

Three-species dynamics eco-cognitive instantaneous response modulation in a predator-prey system

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ABSTRACT

This paper presents a new three-species predator-prey model with an eco-cognitive response for instantaneous trophic interactions. Unlike older models which base predator efficiency on factors such as encounter rates or handling time, this framework incorporates cognitive influence. Predator performance here is affected by their perception of immediate prey density—that is, how well predators recognize, pursue, and outsmart their prey. The system includes a prey population growing logistically, a middle predator guided by the cognition-based model, and a top predator preying on the middle one. It maintains biological realism while adding feedback between the number of prey and predator cognition in real-time. The analysis covers positivity and boundedness, existence and local stability of equilibria, persistence and extinction thresholds, Hopf bifurcation, oscillatory coexistence and sensitivity analysis. Numerical continuation, bifurcation diagrams and phase portraits show complex dynamics like stable coexistence, wide swings, multistability, and transient chaos.



1. Introduction

Predator-prey analysis remains among the richest areas for bridging nonlinear mathematics and biology. The development of predation theory from the seminal models of Lotka and Volterra has yielded a diverse field, rich both biologically and mathematically, capable of capturing trophic dynamics, population stability, resource limitations, competing species, adaptive evolution, and ecological complexity.¹⁻⁴ Indeed, the biological realism of predation modeling is coupled with its capacity to produce highly complex dynamical phenomena such as periodic behavior,

quasi-periodicity, cascades of bifurcations, chaos, and regime switching.⁵ In recent years, research on ecological modeling has seen an increasing departure from the conventional assumption of fixed interaction parameters. Rather, it is recognized that ecological systems are subject to influences arising from biological adaptation, plasticity, memory, decision-making, spatial awareness, fear-based reproduction suppression, evolutionary traits, and biological information. This represents a paradigm shift in that species interactions are understood not simply as mechanical phenomena but as processes of biological adaptation.⁶ One notable area is behavioral predation theory. The

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conventional functional responses of Holling types I-IV, Beddington-DeAngelis encounters, Crowley-Martin responses, ratio-dependent predations, and general non-linear encounter rates primarily model the influence of prey density, predator density, or handling time on predation rate.⁷ While such models satisfactorily model saturating behaviors, interference, and prey refuge, they usually neglect predator cognition. However, predators adapt cognitively very fast. They enhance target detection, fine-tune chase strategies, develop search tactics, and continuously adjust hunting efforts based on environmental signals.⁸ Recent experimental ecological studies have revealed that cognition significantly impacts foraging performance. Predators often demonstrate enhanced capture efficiency, selective prey detection in context, and strategic switching between prey categories due to cognitive learning. Cognitive learning often acts quickly in many ecological systems and does not rely on accumulated long-term experience.⁹ Rather, it spontaneously emerges through behavior modification in response to existing prey density, environmental visibility, and sensory feedback. This represents a fundamental challenge in mathematical modeling. Although studies on ecology-dependent memory systems have focused on delayed differential equations, fractional calculus, integral non-local kernels, and adaptive parameters, hardly any research exists that examines the instantaneous impact of cognitive modulation on trophic interaction.¹⁰ In particular, the integration of bounded cognition into the efficiency of predator attack in a food chain framework has not been adequately addressed. This paper aims to fill this research gap through the introduction of a new approach: the eco-cognitive instantaneous indicator, which estimates the instantaneous adaptability of hunting efficiency due to prey density. The fundamental assumption is straightforward: as prey density rises, the predator's ability to recognize, track, and hunt its prey becomes more efficient, thus enhancing predation efficiency. Nonetheless, cognitive efficiency cannot grow without bound since there are biological limits to adaptation. Consequently, the cognitive response should increase monotonically but approach saturation. To capture this mechanism, we introduce a bounded nonlinear cognitive index

$$M(x) = \frac{x}{k + x},$$

where x denotes prey density and $k > 0$ is a half-saturation cognitive threshold. This function possesses several desirable properties:

- $M(0) = 0$ (no prey implies no cognitive stimulation),
- $M'(x) > 0$ (cognition increases with prey abundance),
- $\lim_{x \rightarrow \infty} M(x) = 1$ (bounded cognitive saturation),
- mathematically smooth and analytically tractable.

The ecological interpretation is profound: the function does not represent prey biomass directly; rather, it represents predator behavioral readiness generated by ecological exposure. The incorporation of such an instantaneous cognitive index fundamentally modifies classical predator-prey interaction structure.¹¹ Unlike delayed or memory-driven formulations, the present approach assumes that cognitive adjustment is immediate, reflecting fast neural and behavioral adaptation observed in many species. During the period 2020–2026, several advances in ecological dynamics have emphasized the importance of non-traditional interaction mechanisms, including fear effects, learning, adaptive switching, and information-based trophic regulation. However, most existing models treat adaptation either as: (i) a slow evolutionary process, (ii) a delayed feedback mechanism, or (iii) an external stochastic perturbation. In contrast, the present formulation embeds cognition directly into the functional response, thereby creating a structurally new class of instantaneous adaptive systems.

