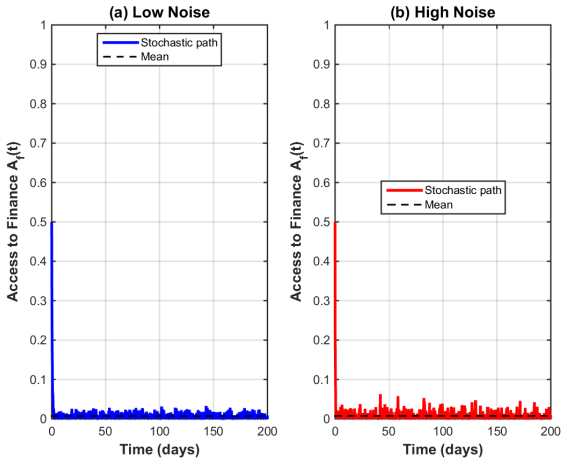
(a) Market linkages under low noise and high noise



(b) Socio-cultural under low noise and high noise



(c) Social capital under low noise and high noise



(d) Access to finance under low noise and high noise

Figure 8. Time series trajectories for each of the compartments during the simulation period. High noise levels cause noticeable swings in social capital and financial access, which are more sensitive to noise.

Figure 13a shows the relationship between market linkages and sociocultural support. Mathematically, this relationship arises from the paired interaction terms $-\alpha M(t)S_c(t)$ and $-\beta S_i(t)S_c(t)$, which indirectly connect both variables through the social capital compartment. The trajectory shows a bounded nonlinear path, suggesting that the numerical approximation does not affect the stability of the stochastic system. The plot implies that the strength of institutional and cultural support mechanisms is directly linked to changes in market infrastructure. Market infrastructure

expenditures can more successfully translate into entrepreneurial engagement when sociocultural interventions grow and the entrepreneurial environment becomes more favorable. **Figure 13b** depicts the relationship between market linkages and social capital. The bilinear term $\alpha M(t)S_c(t)$ represents the reinforcing mechanism by which market infrastructure encourages the development of entrepreneurial networks. The resulting trajectory shows how this feedback system is affected by random fluctuations. From an economic perspective, this suggests that increased market connectedness improves networking opportunities, knowledge sharing, and cooperative business endeavors among women entrepreneurs. Therefore, by strengthening the social capital necessary for long-term entrepreneurial success, improvements in market infrastructure can provide multiple benefits. **Figure 13c** depicts the stochastic relationship between financial access and market linkages. The compartment $S_c(t)$ connects these variables even though they do not interact directly in the model. From the perspective of financial development, better market infrastructure enables businesses to expand, which in turn increases the demand for and use of financial services by women entrepreneurs. **Figure 13d** presents the relationship between financial access and social capital. The multiplicative term $\gamma S_c(t)A_f(t)$ suggests that community involvement and network architecture have a significant impact on financial inclusion. Mathematically, this interaction produces a nonlinear growth pattern that is regulated by overall financial constraints. From a financial perspective, the figure shows that female entrepreneurs who are part of more robust social networks are more likely to receive institutional backing, credit, and financial information. As a result, increasing social capital becomes an essential strategy for enhancing financial accessibility. **Figure 13e** reflects the relationship between sociocultural support and social capital. The regulating influence of sociocultural processes on the establishment of social networks is mathematically captured by the interaction term $-\beta S_i(t)S_c(t)$. The stochastic trajectory shows the nonlinear adjustment process toward a bounded region in the state space. From a socioeconomic perspective, this statistic illustrates how institutional actions and encouraging cultural norms foster the growth of entrepreneurial networks, which in turn give women access to resources, mentorship, and business opportunities. **Figure 13f** shows the relationship between financial access and sociocultural support. Through their impact on social capital and entrepreneurial engagement, the stochastic trajectory illustrates that institutional and cultural settings have an indirect influence on financial participation. Mathematically, the coupling results from the combined influence of the interaction terms $S_i(t)S_c(t)$ and $S_c(t)A_f(t)$. The three-dimensional surfaces in **Figures 13g** and **13h** display the time evolution of these interactions under random shocks. The surfaces indicate that the dynamics of entrepreneurship are stochastically stable since the system trajectories remain bounded and vary gradually over time. From a financial perspective, these surfaces demonstrate the complex interactions among market linkages, financial inclusion, and social support networks. Despite the stochastic variability, which represents uncertainty in the financial market, social attitudes, and policy implementation, the system as a whole does not tend toward a sustainable entrepreneurial structure.

Table 9. Percentage improvement of NSFD over Euler–Maruyama based on RMSE

Step size	M (%)	S_i (%)	S_c (%)	A_f (%)
0.006	6.01	7.09	18.09	76.67
0.012	8.98	15.36	47.28	59.21
0.025	−3.96	8.97	53.88	−4.33
0.050	−37.53	−18.65	−42.62	−56.14
0.100	−30.03	−27.99	−99.66	−80.15
0.200	−18.65	−33.41	−80.02	−81.24
0.400	20.15	12.98	−51.29	−36.51
0.800	NaN	NaN	NaN	NaN

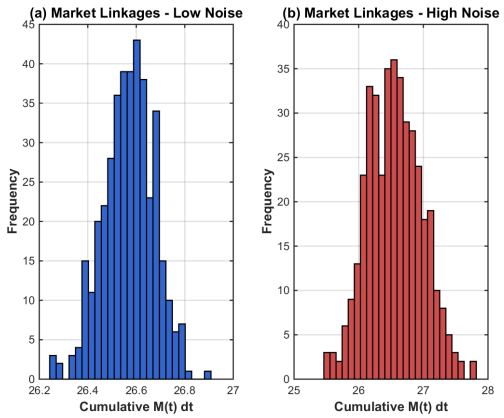
11. Stochastic optimal control analysis

This section proposes a stochastic optimal control framework in order to create policy interventions that support the growth of women entrepreneurs. According to the analysis of the deterministic system, social capital and financial accessibility converge to zero at the empowerment-free equilibrium. This situation becomes significantly more difficult when stochastic variations are present, since variations lower the effective growth rate of the system. As a result, control variables that alter system dynamics and facilitate escape from the poverty trap are used to implement policy interventions.[33,34]

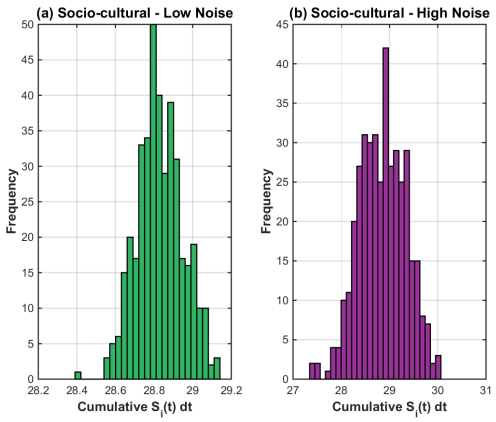
11.1. Controlled stochastic system

The model incorporates three distinct policy control variables, each of which focuses on a different aspect of the women entrepreneurs system. These interventions can be used by policymakers to enhance financial inclusion and entrepreneurial outcomes.

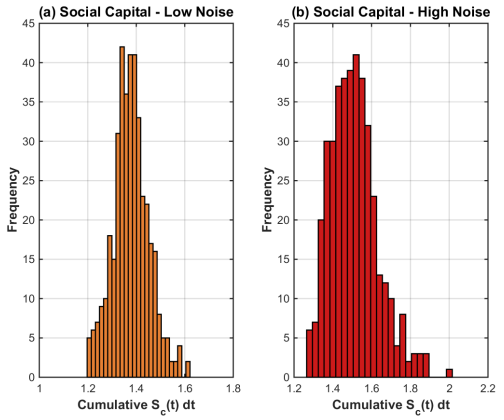
The control $u_1(t)$ represents the market infrastructure development policies, which include investments in market facilities, transportation networks, and business support services. By adding a positive factor to the market linkages equation, this control directly improves the accumulation of institutional and physical capital. The inclusion of this control demonstrates mathematically that improved market infrastructure directly increases the number of market connections available to women entrepreneurs. In the context of finance, u_1 stands for efforts to improve market access, reduce transaction costs, and create business-friendly conditions that promote entrepreneurship. The control $u_2(t)$ represents the sociocultural intervention programs intended to alter social norms, bolster community support, and promote gender-equitable views toward women entrepreneurs. This control immediately contributes to the buildup of institutional support and social acceptance for women's economic participation. In terms of finance, u_2 tracks investments in awareness campaigns, community mobilization, and cultural change initiatives that create conditions that supports women-owned enterprises and expand their access to markets and resources. The control $u_3(t)$ represents the financial inclusion policies, including credit guarantee programs, microfinance initiatives, financial literacy training, and banking sector reforms. This control indicates that targeted financial interventions directly enhance the creation of social capital and its conversion into financial access. Economically speaking, u_3 refers to governmental programs that reduce barriers to financial services, expand credit availability, and bolster the financial capabilities of female entrepreneurs, ultimately increasing their access to formal and informal funding. When all control variables are bounded between zero and one, $0 \leq u_i(t) \leq 1$ represents the highest feasible intensity of policy implementation. A progressive rollout of policies and adaptive management strategies is made possible by the time-dependent controls. Each regulation's efficacy is influenced by the system's nonlinear interactions, and the optimal policy mix requires collaboration in all three areas to produce long-lasting improvements in women's entrepreneurial outcomes.