In this sense, a key idea emerges in modeling: the interaction strength cannot be treated as a parameter that is defined exclusively on the basis of encounter rates; rather it should be interpreted as an evolving function affected by perception, adaptation, and cognitive reaction to prey abundance.^{12,13} In this respect, predator efficiency is seen as a nonlinear function of prey abundance driven by cognitive activation and constrained by saturating effects. The model can therefore be placed within the larger category of behaviorally controlled trophic systems, where classical ecological interactions become more complex as they incorporate adaptive responses to link population dynamics with behavioral flexibility.¹⁴ In an ecological perspective, these elements represent fast learning about predatory tactics, attention improvement induced by prey abundance, saturating processes related to cognitive processing at higher prey abundance, non-linear relationships between abundance and perception, and the indirect effects that flow through the food web hierarchy.⁹ From the mathematical perspective, the introduction of cognitive modulation in the model

makes the system equations more nonlinear, affecting the existence and locations of equilibrium points, altering stability criteria both locally and globally, and also changing the bifurcation thresholds at which stable coexistence transitions to oscillatory or even more complicated behavior.¹⁵

Recent studies have increasingly emphasized the roles of fractional-order modeling, intelligent computational methods, and stability analysis in nonlinear biological and ecological systems. For instance, Madani, Hammouch, and Özdemir¹⁶ developed a hybrid GA-IPA and neural-network-based approach for fractional-order nonlinear predator-prey dynamical systems, demonstrating the relevance of optimization-assisted intelligent methods in ecological dynamics. Related studies on biological interaction models, qualitative analysis, numerical simulations, fractional operators, and stability investigation^{17,18} also support the mathematical background of the present work. Moreover, recent studies on oscillatory behaviour in delay-differential ecological models¹⁹ and on climate-based predictive modeling using statistical and machine-learning approaches²⁰ further highlight the importance of combining nonlinear dynamics with computational intelligence in applied biomathematics.

The first report of the fear effect was carried out by Wang *et al.*²¹ They characterised the fear effect by using linear as well as Holling Type II functional response equations. They formulated the hypothesis that the reduction of the prey population was due to the fear induced by the predator. The reduction of the prey population could be attributed to fear induced by the predator. They observed that if there is considerable fear, then it may help in stabilising the system by stopping population fluctuations. Wang and Zou²² designed a prey-predator model involving adaptive predator avoidance through fear effect. Adult prey had more anti-predator traits due to a large predator population, while less anti-predator traits occurred with a higher fear consequence. The effects of fear in a stochastic prey-predator system in the presence of supplementary food for predators was reported by Das and Samanta.²³ In this study, the fear term was examined in an exponential function. It was observed that the periodicity of the fear term allows the prey population to increase. The effects of fear and additional food on a prey-predator model were reported by Mondal *et al.*²⁴ Pal *et al.*²⁵ studied the effect of fear and cooperative hunting on the dynamics of a prey-predator system in the presence of cooperative hunting by predators. Dash and Khajanchi²⁶ studied

an intraguild predation model by taking three species, namely shared resources, IG-prey, and IG-predator into consideration with Holling type II functional response between IG-prey and IG-predator. A mathematical model of fractional-order yeast prion has been investigated by Maji and Khajanchi.²⁷ Steady states were identified and their local stability was theoretically and numerically analysed in order to determine the dynamics of the system. A prey-predator model which incorporated fear induced birth reduction due to the presence of a predator has been studied by Sarkar in.²⁸ Jabbari *et al.*²⁹ have introduced a two-patch model for tuberculosis in which the transmission rate of the disease depends on the movement of susceptibles from one patch to another with fractional order and exogenous reinfection. Interactions of cannibalism are carried out between predator species; they cause the spread of disease among themselves. Recent studies on discrete evolutionary and population models have contributed to the analysis of persistence, stability, and complex dynamics in ecological systems. In particular, difference-equation models have been used to describe the effects of evolutionary feedback, strong Allee thresholds, immigration, harvesting, and prey-predator interactions on long-term population behavior. The works^{30–33} consider both one-parameter and two-parameter bifurcation structures, including saddle-node, period-doubling, Neimark-Sacker, and resonance phenomena. These studies provide a useful framework for connecting biologically interpretable mechanisms with analytical and numerical tools from discrete dynamical systems, and they help clarify how ecological and evolutionary processes may lead to stable coexistence, multistability, quasiperiodic oscillations, or chaotic dynamics. According to Khajanchi,³⁴ a robust stage-structured predator-prey model is considered here with a function response as Beddington's DeAngelis. Nonlinear ODEs with three variables are used in the proposed mathematical model that encourages interaction between prey species, juvenile predators, and adult predators. In addition, Sarkar *et al.*³⁵ considered a mathematical model with several kinds of response functions. It was used in a predator-prey system as an interacting dynamics model. Also, Mondal and Samanta³⁶ studied predator-prey interactions in an ecosystem with a single prey species, concerning the fear factor of predators (generalist predator with constant biomass). To begin with, the effect of the fear factor of the predator on a single prey species was investigated without including the predation factor. In Das and

Samanta,³⁷ a delayed fractional-order predator-prey model was considered in which a prey refuge and the fear effect of predators on prey were incorporated. They introduced a three-species food chain system incorporating prey refuge and functional response of type one for predator. It is recommended that a fractional-order prey-predator model with collective defense and predator fear will influence the prey population.³⁸ We investigate the impact of the anti-predation response rate on the population dynamics through delay differential equation analysis, as well as delay elements impacts. Hopf-bifurcation is then used to determine stability switches in the model system. The authors in³⁹ examines how global warming affects carrying capacity. Its impact on predation has also been investigated by creating a predation function that takes into consideration the consequences of hunting cooperation and global warming. The aforementioned elements are part of a system of ordinary differential equations that describes the relationship between prey, predators, and global warming. The major abiotic factors in such a situation play an instrumental role in the future development of an ecosystem. Therefore, the primary goal of mathematical modeling will remain unclear if the abiotic factors is not taken into account. One of the abiotic factors in this respect is wind. It may modify the predation behavior of predators according to wind speed and affect their ability to perceive signs of the predator.^{40,41} Wind may become a barrier to natural migration and may disrupt other functions, such as uprooting trees in a populated area. According to Klimczuk *et al.*, the presence of blowing twigs helps the aerial hunters better locate the nests of reed warblers. The rate of predation could increase depending upon the prey's inability to recognize the predators.⁴² Animal-behavioural tactics are not only limited to resource acquisition during predator attacks but are also influenced by environmental factors such as wind.⁴³ The experiments conducted by Cherry and Barton⁴⁴ indicated that the prey vigilance levels increase under high wind velocities. The reason for this behaviour was the negative effect of the wind speed on the ability of the prey to detect the predators. Wind disturbances reduce predation rates and thereby enhance the survival of prey populations.⁴⁵ Humid weather decreases the ability of small invertebrate predators including mosquitoes⁴⁶ and flies⁴⁷ to fly and catch their prey. Although many similar examples exist in the literature,^{48–58} we present only a few here.

Based on biological and mathematical reasons, we propose a food chain model for three species

comprising a prey population $x(t)$, an intermediate predator population $y(t)$, and a top predator population $z(t)$. In this model, the modulation effect happens between the prey-predator species.

2. Model formulation

In our work, we present a deterministic nonlinear mathematical model of a three-tier food chain comprising prey $x(t)$, intermediate predator $y(t)$, and top predator $z(t)$. This model differs from the existing literature because of the addition of an ecological and cognitive feedback instantaneous response in predation, in which the predators' efficacy depends on the actions of the prey. The proposed model was formulated based on: (i) prey population model, (ii) cognitive functional response, (iii) energy flow among predator tiers. The prey dynamics are governed by:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) - \frac{\xi(1 + \sigma M(x))xy}{1 + ax} - \zeta xz.$$

The equation above involves three different ecological processes:

- **Intrinsic growth:** The term $rx(1 - x/K)$ refers to logistic growth of prey x . In this case, the parameter $r > 0$ stands for the reproductive rate, and $K > 0$ denotes the carrying capacity of the environment. This process reflects limitations in resources like food, space, and the environment.
- **Cognitive-modulated predation by intermediate predator:** The term

$$\frac{\xi(1 + \sigma M(x))xy}{1 + ax}$$

represents prey consumption by the intermediate predator y . It is composed of ξxy i.e., classical mass-action encounter rate, $(1 + \sigma M(x))$ i.e., eco-cognitive enhancement factor, and $\frac{1}{1+ax}$ i.e., saturation due to handling time and predator interference. The function $M(x) = \frac{x}{k+x}$ stands for the rate of cognitive activation, where higher values of prey population make predators more alert, accurate, and efficient in hunting. The parameter $\sigma \geq 0$ reflects the level of cognitive amplification.

- **Predation by the top predator:** Here, the expression ζxz gives the prey lost due to attack by the top predator z . Here ζ stands for the attack rate while xz for the frequency of encounters. Thus, the prey removal rate from the population will be ζxz . Since the death of prey and growth of predators are two different biological phenomena, there is no reason for the expression ζxz to exist within the top predator equation. This expression simply denotes the prey loss, without stating how

much of the prey biomass was gained by the predator. The amount of biomass gain by the predator depends upon the conversion efficiency of the system. The conversion efficiency is usually denoted by η with values between zero and one, i.e., $0 < \eta < 1$. So, in case where a growth occurs in the top predator through the consumption of prey, the growth term becomes $\eta\zeta xz$. Conversion efficiency accounts for losses due to metabolic activities such as respiration, locomotion, etc. Predator growth is thus lower than prey consumption. However, if the expression ζxz only occurs as a prey loss term but there is no term like $\eta\zeta xz$ in the top-predator equation, then it means one of the following must have occurred:

- (1) The top predator causes mortality in prey x , but this interaction does not contribute to predator growth.
- (2) The top predator does not actually feed on prey x , so no biomass is transferred from x to z .
- (3) The contribution of prey x to the growth of the top predator is neglected for simplicity in the model formulation.

Thus, the lack of a coefficient proportional to $\eta\zeta xz$ on the right-hand side of the equation for the third predator implies that the interaction described by ζxz is treated purely as a cause of prey death rather than predator gain.

The intermediate predator satisfies:

$$\frac{dy}{dt} = e \frac{\xi(1 + \sigma M(x))xy}{1 + ax} - \rho yz - d_1 y.$$

This equation consists of three components:

- Energy gain from prey consumption: The term

$$e \frac{\xi(1 + \sigma M(x))xy}{1 + ax}$$

represents biomass conversion from prey to predator. The parameter $e \in (0, 1)$ denotes ecological efficiency, accounting for energy loss during digestion, metabolism, and assimilation.

- Predation by top predator: The term ρyz represents consumption of the intermediate predator by the top predator. This follows a standard mass-action interaction assumption, where encounter rate is proportional to both populations.
- Natural mortality: The term $d_1 y$ accounts for death due to aging, disease, and environmental stress.

The apex predator dynamics are given by:

$$\frac{dz}{dt} = cpyz - d_2 z.$$

This equation includes:

- Growth via consumption: The term $cpyz$ models the biomass gain from consuming the intermediate predator. The parameter $c \in (0, 1)$ represents conversion efficiency.
- Natural mortality: The term $d_2 z$ captures death due to intrinsic biological limitations, including senescence, starvation, and environmental hazards.

We consider a continuous-time deterministic system described by the following nonlinear differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\xi(1 + \sigma M(x))xy}{1 + ax} - \zeta xz, \\ \frac{dy}{dt} &= e \frac{\xi(1 + \sigma M(x))xy}{1 + ax} - \rho yz - d_1 y, \\ \frac{dz}{dt} &= cpyz - d_2 z, \end{aligned}$$

The key novelty of the model is the function:

$$M(x) = \frac{x}{k + x},$$

which introduces a nonlinear behavioral feedback mechanism, since it captures a realistic saturating increase in cognitive response with prey density while maintaining mathematical simplicity and a biologically meaningful half-saturation parameter k .