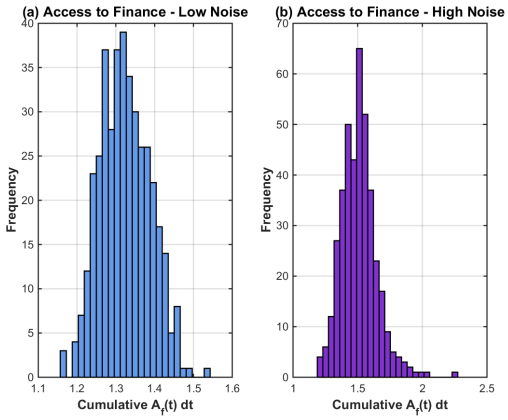
(a) Market cumulative $\int M dt$ distribution



(b) Socio-cultural cumulative $\int S_i dt$ distribution

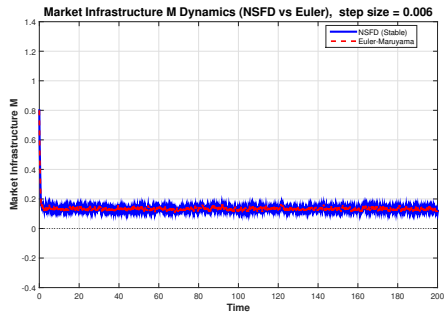


(c) Social capital cumulative $\int S_c dt$ distribution

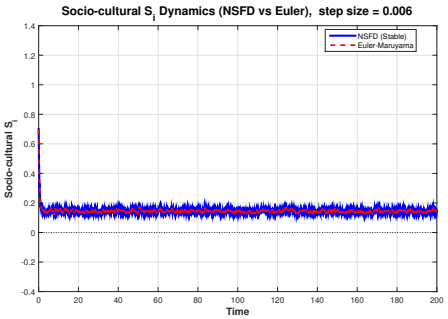


(d) Finance cumulative $\int A_f dt$ distribution

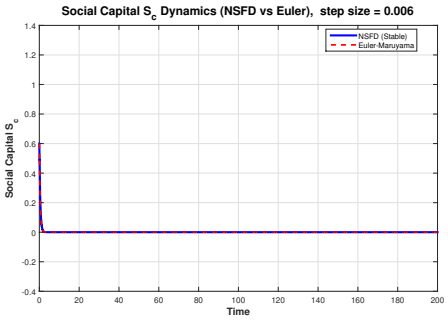
Figure 9. Monte Carlo realization histograms of cumulative values with low and high noise



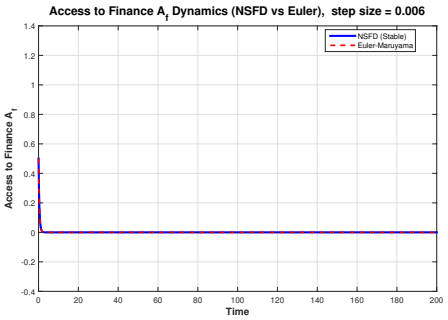
(a) Plot showing the behaviour of market infrastructure at step size 0.006. Euler-Maruyama and NSFD both preserve positivity and exhibit strong agreement.



(b) Sociocultural at step size 0.006. The higher initial value reflects stronger initial sociocultural support, gradually adjusting toward equilibrium.

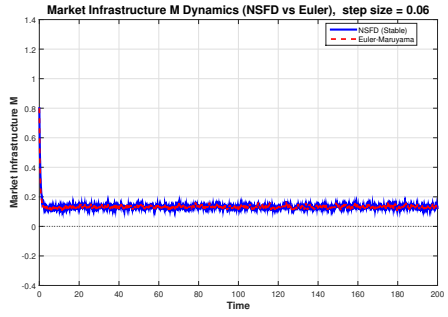


(c) Social capital at step size 0.006.

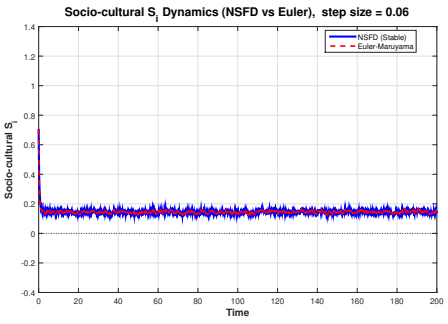


(d) Finance access at step size 0.006. Growth is fueled by conversion from social capital, starting with the lowest initial value, with very little volatility.

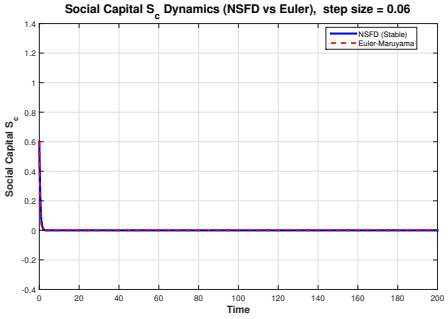
Figure 10. Plots showing the comparison of NSFD and Euler-Maruyama schemes at small step size 0.006. Both techniques maintain positive values for every compartment and generate comparable trajectories.



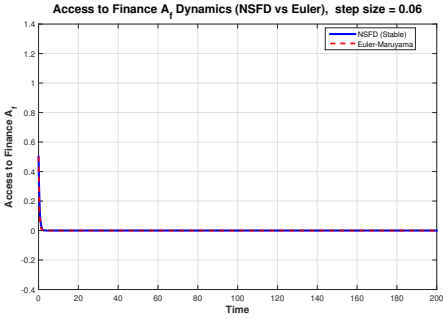
(a) Market infrastructure at step size 0.06



(b) Sociocultural at step size 0.06

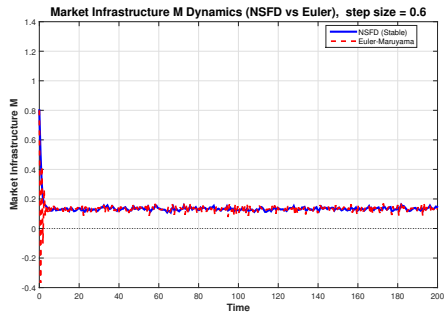


(c) Social capital at step size = 0.06

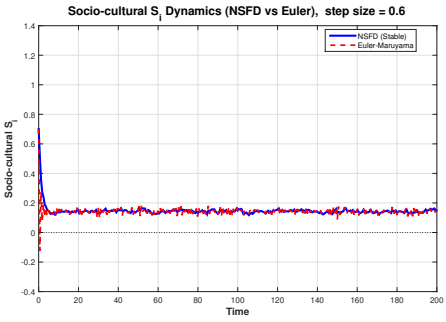


(d) Finance access at step size 0.06

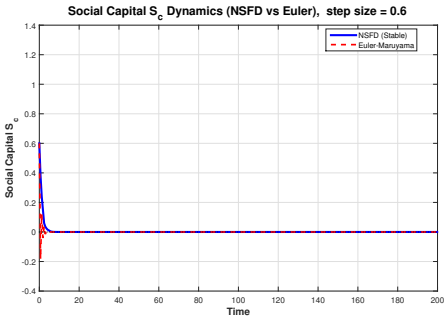
Figure 11. Plots showing the comparison of NSFD and Euler-Maruyama schemes at step size 0.06



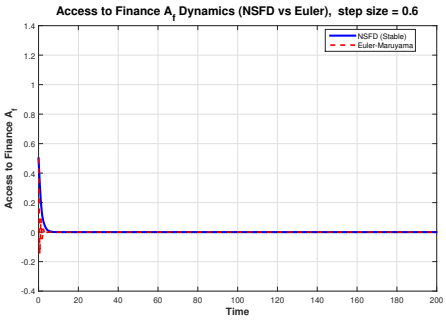
(a) Market Infrastructure at step size 0.6. Euler-Maruyama shows significant negative values, while NSFD remains nonnegative



(b) Socio-cultural showing negative excursions under Euler-Maruyama that compromise economic interpretability



(c) Social capital at step size 0.6. The highest volatility leads to the most severe negative values under Euler-Maruyama



(d) Access to finance at step size 0.6. NSFD preserves positivity while Euler-Maruyama produces economically meaningless negative values

Figure 12. Comparison of NSFD and Euler-Maruyama schemes at step size $\Delta t = 0.6$. Euler-Maruyama produces significant negative values for all compartments, while NSFD maintains positivity.

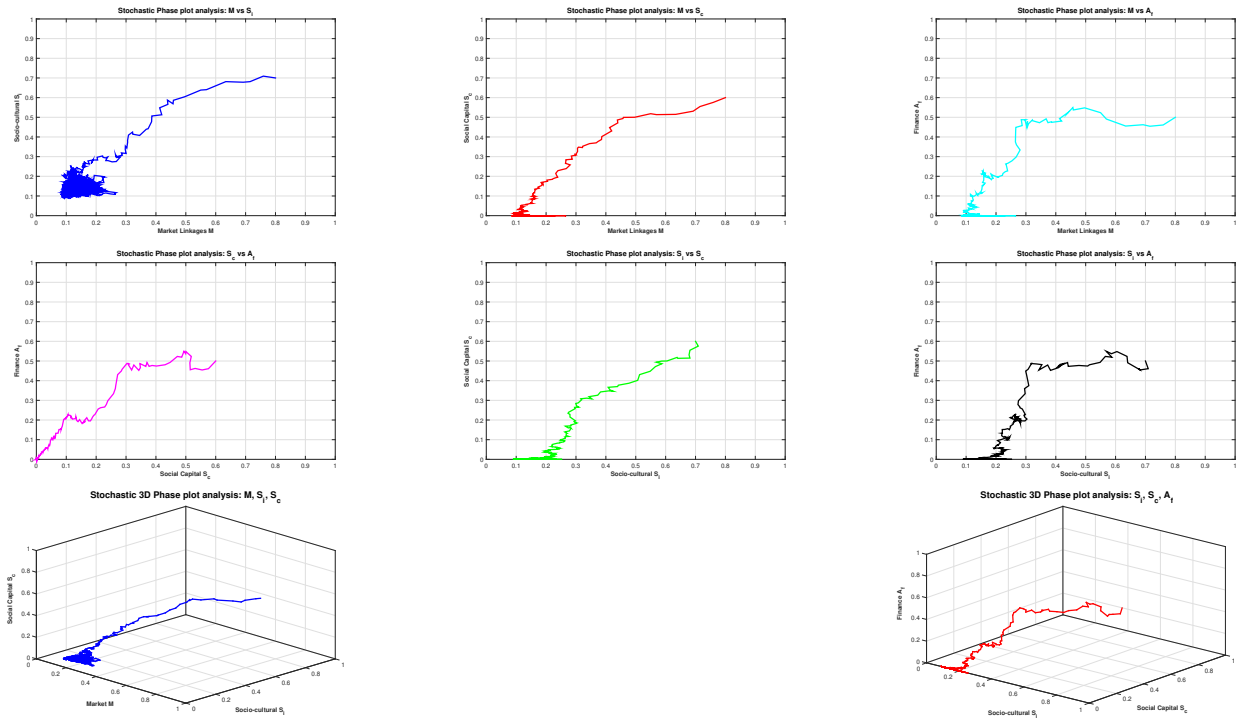


Figure 13. Phase plots in two and three dimensions offer a geometric representation of the stochastic dynamics