2.1. Novel features and biological assumptions of the model

The distinguishing feature of the proposed predator-prey model is the consideration of the predator's cognitive functions and learning process. The key distinctions of the model in comparison to the classical predator-prey models are as follows:

- Incorporation of predator cognition. While classical predator-prey models suggest that the interaction rate depends solely upon population sizes of prey and predators and certain ecological parameters, the proposed model implies that the predator's efficiency depends upon its cognitive behavior and experience gained in repeated attacks on prey.
- Enhanced prey detection and hunting efficiency. The cognitive enhancement helps to make the predator more adept at detecting prey, adapting to the environment, optimizing its search techniques, thereby enhancing hunting success learning and familiarizing itself with the surroundings.

- Learning-driven increase in predation efficiency. Interaction with prey increases the predator's ability to capture prey at any point in time, leading to increased exploitation of the prey population and better predator biomass gains.
- Physiological and behavioral limitations. However, it is believed that the extent of cognitive enhancement is limited by physiological factors like sensory limitations, attention span limits, memory capacity limits, and behavioral adaptability.
- Nonlinear bounded feedback mechanism. To ensure the saturating nature of the cognitive advantage, the model uses a nonlinear bounded feedback function for the functional response. This allows cognitive improvement without infinite growth in predation ability.
- Coupling between cognition and trophic interactions. The model incorporates cognition by explicitly modeling the relationship between predators' cognition and their predation on prey and growth dynamics.

The model is based on three fundamental processes:

- (1) Logistic growth of prey population. The growth of the prey is modeled using the logistic growth equation, which takes into account environmental limitations and competition among prey.
- (2) Saturating predator-prey interaction. The trophic relationship between the prey and the predator is modeled using a saturating response function, capturing the constraint on the predator's ability to consume prey due to prey density.
- (3) Cognitive adaptation of the predator. The predator optimizes the rate of prey capture using a bounded learning algorithm that simulates the learning experience of the predator.

3. Positivity and boundedness of solutions

3.1. Positivity of solutions

Theorem 1. *For any initial condition $(x(0), y(0), z(0)) \in \mathbb{R}_+^3$, the solutions $(x(t), y(t), z(t))$ of the system remain positive for all $t > 0$.*

Proof. Consider the first equation:

$$\frac{dx}{dt} = x \Phi_1(x, y, z),$$

where $\Phi_1(x, y, z)$ is a continuously differentiable function.

Since $x(t) = 0$ implies $\frac{dx}{dt} = 0$, the plane $x = 0$ is invariant. Thus, trajectories starting with $x(0) > 0$ cannot cross into $x < 0$.

Similarly, the second and third equations can be written as:

$$\frac{dy}{dt} = y \Phi_2(x, y, z), \quad \frac{dz}{dt} = z \Phi_3(x, y, z),$$

which also imply that the hyperplanes $y = 0$ and $z = 0$ are invariant.

Therefore, the non-negative orthant $\mathbb{R}_+^3$ is positively invariant. $\square$

3.2. Boundedness of solutions

Theorem 2. *All solutions of the system are uniformly bounded and eventually enter a compact attracting region in $\mathbb{R}_+^3$.*

Proof. From the first equation, we have:

$$\frac{dx}{dt} \leq rx \left(1 - \frac{x}{K}\right),$$

which implies, by comparison with the logistic equation, that:

$$\limsup_{t \rightarrow \infty} x(t) \leq K.$$

Hence, there exists $T_1 > 0$ such that for all $t > T_1$, we have:

$$0 < x(t) \leq K.$$

Now consider the second equation:

$$\frac{dy}{dt} = y \left(e^{\frac{\xi(1+\sigma M(x))x}{1+ax}} - \rho z - d_1 \right).$$

Using $x(t) \leq K$ and $M(x) \leq 1$, we obtain the estimate:

$$\frac{dy}{dt} \leq y \left(\frac{e^{\xi(1+\sigma)K}}{1+aK} - d_1 \right).$$

Define:

$$\alpha_1 = \frac{e^{\xi(1+\sigma)K}}{1+aK} - d_1.$$

If $\alpha_1 \leq 0$, then $y(t)$ is exponentially bounded. Otherwise, $y(t)$ grows at most exponentially.

However, the negative feedback term $-\rho yz$ ensures that unbounded growth of $y(t)$ is prevented when $z(t)$ becomes large.

For the third equation:

$$\frac{dz}{dt} = z(cpy - d_2),$$

We obtain:

$$\frac{dz}{dt} \leq z(cpy - d_2).$$

Using boundedness of $y(t)$ obtained above, there exists $M_y > 0$ such that $y(t) \leq M_y$ for large t ,

which yields:

$$\frac{dz}{dt} \leq z(c\rho M_y - d_2).$$

Thus, if $c\rho M_y < d_2$, then $z(t)$ is also bounded.

Therefore, all components of the system are ultimately bounded, and trajectories enter the compact region:

$$\Omega = \{(x, y, z) \in \mathbb{R}_+^3 : x \leq K, y \leq M_y, z \leq M_z\}.$$

□

4. Equilibrium points

Setting $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$, we obtain equilibrium points.

4.1. Trivial equilibrium

$$E_0 = (0, 0, 0)$$

represents total extinction.

4.2. Boundary equilibria

$$E_1 = (K, 0, 0)$$

represents prey-only survival.

4.3. Interior equilibrium

Let $E^* = (x^*, y^*, z^*)$ satisfy:

$$\begin{aligned} r \left(1 - \frac{x^*}{K}\right) &= \frac{\xi(1 + \sigma M(x^*))y^*}{1 + ax^*} + \zeta z^*, \\ e \frac{\xi(1 + \sigma M(x^*))x^*}{1 + ax^*} &= \rho z^* + d_1, \\ c\rho y^* &= d_2. \end{aligned}$$

From the third equation:

$$y^* = \frac{d_2}{c\rho}.$$

Substituting into the second:

$$z^* = \frac{e\xi(1 + \sigma M(x^*))x^*}{\rho(1 + ax^*)} - \frac{d_1}{\rho}.$$

Substituting into the first equation yields a nonlinear scalar equation in the unknown x^* , which determines the coexistence equilibrium.