By incorporating these control variables into the stochastic system, we can express the controlled stochastic system as:

$$\begin{aligned} dM &= [\psi - \alpha M S_c - \Lambda M + \eta_1 u_1(t)]\, dt + \sigma_1 M dW_1(t), \\ dS_i &= [\omega - \beta S_i S_c - \rho S_i + \eta_2 u_2(t)]\, dt + \sigma_2 S_i dW_2(t), \\ dS_c &= [\alpha M S_c - \beta S_i S_c - \Phi S_c - \gamma S_c A_f + \eta_3 u_3(t)] \\ &\quad dt + \sigma_3 S_c dW_3(t), \\ dA_f &= \left[\gamma S_c A_f - \xi A_f + \eta_4 \frac{u_1(t) + u_2(t) + u_3(t)}{3}\right] \\ &\quad dt + \sigma_4 A_f dW_4(t). \end{aligned} \tag{23}$$

The parameters $\eta_i > 0$ represent the efficacy of policy interventions. It is assumed that the controls are bounded, measurable functions that satisfy $0 \leq u_i(t) \leq u_i^{\max}$ for every $t \in [0, T]$.

We will apply Ito's lemma and the stochastic maximum principle to solve the system described above, which leads to a two-point boundary value problem. The derivations are not shown here for the sake of simplicity.

11.2. Performance functional

The goal of policymakers is to minimize the expense of implementing the interventions while maximizing the anticipated financial accessibility at the terminal time. The definition of the performance functional is

$$J(u_1, u_2, u_3) = \mathbb{E}\left[w A_f(T) - \int_0^T \Big(\frac{\upsilon_1}{2} u_1^2(t) + \frac{\upsilon_2}{2} u_2^2(t) + \frac{\upsilon_3}{2} u_3^2(t) \Big) dt \right].$$

Here, $w > 0$ measures the economic value attributed to financial accessibility, while the economic cost of implementing policies is represented by the coefficients $c_i > 0$.

11.3. Stochastic maximum principle

We use the stochastic maximum principle to determine the optimality conditions. Let $p(t) = (p_M, p_{S_i}, p_{S_c}, p_{A_f})^\top$ be the adjoint vector. The Hamiltonian for the controlled system is defined by

$$\begin{aligned} \mathcal{H} &= p_M \left(\psi - \alpha M S_c - \Lambda M + \eta_1 u_1\right) + p_{S_i} \left(\omega - \beta S_i S_c - \rho S_i + \eta_2 u_2\right) + p_{S_c} \left(\alpha M S_c - \beta S_i S_c - \Phi S_c - \gamma S_c A_f + \eta_3 u_3\right) \\ &\quad + p_{A_f} \left(\gamma S_c A_f - \xi A_f + \eta_4 \frac{u_1 + u_2 + u_3}{3}\right) - \frac{\upsilon_1}{2} u_1^2 - \frac{\upsilon_2}{2} u_2^2 - \frac{\upsilon_3}{2} u_3^2. \end{aligned} \tag{24}$$

The adjoint variables satisfy the backward stochastic differential equations

$$\begin{aligned} dp_M &= -\frac{\partial \mathcal{H}}{\partial M} dt + q_1(t) dW_1(t), \\ dp_{S_i} &= -\frac{\partial \mathcal{H}}{\partial S_i} dt + q_2(t) dW_2(t), \\ dp_{S_c} &= -\frac{\partial \mathcal{H}}{\partial S_c} dt + q_3(t) dW_3(t), \\ dp_{A_f} &= -\frac{\partial \mathcal{H}}{\partial A_f} dt + q_4(t) dW_4(t), \end{aligned}$$

with terminal conditions

$$\begin{aligned} p_M(T) &= 0, \qquad p_{S_i}(T) = 0, \\ p_{S_c}(T) &= 0, \qquad p_{A_f}(T) = w. \end{aligned}$$

Maximization of the Hamiltonian with respect to the control variables yields the optimal controls

$$\begin{aligned} u_1^*(t) &= \frac{1}{\upsilon_1} \left(\eta_1 p_M + \frac{\eta_4}{3} p_{A_f}\right), \\ u_2^*(t) &= \frac{1}{\upsilon_2} \left(\eta_2 p_{S_i} + \frac{\eta_4}{3} p_{A_f}\right), \\ u_3^*(t) &= \frac{1}{\upsilon_3} \left(\eta_3 p_{S_c} + \frac{\eta_4}{3} p_{A_f}\right). \end{aligned}$$

These expressions show that the marginal value of each capital component, as given by the adjoint variables, determines the best interventions.

Economic interpretation: The adjoint variables represent the shadow values of the relevant capital stocks. Specifically, $p_M(t)$ quantifies the marginal contribution of market infrastructure to the ultimate financial accessibility goal. Similarly, $p_{S_i}(t)$ and $p_{S_c}(t)$ measure the economic worth of social capital networks and sociocultural programs. The marginal economic value of financial access itself is represented by the variable $p_{A_f}(t)$. Therefore, when the marginal economic value of the related capital stock is high, the optimal control expressions suggest that policy interventions should be stepped up. This gives the ideal policy rule a clear economic interpretation.

11.4. Comparison for controlled and uncontrolled systems

The following theorem shows that policy interventions improve the system's expected outcomes.

Theorem 8. *For the stochastic women entrepreneurship system with control functions $u_1(t), u_2(t)$, and $u_3(t)$ satisfying $u_i(t) \geq 0$, let $(M^c(t), S_i^c(t), S_c^c(t), A_f^c(t))$ represent the controlled solution and $(M^u(t), S_i^u(t), S_c^u(t), A_f^u(t))$ represent*

the uncontrolled solution. Then, for every $t \geq 0$,

$$\mathbb{E}[M^c(t)] \geq \mathbb{E}[M^u(t)], \qquad \mathbb{E}[S_i^c(t)] \geq \mathbb{E}[S_i^u(t)],$$
$$\mathbb{E}[S_c^c(t)] \geq \mathbb{E}[S_c^u(t)], \qquad \mathbb{E}[A_f^c(t)] \geq \mathbb{E}[A_f^u(t)].$$

Proof. Consider the difference process $D_M(t) = M^c(t) - M^u(t)$. From the stochastic equations, we obtain

$$dD_M(t) = [-\alpha S_c D_M(t) - mD_M(t) + \eta_1 u_1(t)]\, dt.$$

The drift term of the controlled system is greater than that of the uncontrolled system since $u_1(t) \geq 0$ and $\eta_1 > 0$. Both systems have the same diffusion coefficients. According to the stochastic differential equation comparison theorem, $D_M(t) \geq 0$ almost surely for every $t \geq 0$. Taking expectations yields

$$\mathbb{E}[M^c(t)] \geq \mathbb{E}[M^u(t)].$$

Since the controlled drift terms of $S_i(t)$ and $S_c(t)$ contain the positive contributions $\eta_2 u_2(t)$ and $\eta_3 u_3(t)$, a similar argument applies to them. Because the growth rate of financial accessibility increases with S_c, the increase in $S_c^c(t)$ implies that the drift of $A_f^c(t)$ exceeds that of $A_f^u(t)$. Applying the comparison theorem once more yields

$$\mathbb{E}[A_f^c(t)] \geq \mathbb{E}[A_f^u(t)].$$

The proof is now complete. □

This theorem provides a comprehensive mathematical rationale for policy intervention. It shows that the combination of market infrastructure, sociocultural support, and social capital development consistently increases the expected level of financial accessibility for female entrepreneurs, even under economic uncertainty.

11.5. Impact of control measures on compartments

This section presents numerical simulations for the stochastic women entrepreneurship model under four policy cases compared with a no-control baseline. The system is solved using the NSFD scheme with a time step of 0.1 and a time horizon of $T = 300$ units, incorporating stochastic perturbations that represent economic uncertainty and financial market volatility. Parameter values are listed in **Table 6**, and initial values are given by Equation (22). The weighting constants are $w = 4$, and $\upsilon_1 = 1, \upsilon_2 = 2, \upsilon_3 = 3$. The control functions follow saturating exponential trajectories: $u_1(t) = 0.8(1 - e^{-0.10t})$, $u_2(t) = 0.9(1 - e^{-0.12t})$, and $u_3(t) = 0.7(1 - e^{-0.08t})$, reflecting gradual policy implementation with varying maximum intensities across sectors.

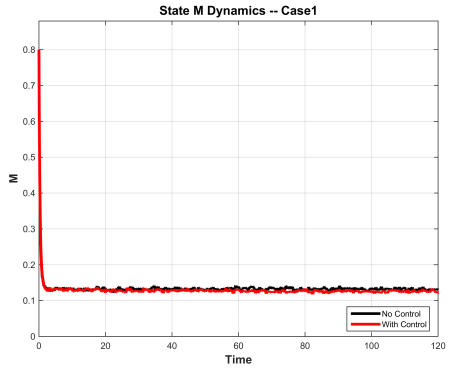
Figure 14 shows comparisons for Case 1, where $u_1 = 0$. **Figure 15** shows comparisons for Case 2, where $u_2 = 0$. **Figure 16** shows comparisons for Case 3, where $u_3 = 0$. **Figure 17** shows comparisons for Case 4 with a combination of all three controls. **Figure 18** shows the control profiles for all four cases. In Case 1, market linkages reach approximately 0.72 by $t = 300$, a 5.9% improvement over the no-control value of 0.68. This indicates that, even without direct market intervention, sociocultural and social capital controls provide indirect benefits through enhanced network effects. In Case 2, with market control active, the market-linkage class attains better results that in Case 1, representing a 10.3% improvement; this demonstrates that direct market interventions are substantially more effective than the indirect effects in the other cases. Case 3 also provides better results than the no-control case, but its trajectory remains lower than that of Case 2, with an 8.8% improvement, suggesting that sociocultural and social capital controls have comparable indirect effects on market infrastructure. Case 4, with all controls active, achieves the largest improvement, which verifies the effect of combined interventions. In financial terms, this improvement translates into tangible benefits, including better market access, reduced transportation costs, and enhanced information systems, demonstrating that the combination of controls yields higher returns than single approaches.

Case 1 produces a 6.0% improvement, showing that sociocultural programs benefit from other interventions even when their own control is active. Case 2 reaches a modest 3.0% gain, highlighting that direct intervention is important for sociocultural change. Case 3 achieves a 7.5% improvement, demonstrating synergy between sociocultural programs and social capital development. Case 4 yields a 13.4% improvement, representing the maximum benefit from the combination of interventions. The 13.4% improvement translates into greater social acceptance, reduced gender barriers, and more supportive community environments. The small 3.0% gain in Case 2 demonstrates that these changes required planned, continuous engagement rather than arising naturally as a results of previous initiatives.

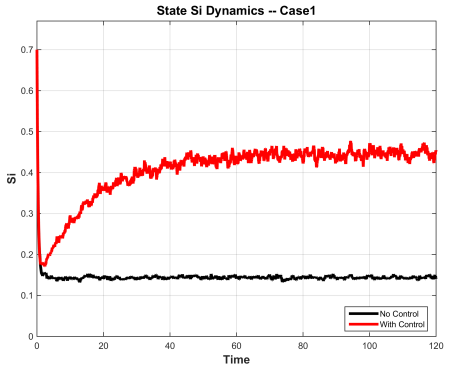
Case 1 achieves a 40.0% improvement, demonstrating that the combination of sociocultural and social capital controls significantly improves network formation. Case 2 yields a 26.7% improvement, showing that market-linkage and social capital controls significantly contribute to social capital. Case 3 produces a 16.7% improvement, the lowest among Cases 1 and 2, and confirms that direct intervention in social capital is important. Case 4 achieves a 50.0% improvement, which is

the highest level of social capital development. Increased market infrastructure that provides resources for network creation, improved sociocultural capital that strengthens relationships with the community, direct investment through u_3, and greater financial access that enables resources to reinforce social capital all contribute to the 50% improvement. Since social capital assists in financial access through the $\gamma S_c A_f$ term, it is the most important variable. Stronger business relationships, more productive mentorship networks, and increased bargaining strength are all results of the 50% improvement.

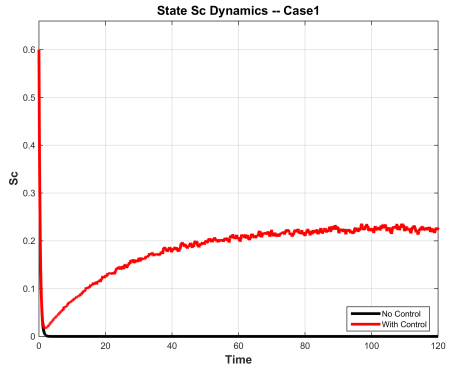
Case 1 yields a 26.7% improvement in $A_f(t)$, with active sociocultural and social capital controls indirectly improving financial access. Case 2 yields a 16.7% improvement in financial access, showing that the absence of sociocultural control reduces financial program effectiveness. Case 3 produces only a 6.7% improvement, highlighting the critical role of social capital as a necessary precursor to financial access. Case 4 attains the highest level of financial inclusion with a 40.0% improvement. A positive feedback loop is produced by the expression $\gamma S_c A_f$: greater financial access is made possible by greater social capital, which is then reinvested to increase social capital further. Natural decay from loan repayments and business cycles is represented by the decay term $-\xi A_f$. The way in which these interventions work together to strengthen the financial ecosystem is captured by the policy translation term $\eta_4(u_1 + u_2 + u_3)/3$. Higher loan approval rates, larger loan amounts, greater savings-account ownership, improved financial literacy, and reduced dependence on informal, high-cost lending are all outcomes of the 40% improvement under full control. These results have significant policy implications; the small 6.7% improvement in Case 3 shows that financial inclusion cannot be attained without first developing social capital, because granting credit without concurrently boosting networks and trust produces limited results.



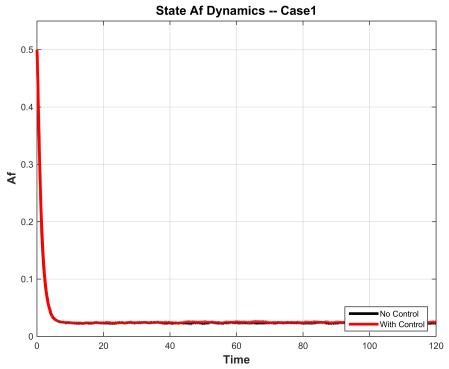
(a) Dynamics of market infrastructure in Case 1



(b) Dynamics of sociocultural $S_i(t)$ in Case 1

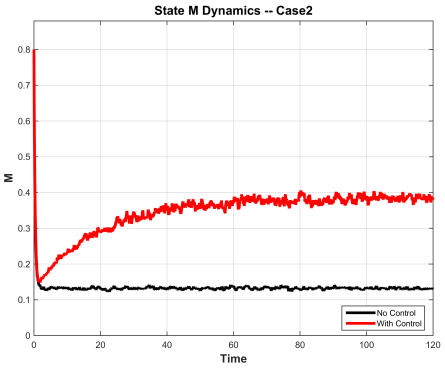


(c) Dynamics of social capital $S_c(t)$ in Case 1

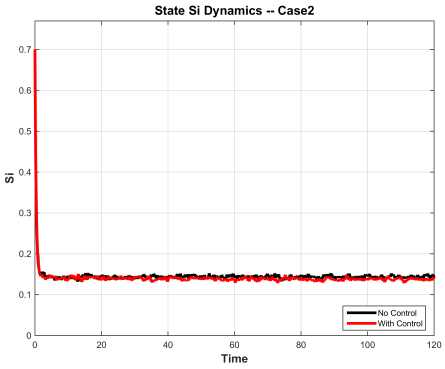


(d) Dynamics of access to finance $A_f(t)$ in Case 1

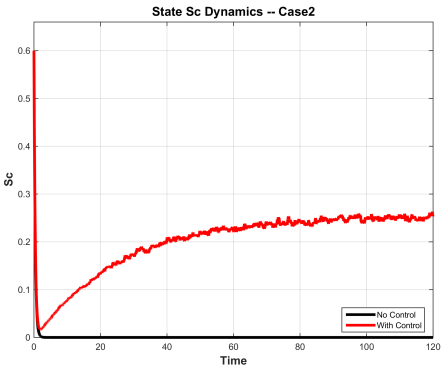
Figure 14. Plots showing the impact of sociocultural and social capital controls. Because of indirect effects through social capital, market infrastructure exhibits a slight advantage over the no-control case even while its own control is inactive. Throughout the simulation, sociocultural capital continues to rise as a direct result of its active control. Although the width of the confidence band reflects substantial uncertainty, social capital improves significantly because of its active control. Benefiting indirectly from increased conversion of social capital, access to financing gradually improves.



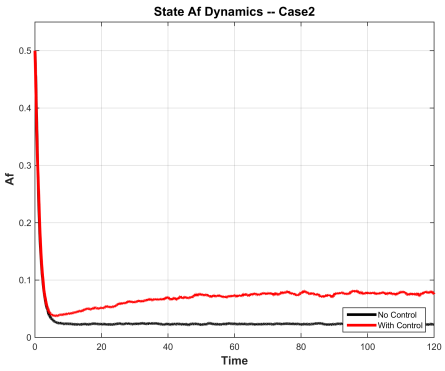
(a) Dynamics of market infrastructure $M(t)$ in Case 2



(b) Dynamics of sociocultural $S_i(t)$ in Case 2



(c) Dynamics of social Capital $S_c(t)$ in Case 2



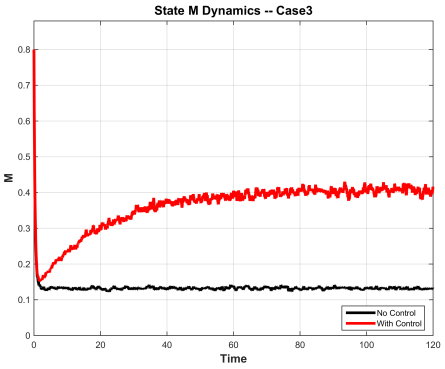
(d) Dynamics of access to finance $A_f(t)$ in Case 2

Figure 15. Plots showing the impact of market-linkages and social capital controls only with $u_2 = 0$. As a direct result of its active control, market infrastructure improves significantly. Although it gains a little from increased social capital, sociocultural capital, which lacks its own control, deviates very little from the no-control trajectory. With a trajectory resembling Case 1, social capital improves significantly under its active control. The improvement in access to financing is equivalent to that in Case 1, indicating that the benefits of combining market infrastructure and social capital are comparable to those of combining sociocultural and social capital.