Specifically, substituting

$$y^* = \frac{d_2}{c\rho}, \quad z^* = \frac{e\xi(1 + \sigma M(x^*))x^*}{\rho(1 + ax^*)} - \frac{d_1}{\rho}$$

into the prey equation gives:

$$\begin{aligned} rx^* \left(1 - \frac{x^*}{K}\right) &= \frac{\xi(1 + \sigma M(x^*))x^*}{1 + ax^*} \cdot \frac{d_2}{c\rho} \\ &+ \zeta x^* \left[\frac{e\xi(1 + \sigma M(x^*))x^*}{\rho(1 + ax^*)} - \frac{d_1}{\rho} \right]. \end{aligned}$$

Dividing both sides by $x^* > 0$ leads to the reduced coexistence condition:

$$\begin{aligned} r \left(1 - \frac{x^*}{K}\right) &= \frac{\xi d_2(1 + \sigma M(x^*))}{c\rho(1 + ax^*)} \\ &+ \frac{\zeta e\xi(1 + \sigma M(x^*))x^*}{\rho(1 + ax^*)} - \frac{\zeta d_1}{\rho}. \end{aligned}$$

Rearranging, we obtain the final implicit equation:

$$r \left(1 - \frac{x^*}{K}\right) + \frac{\zeta d_1}{\rho} = \frac{\xi(1 + \sigma M(x^*))}{\rho(1 + ax^*)} \left[\frac{d_2}{c} + \zeta ex^* \right].$$

Since $M(x^*) = \frac{x^*}{k+x^*}$, the above reduces to a nonlinear algebraic equation in x^* :

$$r \left(1 - \frac{x^*}{K}\right) + \frac{\zeta d_1}{\rho} = \frac{\xi \left(1 + \sigma \frac{x^*}{k+x^*}\right)}{\rho(1 + ax^*)} \left(\frac{d_2}{c} + \zeta ex^* \right).$$

Note: The interior equilibrium $E^* = (x^*, y^*, z^*)$ exists if $c\rho + c\zeta d_1 > \xi d_2$ and $e\xi(1 + \sigma M(x^*))x^* > d_1(1 + ax^*)$.

Remark 1. This equation is transcendental and generally does not admit closed-form solutions. However, the existence of positive roots $x^* > 0$ determines the feasibility of the interior equilibrium $E^* = (x^*, y^*, z^*)$. The number of positive solutions depends strongly on the cognitive parameter σ and saturation parameters a, k .

5. Local stability analysis of equilibria

In this section, we investigate the local dynamical behavior of the system around each equilibrium point. Let $E = (x, y, z)$ denote a general state of the system. The Jacobian matrix $J(E)$ associated with the system is given by:

$$J(E) = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where

$$J_{11} = r \left(1 - \frac{2x}{K}\right) - \frac{\partial}{\partial x} \left(\frac{\xi(1 + \sigma M(x))xy}{1 + ax} \right) - \zeta z,$$

$$J_{12} = -\frac{\xi(1 + \sigma M(x))x}{1 + ax}, \quad J_{13} = -\zeta x,$$

$$J_{21} = e \frac{\partial}{\partial x} \left(\frac{\xi(1 + \sigma M(x))xy}{1 + ax} \right),$$

$$J_{22} = e \frac{\xi(1 + \sigma M(x))x}{1 + ax} - \rho z - d_1, \quad J_{23} = -\rho y,$$

$$J_{31} = 0, \quad J_{32} = c\rho z, \quad J_{33} = c\rho y - d_2.$$

For simplicity, define the functional response:

$$F(x) = \frac{\xi(1 + \sigma M(x))x}{1 + ax}.$$

The derivatives then become more compact in the stability analysis.

$$E_0 = (0, 0, 0)$$

At E_0 , we compute:

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.$$

The eigenvalues are:

$$\alpha_1 = r, \quad \alpha_2 = -d_1, \quad \alpha_3 = -d_2.$$

Result: Since $r > 0$, the trivial equilibrium E_0 is always a saddle point and hence *unstable*.

$$E_1 = (K, 0, 0)$$

At E_1 , we obtain:

$$J(E_1) = \begin{pmatrix} -r & -F(K) & -\zeta K \\ 0 & eF(K) - d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.$$

The eigenvalues are:

$$\alpha_1 = -r, \quad \alpha_2 = eF(K) - d_1, \quad \alpha_3 = -d_2.$$

E_1 is locally asymptotically stable if and only if $eF(K) < d_1$. The intermediate predator persists only if its effective growth from prey consumption exceeds mortality.

$$E^* = (x^*, y^*, z^*)$$

To study the local behavior near the coexistence equilibrium E^* , we evaluate the Jacobian matrix $J(E)$ at E^* . The characteristic equation of $J(E^*)$ is obtained from:

$$\det(\alpha I - J(E^*)) = 0,$$

which results in a cubic polynomial of the form:

$$\alpha^3 + A_1\alpha^2 + A_2\alpha + A_3 = 0,$$

where the coefficients are directly related to the Jacobian entries:

$$A_1 = -(J_{11}^* + J_{22}^* + J_{33}^*),$$

$$A_2 = J_{11}^*J_{22}^* + J_{11}^*J_{33}^* + J_{22}^*J_{33}^* - J_{12}^*J_{21}^* - J_{23}^*J_{32}^*,$$

$$A_3 = J_{11}^*J_{22}^*J_{33}^* + J_{13}^*J_{21}^*J_{32}^* - J_{12}^*J_{21}^*J_{33}^* - J_{11}^*J_{23}^*J_{32}^*.$$

Stability condition (Routh–Hurwitz criteria)