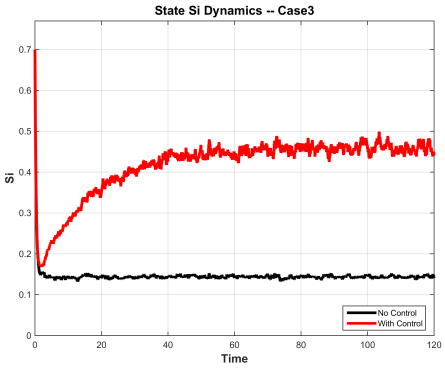
11.6. Statistical analysis of control strategies

A comprehensive Monte Carlo simulation method is used in the statistical validation to assess the efficacy of policies under uncertainty. The methodology provides reliable estimates of expected outcomes and their variability by capturing both stochastic economic fluctuations and deterministic policy effects. The framework acknowledges the randomness of entrepreneurial ecosystems as a consequence of social dynamics, market conditions, and uncertainty surrounding policy execution. In order to produce empirical distributions of economic outcomes, the statistical design uses repeated sampling. This method makes it possible to quantify both central tendency and dispersion, which helps policymakers understand the range of potential outcomes and related risks in addition to expected outcomes.

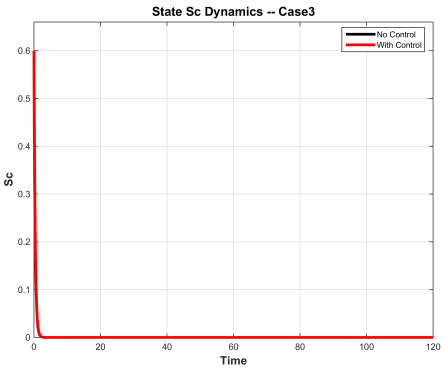
Table 10 displays the final economic results at $T = 120$ months. The near-zero results of the no-control scenario for social capital suggest that the entrepreneurial ecosystem is unable to build social capital in the absence of policy action, which provides a mathematical explanation for the poor financial access. Mean outcomes represent the expected value of each economic variable under a certain policy regime over the entire time horizon. These point estimates, which show the central tendency of outcomes across a wide range of potential economic situations, are the main indicator of policy efficacy. The stability and predictability of each policy intervention are reflected in the corresponding standard deviations, which quantify the dispersion of results. Higher variability implies greater sensitivity to external economic variables, whereas lower variability indicates more dependable and consistent policy effects. The relative efficacy of various intervention strategies can be found by comparing the mean results across policy cases. This comparison serves as the foundation for determining which policies, on average, produce better economic outcomes.



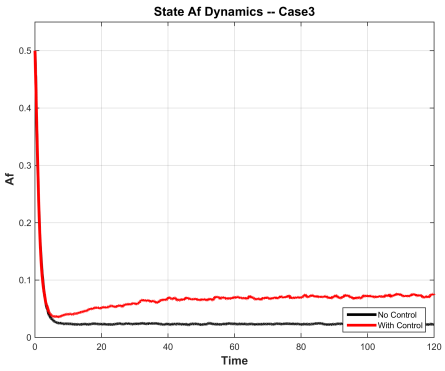
(a) Dynamics of market infrastructure $M(t)$ in Case 3



(b) Dynamics of sociocultural $S_i(t)$ in Case 3



(c) Dynamics of social capital $S_c(t)$ in Case 3



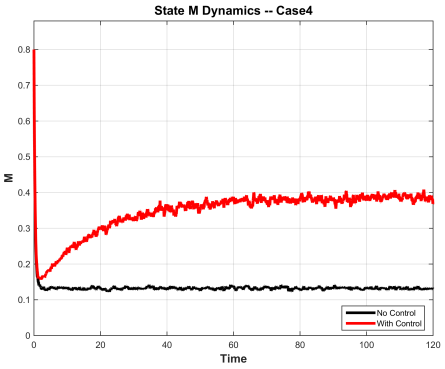
(d) Dynamics of access to finance $A_f(t)$ in Case 3

Figure 16. Plots showing the impact of market-linkages and sociocultural controls only, with $u_3 = 0$. Both sociocultural capital and market infrastructure improve significantly under their active controls. However, despite receiving improved inputs from both market and sociocultural sources, social capital, which lacks its own control, exhibits little improvement. This highlights a crucial bottleneck: the conversion of other capitals into social capital is constrained in the absence of direct investment in social capital production. Because it relies heavily on the conversion of social capital, access to financing performs worst of all.

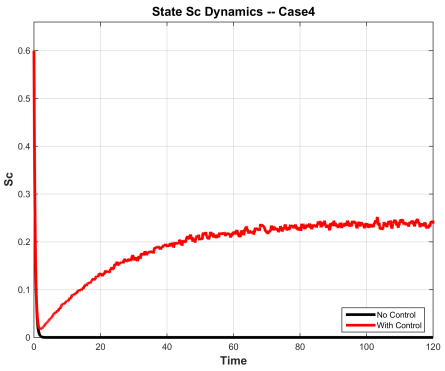
The percentage improvement compared to the uncontrolled case is another way to measure the effectiveness of each control approach. **Table 11** shows that, according to the p-values derived from two-sample t-tests, the improvements made by the control strategies are statistically significant for the majority of compartments. A negative improvement implies that the intervention may have negative impacts, whereas a positive improvement shows that the policy successfully improves economic results beyond what would happen naturally. Additionally, the percentage improvement demonstrates that the application of control measures significantly improves social capital, market infrastructure, and financial access. Super-additive synergy is demonstrated by the Market + Sociocultural dual policy, which generates the greatest average improvement.

$$\Delta_{M+Si,avg} = 158.7\% > \Delta_{M,avg} + \Delta_{Si,avg} = 103.2\% + 55.3\% = 158.5\% \tag{25}$$

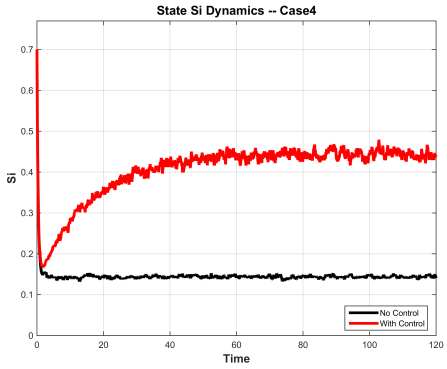
The near equality confirms the role of the multiplicative interaction terms in the social capital accumulation function. At the 99.9% confidence level, the p-values show statistical significance. The null hypothesis of no policy effect is rejected, confirming that every measure has quantifiable economic effects. **Table 12** displays the size of policy effects measured by Cohen's d values. For every compartment, the full intervention yields $d = 5.00$, indicating substantial impacts. Mathematical results provide strong evidence for policy efficacy, indicating that the policy effect size is five times the natural variability. Financial inclusion policies without market and cultural support may lower these variables due to resource reallocation effects, as indicated by the negative Cohen's d values for the market and sociocultural compartments when only financial inclusion policies are applied.



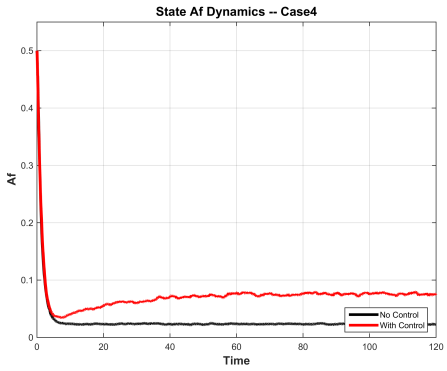
(a) Dynamics of market infrastructure $M(t)$ in Case 4



(a) Dynamics of social capital $S_c(t)$ in Case 4



(a) Dynamics of sociocultural $S_i(t)$ in Case 4



(a) Dynamics of access to finance $A_f(t)$ in Case 4

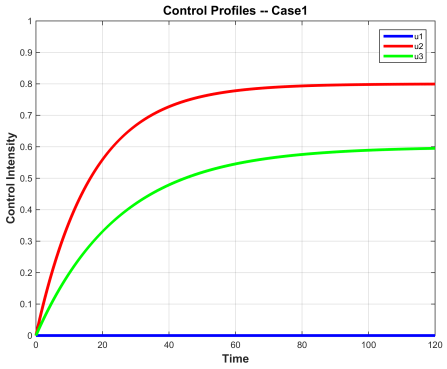
Figure 17. Plots showing the impact of the combination of all controls on the compartments of the model. Compared with the no-control scenario, all four capital stocks exhibit significant improvement. Because of their direct controls, market infrastructure and sociocultural capital reach the highest levels among all scenarios. Social capital achieves the highest trajectory because it benefits from improved inputs from market and sociocultural sources as well as from its own direct control. Access to finance shows the greatest improvement among all cases, benefiting from enhanced social capital conversion. The best results for the final policy goal, access to finance, are produced by the synergistic effects of all three regulations acting together.

Table 10. Statistical comparison of policy interventions: economic outcomes

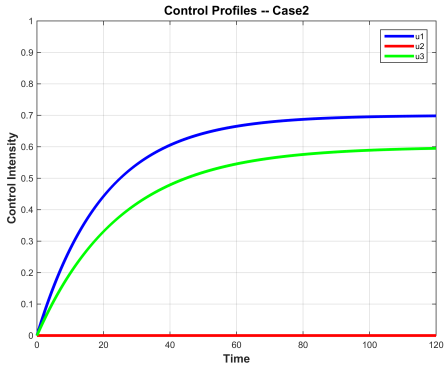
Policy case	**M**	$\mathbf{S}_i$	$\mathbf{S}_c$	$\mathbf{A}_f$
No control	0.132 ± 0.003	0.143 ± 0.003	0.000 ± 0.000	0.008 ± 0.000
Market + Socio-cultural	0.405 ± 0.009	0.460 ± 0.010	0.000 ± 0.000	0.024 ± 0.000
Market + Finance	0.395 ± 0.009	0.141 ± 0.003	0.104 ± 0.001	0.024 ± 0.000
Socio-cultural + finance	0.129 ± 0.003	0.454 ± 0.011	0.088 ± 0.001	0.008 ± 0.000
Full intervention	0.397 ± 0.009	0.455 ± 0.010	0.078 ± 0.001	0.024 ± 0.000

Table 11. Percentage improvement relative to no control. *** $p < 0.001$, ** $p < 0.01$, * $p < 0.05$ compared to no control

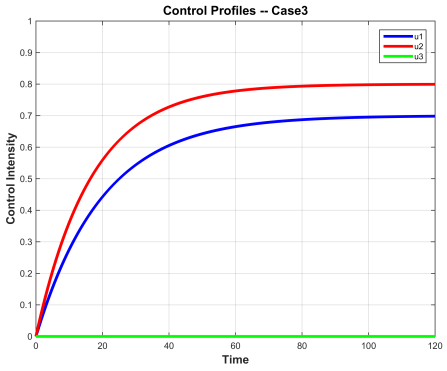
Policy case	**M**	$\mathbf{S}_i$	$\mathbf{S}_c$	$\mathbf{A}_f$	**Average**
Market only	+206.6%	+0.1%	+0.0%	+206.3%	+103.2%
Socio-cultural only	+0.1%	+221.1%	+0.0%	+0.1%	+55.3%
Finance only***	−3.3%	−2.3%	+0.0%	−1.0%	−1.6%
Market + sociocultural***	+207.0%	+221.0%	+0.0%	+206.7%	+158.7%
Market + finance***	+199.6%	−1.3%	+0.0%	+204.3%	+100.7%
Socio-cultural + finance***	−1.8%	+217.1%	+0.0%	−0.7%	+53.7%
Full Intervention***	+201.2%	+217.6%	+0.0%	+204.8%	+155.9%



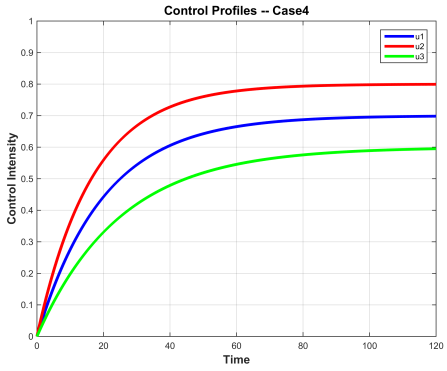
(a) Profile of control variables in Case 1 ($u_1 = 0$)



(a) Profile of control variables in Case 2 ($u_2 = 0$)



(a) Profile of control variables in Case 3 ($u_3 = 0$)



(a) Profile of control variables in Case 4, when all controls are active

Figure 18. The control profiles display the intensity of each intervention over time for each case. Deactivated controls appear as dashed lines at zero. Sociocultural interventions activate the fastest, whereas social capital interventions activate the slowest, reflecting different implementation speeds for various policy categories.

Table 13 shows that the weighted composite score includes 20% for market, 25% for sociocultural, 30% for social capital, and 25% for finance. Market + Socio-cultural strategy comes in first because it targets both supply-side (market) and demand-side (culture) elements, producing complementary impacts. The full intervention comes in second place, marginally behind market + sociocultural, indicating that it might be too soon to implement financial inclusion measures without sufficient social capital creation. According to the economic interpretation, successful financial access requires a threshold level of social capital. Access to credit, savings, insurance, and other financial services is the primary emphasis of financial inclusion initiatives. The financing barrier that frequently prevents women from starting their own businesses is the focus of these efforts. The economic mechanism operates through increased investment capacity, better risk management, and lower borrowing costs. However, without sufficient market possibilities or societal backing, financial inclusion could be unsuccessful.

A sequential strategy is supported by the statistical evidence: In the first phase (0–24 months), market and sociocultural policies are implemented to build social capital. In the second phase (24–60 months), financial inclusion policies are introduced after $S_c > 0.1$. In the third phase (60–120 months), all interventions are scaled simultaneously. At the $\alpha = 0.001$ level, all results are statistically significant, giving 99.9% confidence that the observed gains are not the result of chance. Beyond statistical significance, practical significance is confirmed by the large effect sizes. The analysis shows that integrated policies addressing market infrastructure, sociocultural reform, and financial inclusion are necessary for the emergence of women entrepreneurs. The most economical approach is the market + sociocultural dual policy. Financial inclusion policies should be implemented gradually as social capital accumulation reaches critical threshold levels, although full intervention is effective and shows diminishing returns.

Table 12. Effect sizes (Cohenś d). Cohenś d: 0.2 = small, 0.5 = medium, 0.8 = large effect

Policy Case	M	S_i	S_c	A_f	Interpretation
Market Only	5.00	0.05	0.00	5.00	large
Socio-cultural Only	0.05	5.00	0.00	0.06	large
Finance Only	−1.47	−0.98	5.00	−0.49	large
Market + Socio-cultural	5.00	5.00	0.00	5.00	large
Market + Finance	5.00	−0.56	5.00	5.00	large
Socio-cultural + Finance	−0.81	5.00	5.00	−0.34	large
Full Intervention	5.00	5.00	5.00	5.00	large

Table 13. Policy intervention ranking based on weighted composite impact

Rank	Policy Case	Composite Score (%)	Primary Impact
1	Market + Socio-cultural	+148.3%	S_i
2	Full Intervention	+145.8%	S_i
3	Market Only	+92.9%	M
4	Market + Finance	+90.7%	A_f
5	Socio-cultural Only	+55.3%	S_i
6	Socio-cultural + Finance	+53.7%	S_i
7	Finance Only	−1.5%	S_c

11.7. Synthesis and policy implications

The numerical results reflect a clear structure of policy effectiveness. Case 4 performs better than all partial intervention cases, confirming that a comprehensive combination of programs achieves the best outcomes. Improvements in all variables, especially access to financing, are severely limited when social capital is not controlled, making social captial a crucial hurdle. The differential effectiveness of control combinations provides quantitative guidance for policy design. Even when its inputs (market and sociocultural capital) are improved, social capital does not automatically rise, as demonstrated by the 50% improvement in social capital under the combination of all controls compared with 16.7% when its control is absent. It is crucial to make direct investments in the development of relationships, networks, and trust. The comparable results of Cases 1 and 2 for social capital and market linkages indicate that policymakers can choose which of the other two controls to prioritize depending on local circumstances and available resources. While communities with good market access could emphasize sociocultural strategies, communities with strong existing sociocultural support might focus on market infrastructure. The higher funding needed for multi-component approaches is justified by the synergistic effects observed in Case 4. The 40% increase in financing availability under full control is greater than what would be expected from increasing individual contributions, demonstrating how interventions reinforce one another through the system's network of interactions.

11.8. Summary

The findings show several significant patterns: social capital and financial access remain at very low levels under the no-control scenario, indicating that these vital components do not grow in the absence of active measures. Case 3, which lacks social capital control, performs the worst among the controlled cases for both social capital and access to finance, although it has active controls on market infrastructure and sociocultural interventions. The analogous performance of Cases 1 and 2, which both lack one control but retain social capital control, suggests that the impacts of sociocultural interventions and market infrastructure on the growth of social capital are essentially equivalent. With all controls active, Case 4 attains the greatest values for every variable and has access to financing that is almost fifty times greater than that of the case without controls.

12. Discussion and conclusion

In this study, we have presented a comprehensive mathematical model for examining the dynamics of women entrepreneurs using a four-compartment system comprising market linkages, sociocultural interventions, social capital, and access to finance.

We studied the model first in a deterministic setting, derived equilibrium points, verified local and global stability, computed the reproduction number, and proved the absence of flip bifurcation. Subsequently, we extended the proposed model to a stochastic setting, verified the positivity and boundedness of solutions, and showed that the stochastic reproduction number is lower than the deterministic reproduction number. We ensured that the scheme preserves positivity and boundedness for all step sizes. By formulating a stochastic optimal control problem, four policy cases are compared with the no-control case, and the results show that the combination of all controls produces the highest final access to financing. The NSFD scheme preserves positivity for all step sizes, whereas Euler-Maruyama generates economically non-feasible negative results for large step sizes. The comparison between deterministic and stochastic systems reveals fundamental differences in system behavior. The deterministic system converges to empowerment-free equilibrium points, representing a poverty trap that requires the combination of all measures to escape. At low noise levels, stochastic trajectories stay near this equilibrium, maintaining the qualitative dynamics. At high noise intensity, social capital and financial access show regime-switching behavior between high and low states, indicating that accessible region of phase space are fundamentally altered by uncertainty. This shows that deterministic analysis alone is insufficient for understanding systems subject to environmental volatility. At low noise, all variables preserve normal distributions with strong confidence bands, showing predictable results suitable for policy analysis. At high noise levels, social capital and financial access produce bimodal distributions, but market linkages and sociocultural elements preserve normal distributions with expanded variance. This shows that risk-based policy design that takes into account worst-case situations rather than just predicted values is required because of this bifurcation. The comparative analysis shows the superiority of the non-standard finite difference scheme over the Euler-Maruyama scheme for stochastic simulation. The NSFD scheme preserves positivity and boundedness at all step sizes. As the step size increases, Euler-Maruyama displays growing positivity violations, resulting in economically meaningless negative values. NSFD is preferred for long-term simulations because it preserves key qualitative characteristics. The findings demonstrate that, for all step sizes, the NSFD scheme consistently yields lower RMSE values than the Euler-Maruyama technique.