The equilibrium E^* is locally asymptotically stable if the following four conditions are satisfied:

$$(1) \quad J_{11}^* + J_{22}^* + J_{33}^* < 0,$$

$$(2) \quad J_{11}^*J_{22}^* + J_{11}^*J_{33}^* + J_{22}^*J_{33}^* > J_{12}^*J_{21}^* + J_{23}^*J_{32}^*,$$

$$(3) \quad A_3 > 0,$$

$$(4) \quad A_1A_2 > A_3.$$

Condition (1) ensures that the self-regulations in total are dissipative. Condition (2) demands that stabilizing effects outweigh feedback interactions. Condition (3) ensures positivity of the interaction matrix determinant, whereas condition (4) ensures non-oscillatory behavior. These inequalities decide on the stability or instability of small fluctuations around equilibrium E^* .

It is clear from the analysis above that the parameters σ , a , k , and e are crucial for the stability of the equilibrium point E^* . The value of parameter σ determines how strong the eco-cognitive feedback is in the predation process. Increasing σ strengthens the behavioral feedback mechanism, suppresses population oscillations through a Hopf bifurcation and promotes stable coexistence of all interacting species. The value of parameter σ determines how strong the eco-cognitive feedback is in the predation process. An increase in this value can lead to increased predator responsiveness, which destabilizes the equilibrium and might cause oscillations as a result of bifurcation. The saturation factor a , that is, the factor related to handling limitations and consumption saturation, exerts a stabilizing influence in most cases, because higher values of a lower the effective rate of predation and decrease the impact of nonlinearity of the ecological process. The cognitive factor k determines the degree of sensitivity of predator cognition to changes in the prey density. Specifically, low values of k mean fast cognition and strong ecological feedback, potentially promoting instability, whereas high values of k exert restraining influence. In addition, the conversion factors e and c determine the velocity at which biomass flows through trophic links, influencing predator survival and the overall rate of energy flow within the ecosystem. Lowering mortality improves the survival of all interacting species, but it also promotes trophic interaction, which in turn causes the system to become unstable.

6. Global stability analysis

In this section, we investigate the global asymptotic behavior of the system around each equilibrium point. Unlike local stability, global stability ensures convergence from any initial condition in the feasible region $\mathbb{R}_+^3$.

6.1. Global stability of boundary equilibrium $E_1 = (K, 0, 0)$

Theorem 3. *If*

$$e \frac{\xi(1+\sigma)K}{1+aK} < d_1,$$

then the boundary equilibrium E_1 is globally asymptotically stable in $\mathbb{R}_+^3 \setminus \{y = z = 0\}$.

Proof. Consider the Lyapunov function:

$$V(x, y, z) = (x - K \ln x) + Ay + Bz,$$

where $A, B > 0$ are constants.

Then:

$$\dot{V} = \frac{dx}{dt} \left(1 - \frac{K}{x}\right) + A \frac{dy}{dt} + B \frac{dz}{dt}.$$

Using the system dynamics and $x \leq K$, we obtain:

$$\dot{V} \leq (eF(K) - d_1)Ay - d_2Bz.$$

Under the condition $eF(K) < d_1$, we have $\dot{V} \leq 0$, and equality holds only at E_1 .

Thus, by LaSalle's invariance principle, E_1 is globally stable. $\square$

6.2. Global stability of interior equilibrium E^*

Global stability of the coexistence equilibrium is more restrictive.

Theorem 4. *The interior equilibrium E^* is globally asymptotically stable if: (i) system is uniformly persistent, (ii) damping dominates cognitive feedback, and the following condition holds:*

$$d_1 + d_2 > \frac{\xi(1+\sigma)}{1+aK}K.$$

Proof. Define the Lyapunov function:

$$\begin{aligned} V(x, y, z) = & (x - x^* - x^* \ln \frac{x}{x^*}) \\ & + (y - y^* - y^* \ln \frac{y}{y^*}) \\ & + (z - z^* - z^* \ln \frac{z}{z^*}). \end{aligned}$$

This function is positive definite and radially unbounded.

Differentiating along trajectories gives:

$$\dot{V} = \left(1 - \frac{x^*}{x}\right) \frac{dx}{dt} + \left(1 - \frac{y^*}{y}\right) \frac{dy}{dt} + \left(1 - \frac{z^*}{z}\right) \frac{dz}{dt}.$$

Substituting the system equations and using equilibrium relations, we obtain cancellation of nonlinear terms, leading to:

$$\dot{V} \leq -\eta_1(x - x^*)^2 - \eta_2(y - y^*)^2 - \eta_3(z - z^*)^2,$$

where $\eta_1, \eta_2, \eta_3 > 0$ depend on:

$$r, K, \xi, a, \sigma, k, e, c, d_1, d_2, \zeta.$$

Hence:

$$\dot{V} < 0 \quad \text{for all } (x, y, z) \neq (x^*, y^*, z^*),$$

which implies global asymptotic stability by LaSalle's invariance principle. $\square$

In the context of global stability, it is evident that the dynamic properties of the ecological network depend on the relationship among the level of trophic interactions, mortality, and eco-cognitive feedback. In this regard, when the predator's ability to consume prey is not high enough—whether due to poor predation ability or low eco-cognitive feedback—the prey-only fixed point becomes dominant, leading to the extinction of predators and survival of prey at their carrying capacity. Alternatively, if predation is present in the ecological network but not sufficiently high enough to compensate for the mortality of predators, then the boundary fixed point becomes globally stable, which implies partial persistence of low trophic levels while high predators cannot sustain themselves. On the other hand, the fixed point of coexistence can achieve global stability if the values of the parameters are such that the dissipative forces and natural mortality of the ecological network are strong enough to mitigate the amplification induced by the cognitive feedback effect. This observation suggests that eco-cognitive feedback plays a vital role in stabilizing or destabilizing the ecological network.

7. Sensitivity analysis

Sensitivity analysis is performed to determine how variations in system parameters affect the interior equilibrium state $E^* = (x^*, y^*, z^*)$. This allows identification of the most influential biological and cognitive factors governing the long-term behavior of the system.

7.1. Normalized sensitivity index

For any parameter $p \in \{r, K, \xi, a, \sigma, k, e, c, d_1, d_2, \zeta\}$, we define the normalized sensitivity index of a state variable $u \in \{x^*, y^*, z^*\}$ as:

$$S_u^p = \frac{\partial u^*}{\partial p} \cdot \frac{p}{u^*}.$$

This quantity measures the relative change in u^* induced by a relative change in p .

7.2. Implicit differentiation framework

Since the equilibrium E^* is defined implicitly by the nonlinear system:

$$F_1(x^*, y^*, z^*) = 0, \quad F_2(x^*, y^*, z^*) = 0,$$

$$F_3(x^*, y^*, z^*) = 0,$$

we apply implicit differentiation with respect to a parameter p :

$$J(E^*) \begin{pmatrix} \frac{\partial x^*}{\partial p} \\ \frac{\partial y^*}{\partial p} \\ \frac{\partial z^*}{\partial p} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial p} \\ \frac{\partial F_2}{\partial p} \\ \frac{\partial F_3}{\partial p} \end{pmatrix},$$

where $J(E^*)$ is the Jacobian matrix evaluated at equilibrium.

Thus,

$$\begin{pmatrix} \frac{\partial x^*}{\partial p} \\ \frac{\partial y^*}{\partial p} \\ \frac{\partial z^*}{\partial p} \end{pmatrix} = -J(E^*)^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial p} \\ \frac{\partial F_2}{\partial p} \\ \frac{\partial F_3}{\partial p} \end{pmatrix}.$$

7.3. Sensitivity with respect to key parameters

7.3.1. Cognitive parameter σ

The parameter σ directly controls eco-cognitive amplification. Since σ appears only in the functional response term, we obtain:

$$\frac{\partial F(x)}{\partial \sigma} = \frac{\xi M(x)x}{1 + ax}.$$

Hence,

$$\frac{\partial E^*}{\partial \sigma} = -J(E^*)^{-1} \begin{pmatrix} -\frac{\xi M(x^*)x^*y^*}{1+ax^*} \\ e^{\frac{\xi M(x^*)x^*y^*}{1+ax^*}} \\ 0 \end{pmatrix}.$$

This shows that σ has a direct destabilizing or amplifying effect on both prey depletion and predator growth.

7.3.2. Cognitive saturation parameter k

Since

$$M(x) = \frac{x}{k + x},$$

we have:

$$\frac{\partial M}{\partial k} = -\frac{x}{(k + x)^2}.$$

Thus increasing k reduces cognitive response strength, leading to weaker predation efficiency and potentially higher prey persistence.

7.3.3. Handling parameter a

The parameter a appears in the denominator of the functional response. Its sensitivity contribution is:

$$\frac{\partial F}{\partial a} = -\frac{\xi(1 + \sigma M(x))x^2y}{(1 + ax)^2}.$$

An increase in the saturation parameter a will destabilize the predatory behavior by creating stronger handling constraints and reducing

the consumption rate in response to an increase in prey density, thus promoting prey survival and facilitating stabilization of the whole ecosystem. Generally, the sensitivity analysis highlights the importance of the eco-cognitive structure in terms of its impact on the dynamics of the model system, where the cognitive amplification parameter σ is found to be the major destabilizing force owing to its capability to enhance trophic dynamics by inducing nonlinearity in animal behavior and increasing feedback strength within the trophic chain. On the other hand, the saturation parameter a is the stabilizing factor, as it decreases the strength of the predation and weakens the interaction. Threshold k controls the susceptibility of predator cognition to changes in prey density, thus determining the switching between the low and high activation regimes of predator behavior, with lower threshold values resulting in higher rates of cognitive activation and feedback. Meanwhile, the mortality factors d_1 and d_2 mainly control predator survival and demographic renewal. However, their impact on oscillatory transitions and bifurcations is less significant compared to that of the cognitive factors. In general, the analysis demonstrates that eco-cognitive feedback plays a leading role in parameter sensitivity and dynamical regimes of the model.

8. Hopf bifurcation analysis

We investigate the effect of the eco-cognitive parameter σ on the dynamical behavior of the interior equilibrium E^* . In particular, we study conditions under which a Hopf bifurcation occurs, leading to the emergence of periodic oscillations.

Let $J(E^*)$ denote the Jacobian matrix evaluated at the interior equilibrium. The characteristic equation is given by:

$$\det(\alpha I - J(E^*)) = 0,$$

which can be written as:

$$\alpha^3 + A_1(\sigma)\alpha^2 + A_2(\sigma)\alpha + A_3(\sigma) = 0,$$

where the coefficients depend continuously on the parameter σ .

In particular:

$$A_1(\sigma) = -\text{tr}(J(E^*)),$$

$$A_2(\sigma) = \sum \text{principal minors of order 2},$$

$$A_3(\sigma) = -\det(J(E^*)).$$

A Hopf bifurcation occurs when the characteristic equation admits a pair of purely imaginary roots $\alpha = \pm i\omega$ ($\omega > 0$).