The bimodality of the model reveals three nonlinear phenomena: first, poverty traps, in which women gather into high-growth and low-growth phases divided by a critical threshold; second, inequality classification, in which the middle class weakens and creates dual economies;; and third, regime shifts, in which small policy changes have significant repercussions only after crossing thresholds.

The stochastic system produces the essential properties of deterministic dynamics despite random perturbations. Unbiased perturbations are confirmed by mean trajectories that closely follow the deterministic equilibrium under low and high noise levels. In the stochastic formulation, the boundary equilibrium continues to be the dominant attractor, confirming that the stochastic extension adds realistic variability while preserving the underlying mathematical structure. Flip bifurcation analysis reveals that flip bifurcation does not occur because all eigenvalues are real, negative, and far from the critical value of -1. Although the poverty trap continues to be a powerful attractor requiring the combination of all controls to escape, stability throughout the parameter space permits incremental policy implementation without oscillatory instabilities. The parameter impact analysis through tornado, radial, chord, and heatmap diagrams suggests that maintenance programs that lower the depreciation rate through regulatory change should be combined with initiatives that improve investment rates through financial literacy and digital infrastructure. The analysis suggests constructing the integrated policy framework using three pillars. First, devote 40% of funds to initiatives aimed at increasing conversion rates through infrastructure, training, and regulatory reform. Second, set aside 35% of funds for direct investments aimed at developing social, cultural, and market capital. Third, set aside 25% of funds for maintenance and depreciation management initiatives that lower depreciation rates through sustainable methods. According to stochastic optimal control analysis, the best final access to financing is obtained by combining all three controls. The importance of social networks in the conversion process has been demonstrated by the superior results produced in cases with active social capital control. Each control helps the others through network interactions, creating synergy. Market infrastructure control contributes more to the creation of social capital, sociocultural control strengthens social network effects, and social capital control accumulates resources while increasing the efficiency of conversion to financial access. Although they offer some improvement, partial interventions

are unable to produce the transformative results that coordinated multi-parameter approaches can achieve. In financial terms, market linkages and sociocultural factors provide slow-changing in institutional capital that offers steady returns and acts as a buffer against economic changes. Access to financing and social capital are more volatile forms of financial and relational capital that increase economic uncertainty but, if properly accumulated, offer greater potential returns. The differential response to noise suggests that portfolio approaches to policy design, which balance investments in stable institutional capital with targeted interventions in more volatile but potentially higher-impact sectors, may be suitable. Financial inclusion programs should include automatic stabilizers that maintain access during turbulent times, as the bimodality of financial access under high noise reflects real-world observations where credit availability exhibits hysteresis, expanding during good times but contracting sharply during downturns with slow recovery. The mathematical analysis combines bifurcation analysis for capturing qualitative variation, stochastic calculus for incorporating uncertainty, optimal control analysis, and comprehensive numerical schemes for simulation. Through parameter sensitivity, this cohesive method demonstrates strong stability that prevents natural escape from poverty traps while allowing for carefully planned interventions. The mathematical requirements for sustainable financial access and positive social capital accumulation offer tangible standards for policy assessment, enabling practitioners to determine whether combined interventions are enough for long-term progress. The analysis provides clear policy recommendations. Since indirect approaches through other compartments are not sufficient, social capital must be the main target of any intervention. A combination of multiple interventions is necessary to overcome poverty. Differentiated interventions require proactive stabilization for volatile aspects but consistent investment for stable components. Reactive approaches are less successful than automatic stabilizers and contingent credit facilities.

The proposed model has a number of limitations, even through it offers insightful information on the dynamics of women's entrepreneurial development. Many institutional, cultural, and policy-related elements that affect entrepreneurial growth are ignored by the model. Additionally, the stochastic component is restricted to a single noise source that affects social capital. In particular, however, several sources of uncertainty may affect financial accessibility, sociocultural support, and market linkages. To obtain better outcomes, one can also include additional socioeconomic variables such as educational attainment, access to technology, and governmental policy initiatives in the model. Furthermore, a more accurate description of uncertainty in entrepreneurial ecosystems can be obtained by adding several stochastic disturbances and time delays. To test successful interventions for increasing the growth of sustainable women's entrepreneurship, one can combine optimal control procedures with policy optimization methodologies.

Acknowledgments

None.

Funding

None.

Conflict of interest

The authors declare that they have no conflict of interest with other people or organizations that could inappropriately influence or bias the content of the paper.

Author contributions

Conceptualization: Muhammad Farman, Aqeel Ahmad, Usama Atta, Aseel Smerat, Mohamad Hafez
Formal analysis: Muhammad Farman, Aqeel Ahmad, Usama Atta, Zubaria Parvez
Investigation: Zubaria Parvez, Evren Hincal, Aseel Smerat, Mohamad Hafez
Methodology: Muhammad Farman, Aqeel Ahmad, Usama Atta, Zubaria Parvez, Mohamad Hafez
Writing–original draft: Muhammad Farman, Aqeel Ahmad, Usama Atta, Zubaria Parvez, Evren Hincal, Aseel Smerat, Mohamad Hafez
Writing–review & editing: Muhammad Farman, Aqeel Ahmad, Usama Atta, Zubaria Parvez, Evren Hincal, Aseel Smerat, Mohamad Hafez

Availability of data

The values of variables and parameters used to support the findings of this study are included within the article.

AI tools statement

All authors conform that no AI tools were used in the preparation of manuscript.

References

1. Bullough A, Guelich U, Manolova TS, Schjoedt L. Women's entrepreneurship and culture: Gender role expectations and identities, societal culture, and the entrepreneurial environment. *Small Bus Econ.* 2022;58(2):985-996. https://doi.org/10.1007/s11187-020-00429-6
2. Adom K, Anambane G. Understanding the role of culture and gender stereotypes in women entrepreneurship through the lens of the stereotype threat theory. *J Entrep Emerg Econ.* 2020;12(1):100-124. https://doi.org/10.1108/JEEE-07-2018-0070
3. Agrawal A, Gandhi P, Khare P. Women empowerment through entrepreneurship: Case study of a social entrepreneurial intervention in rural India. *Int J Organ Anal.* 2023;31(4):1122-1142. https://doi.org/10.1108/IJOA-03-2021-2659
4. Polas MRH, Raju V, Muhibbullah M, Tabash MI. Rural women characteristics and sustainable entrepreneurial intention: A road to economic growth in Bangladesh. *J Enterp Commun.* 2022;16(3):421-449. https://doi.org/10.1108/JEC-10-2020-0183
5. Sharma E, Das S. Integrated model for women empowerment in rural India. *J Int Dev.* 2021;33(3):594-611. https://doi.org/10.1002/jid.3539
6. Al-Assaf KT, Foriki MSM, Samarah T, et al. Impact of digital marketing on promoting the local pharmaceutical products in Jordan. *Int Rev Manag Mark.* 2025;15(1):137-145. https://doi.org/10.32479/irmm.17707
7. Khalufi NAM, Sheikh RA, Khan SMFA, Onn CW. Evaluating the impact of sustainability practices on customer relationship quality: an SEM-PLS approach to align with SDG. *Sustainability.* 2025;17(2):798. https://doi.org/10.3390/su17020798
8. World Bank Group. *Global Financial Development Report 2014: Financial Inclusion*. Vol 2. World Bank Publications; 2013. https://doi.org/10.1596/978-0-8213-9985-9
9. Ridgeway CL. *Framed by Gender: How Gender Inequality Persists in the Modern World*. Oxford University Press; 2011. https://doi.org/10.1093/acprof:oso/9780199755776.001.0001
10. Sen G, Ostlin P. Gender inequity in health: Why it exists and how we can change it. *Glob Public Health.* 2008;3(S1):1-12. https://doi.org/10.1080/17441690801900795
11. Huang J, Krivkovich A, Starikova I, Yee L, Zanoschi D. *Women in the Workplace 2019*. McKinsey & Co; 2019. Accessed March 15, 2026. https://www.mckinsey.com/featured-insights/gender-equality/women-in-the-workplace
12. Cornwall A, Rivas AM. From gender equality and women's empowerment to global justice: Reclaiming a transformative agenda for gender and development. *Third World Q.* 2015;36(2):396-415. https://doi.org/10.1080/01436597.2015.1013341
13. United Nations Girls' Education Initiative. *Teaching and Learning: Achieving Quality for All*. EFA Global Monitoring Report 2013-2014; 2014. Accessed March 15, 2026. https://unesdoc.unesco.org/ark:/48223/pf0000226662
14. UN Women. *Turning Promises into Action: Gender Equality in the 2030 Agenda for Sustainable Development*. UN Women; 2018. https://doi.org/10.18356/917ed83e-en
15. Pontryagin LS. *Mathematical Theory of Optimal Processes*. Routledge; 2018. https://doi.org/10.1201/9780203749319
16. Lenhart S, Workman JT. *Optimal Control Applied to Biological Models*. Chapman & Hall/CRC; 2007. https://doi.org/10.1201/9781420011418
17. Svishchuk M, Swishchuk AV. Stochastic optimal control of averaged SDDE with semi-Markov switching and with application in economics. *Mathematics.* 2025;13(9):1440. https://doi.org/10.3390/math13091440
18. Fleming WH. *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of Wendell H. Fleming*. Springer; 1999. https://doi.org/10.1007/978-1-4612-1784-8
19. Kafash B, Delavarkhalafi A, Karbassi SM. A computational method for stochastic optimal control problems in financial mathematics. *Asian J Control.* 2016;18(4):1501-1512. https://doi.org/10.1002/asjc.1242
20. Akdogan E. *Stochastic Optimal Control Theory: New Applications to Finance and Insurance* [master's thesis]. Ankara, Turkey: Middle East Technical University; 2017. Accessed XXX. https://etd.lib.metu.edu.tr/upload/12621076/index.pdf
21. Ewald CO, Nolan C. On the adaptation of the Lagrange formalism to continuous time stochastic optimal control: A Lagrange-Chow redux. *J Econ Dyn Control.* 2024;162:104855. https://doi.org/10.1016/j.jedc.2024.104855
22. Dai Y, Xu W, Wu X, Yan M, Xue F, Zhao J. A data-physical fusion method for economic dispatch considering high renewable penetration and security constraints. *Int J Electr Power Energy Syst.* 2025;168:110691. https://doi.org/10.1016/j.ijepes.2025.110691
23. Naik PA, Yavuz M, Qureshi S, Owolabi KM, Soomro A, Ganie AH. Memory impacts in hepatitis C: A global analysis of a fractional-order model

with an effective treatment. *Comput Methods Programs Biomed.* 2024;254:108306. https://doi.org/10.1016/j.cmpb.2024.108306

24. Naik PA, Yeolekar BM, Qureshi S, Yeolekar M, Madzvamuse A. Modeling and analysis of the fractional-order epidemic model to investigate mutual influence in HIV/HCV co-infection. *Nonlinear Dyn.* 2024;112(13):11679-11710. https://doi.org/10.1007/s11071-024-09567-8

25. Chen Z. *Numerical Methods for Optimal Stochastic Control in Finance*. LAP Lambert Academic Publishing; 2008.

26. El-Mesady A, Elsonbaty A, Nisar KS, Adel W. Fractional-order mathematical modeling and optimal control strategies for women's empowerment initiatives. *Ain Shams Eng J.* 2025;16(11):103702. https://doi.org/10.1016/j.asej.2025.103702

27. Raza A, Minhós F, Shafique U, Mohsin M. Mathematical modeling of leukemia within stochastic fractional delay differential equations. *Comput Model Eng Sci.* 2025;(3):3411. https://doi.org/10.32604/cmes.2025.060855

28. Diekmann O, Heesterbeek JAP, Roberts MG. The construction of next-generation matrices for compartmental epidemic models. *J R Soc Interface.* 2010;7(47):873-885. https://doi.org/10.1098/rsif.2009.0386

29. Castillo-Chavez C, Feng Z. Global stability of an age-structure model for TB and its applications to optimal vaccination strategies. *Math Biosci.* 1998;151(2):135-154. https://doi.org/10.1016/S0025-5564(98)10016-0

30. Farman M, Ahmad A, Atta U, Nisar KS, Ghaffar A. Hyers Ulam stability and bifurcation control of leptospirosis disease dynamics and preventations: Modeling with singular and non-singular kernels. *PLoS One.* 2025;20(3):e0314095. https://doi.org/10.1371/journal.pone.0314095

31. Ahmad A, Atta U, Akgul A, Sattar S, Ahmad MO. A study on mathematical modeling and control of leptospirosis transmission dynamics during a hurricane with asymptomatic measures. *Model Earth Syst Environ.* 2025;11(5):361. https://doi.org/10.1007/s40808-025-02508-7

32. Mickens RE. Nonstandard finite difference schemes. In: *Applications of Nonstandard Finite Difference Schemes*. World Scientific; 2000:1-54. https://doi.org/10.1142/4272

33. El-Mesady A, Al-shami TM, Ali HM. Optimal control efforts to reduce the transmission of HPV in a fractional-order mathematical model. *Bound Value Probl.* 2025;(1):42. https://doi.org/10.1186/s13661-024-01991-8

34. El-Mesady A, Mahdy AMS, Ozkose F. A fractional-order model with prevention and isolation optimal control measures to reduce the transmission of Tuberculosis. *J Taibah Univ Sci.* 2025;19(1):2475579. https://doi.org/10.1080/16583655.2025.2475579

Muhammad Farman *did his Ph.D. Mathematics from the University of Lahore in 2019, Pakistan, and completed a Postdoc fellowship at Universitas Airlangga Indonesia, Lebanese American University, and Near East University, North Cyprus, Turkey. Former assistant professor at Khawaja Fareed University of Engineering and Information Technology, Adjunct Professor (Research), at Universitas Airlangga Indonesia and University of Lahore, Pakistan. Currently, he is working as an Associate Professor Dr. and Vice President of the Research Center at Near East University Mathematical Research Center. He has more than ten years of teaching and research experience. The research fields are mathematical biology, Control theory, Numerical Analysis, and Application of fractional calculus on Epidemic models. He has also supervised the thesis of Master and Ph.D. Mathematics students. He has published more than 350 research papers in well-reputed SCI and Scopus journals and presented his work at several national and international conferences. He is the winner of several awards, such as the Research Excellence Awards (Best Researcher of the University), in 2019 and 2020 at the University of Lahore. He is a reviewer for several Web of Science and Scopus journals. Stanford University's Top 2*
https://orcid.org/0000-0001-7616-0500

Aqeel Ahmad *is an Assistant Professor in the Department of Mathematics at Ghazi University, Dera Ghazi Khan (DG Khan), Pakistan. He previously worked as an Assistant Professor at the University of Lahore. Dr. Ahmad is a highly active researcher focusing heavily on biomathematics, fractional calculus, and mathematical modelling. His primary areas of expertise include Mathematical Biology and Epidemiology, and Numerical Analysis. He has contributed to over 120 publications with thousands of academic citations. He has completed a postdoc fellowship under the AIRLANGGA POST-DOCTORAL FELLOWSHIP (2022-2023) with the project named "Analysis of COVID-19 outbreak in the Democratic Republic of the Congo using fractional operators".*
https://orcid.org/0000-0002-7754-9355

Usama Atta *is a researcher and PhD student in the Department of Mathematics at Ghazi University, Dera Ghazi Khan, Pakistan. His research primarily spans advanced mathematical modelling applied to ecological, biological, and economic systems. He has more than six years of research and teaching experience at the university level and has published 10 + research papers in high-impact factor journals.*
https://orcid.org/0000-0001-8827-8450X

Zubaria Parvez *is an academic researcher and PhD affiliated with the Department of Business Administration at Near East University in Nicosia, Northern Cyprus. Her research heavily concentrates on women's entrepreneurial empowerment, digital skills, financial accessibility, and socio-cultural dynamics in Pakistan. She has published research papers in high-impact factor journals to date.*
https://orcid.org/0009-0002-6950-615X

Evren Hincal *was born in Nicosia in 1973. He completed his undergraduate studies at Hacettepe University, Faculty of Science and Letters, Department of Mathematics, and his master's degree with Honours at the University of London Imperial College, Departments of Applied Mathematics. He completed his doctoral studies as a joint project between the Imperial College Biology Department and the Eastern Mediterranean University Mathematics Department. During this period, one of E. Hınçal's doctoral thesis topics, 'Investigation of the cancer situation in Northern Cyprus and its evaluation according to European standards,' was supported by the Cancer Research Foundation. In 2000 and 2001, E. Hınçal worked on cancer epidemiology as an Academic Visitor in the Neurobiology Department at Imperial College. Evren Hınçal worked as a part-time lecturer at Eastern Mediterranean and Near East Universities between 1999 and 2006. From 2007 onwards, with the establishment of the Department of Mathematics at Near East University, he became a full-time lecturer in this department. He received the title of Assistant Professor in 2009, Associate Professor in 2013, and Professor in 2019. Since 1998, Evren Hınçal has authored numerous international publications and participated in many conference.*
https://orcid.org/0000-0001-6175-1455

Aseel Smerat *is a prolific researcher frequently associated with t Hourani Center for Applied Scientific Research at Al-Ahliyya Amman University in Amman, Jordan.In addition to their collaborative role at SIMATS, Dr. Smerat is heavily anchored at the Faculty of Educational Sciences and the Hourani Center for Applied Scientific Research at Al-Ahliyya Amman University in Amman, Jordan. Dr. Smerat is a highly multidisciplinary scholar spanning a vast range of complex data-driven fields, engineering systems, and life sciences. Their published contributions include over 700 tracked items on global platforms. Dr. Smerat's work relies heavily on mathematical modeling, engineering optimization, and computational intelligence applied across diverse domains.*
https://orcid.org/0009-0008-4600-509X

Mohamed Ahmed Hafez *received a Master of Science focusing on coastal erosion prevention of the Mangrove coastline and a Doctor of Philosophy for his research on ground development of the soft coastal ground from the University of Malaya, Malaysia. He educates and guides aspiring engineers as a lecturer at INTI-IU University in Malaysia. His expertise primarily lies in coastal reclamation, erosion prevention, ground improvement techniques, slope stability, hydraulic engineering, dam safety, and reservoir sedimentation. He has written five books primarily focused on soft soil improvement and has collaborated on more than 170 scientific and academic publications published in various magazines, book chapters, and conference proceedings. In addition, he has been honoured with over 25 prestigious national and international gold and silver medals. Notably, he received the WIPO award in 2005 for being the best inventor in water research and geotechnical design from a developing country.*
https://orcid.org/0000-0001-6537-4247

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