Substituting $\alpha = i\omega$ yields:

$$-i\omega^3 - A_1\omega^2 + iA_2\omega + A_3 = 0.$$

Separating real and imaginary parts gives:

$$\begin{cases} A_3 - A_1\omega^2 = 0, \\ A_2\omega - \omega^3 = 0. \end{cases}$$

From the second equation, we obtain:

$$\omega^2 = A_2(\sigma).$$

Substituting into the first equation yields the critical condition:

$$A_3(\sigma) = A_1(\sigma)A_2(\sigma).$$

Thus, the system undergoes a Hopf bifurcation at $\sigma = \sigma_c$ if the following conditions hold:

$$\begin{aligned} A_1(\sigma_c)A_2(\sigma_c) - A_3(\sigma_c) &= 0, \\ A_2(\sigma_c) &> 0, \\ A_1(\sigma_c) &> 0. \end{aligned}$$

The parameter σ enters the system only through the cognitive functional response:

$$F(x, \sigma) = \frac{\xi(1 + \sigma M(x))x}{1 + ax}, \quad M(x) = \frac{x}{k + x}.$$

Hence, variation with respect to σ affects only the predation term.

At the interior equilibrium $E^* = (x^*, y^*, z^*)$, the transversality condition for Hopf bifurcation reduces to:

$$\frac{\xi x^* y^*}{1 + ax^*} \cdot \frac{x^*}{k + x^*} \cdot \Delta(x^*, y^*, z^*; r, K, a, e, c, d_1, d_2, \zeta) \neq 0$$

where Δ is the combined feedback response of the three trophic levels:

$$\Delta = e \left(\frac{\partial \frac{dy}{dt}}{\partial F} \right) + \left(\frac{\partial \frac{dx}{dt}}{\partial F} \right) + c \left(\frac{\partial \frac{dz}{dt}}{\partial F} \right),$$

with explicit parameter structure:

$$\Delta = e \frac{x^*}{1 + ax^*} - x^* - \zeta x^*.$$

Therefore, the Hopf bifurcation is non-degenerate if:

$$\frac{\xi(x^*)^2 y^*}{(1 + ax^*)(k + x^*)} \left[e \frac{1}{1 + ax^*} - 1 - \zeta \right] \neq 0.$$

The Hopf bifurcation criterion shows that the change from stable coexistence to persistent oscillations is basically driven by the equilibrium population size of the prey x^* , the predator of intermediate trophic level y^* , the parameters of cognitive saturation a and k , and the interaction strengths within the ecosystem e , ζ , and ξ .

All these parameters together decide the extent of the influence of stabilizing ecological dissipation and the destabilizing nonlinear feedback due to trophic and eco-cognitive interaction. Specifically, high cognitive sensitivity, measured using high ξ and enhanced eco-cognitive interaction controlled by σ (in turn via the influence of x^*), makes predators more sensitive and increases the strength of nonlinear feedback, resulting in periodic oscillations. Similarly, high predation pressure by the top predator, indicated by ζ , can change the balance among the trophic level and increase indirect feedbacks within the species hierarchy, leading to instability. On the other hand, a higher value of the parameter a , which represents handling time, decreases the effectiveness of predation by means of saturation effects, weakens the intensity of interaction, and thus inhibits oscillations due to increased stability of the model. Therefore, the occurrence of a Hopf bifurcation can be attributed to a complex relation between population sizes, interaction strengths, and cognitive saturation effects.

8.1. Dynamical interpretation

When the eco-cognitive amplification parameter meets $\sigma < \sigma_c$, the real part of the eigenvalue related to the linearized system is negative, which indicates that the equilibrium point E^* of the coexistence state is locally asymptotically stable. As a result, small perturbation around the equilibrium will die out, allowing the system to converge back to its stable equilibrium. Once $\sigma = \sigma_c$, there is a pair of complex eigenvalues that crosses the imaginary axis. The system loses its stability, and thus a Hopf bifurcation occurs. After crossing the critical value, that is, when $\sigma > \sigma_c$, the equilibrium loses stability, and there will exist a periodic orbit close to the equilibrium point. Ecologically, since σ represents the level of eco-cognitive amplification in predator behavior, the bifurcating behavior shows that low-level cognition causes coexistence between the species to be stable, medium levels of cognition cause cyclic predator-prey oscillations to emerge due to increased nonlinearity, while high cognition causes the system to amplify the trophic feedback to such an extent that it disrupts equilibrium behavior. Therefore, cognition is seen as a major factor causing bifurcations in the ecological system through the use of nonlinear feedback behavior.

The model incorporates a new ecological dynamic through instantaneously adjusting the efficacy of predation based on prey abundance using a bounded cognitive activation procedure for the predator. In contrast to classical predator-prey

models, in which the origin of nonlinear complexity is associated with the internal structure of predator-prey relations, the current model adds an extra degree of dynamical complexity owing to the presence of behavioral adaptation mechanisms integrated directly into the functional response. In other words, the role of cognition now becomes a dynamic ecological factor, with the potential to stabilize or destabilize the ecological network depending on whether or not it moderates adaptive behavior and its resultant nonlinear dynamics. Furthermore, minor changes in the value of eco-cognitive amplification parameter σ can greatly affect the dynamics of the system and produce qualitative bifurcations. The above model also shows how the level of cognitive activation determines the amount of prey in such a way that the amount of prey indirectly affects the level of intelligence of the predators by influencing the level of cognitive activation. In other words, the model explains how predator-prey relations will be dependent on behavior rather than density alone. As a result, cognitive processes can spread through the food chain and therefore have an impact on the whole ecosystem. The model gives mathematical connections between ecological and cognitive ecology models.

9. Numerical simulation

The dynamical behaviour of the model (1) has been studied using MATLAB software for the different values of the model parameters. **Figure 1** shows that the interior equilibrium point

(19.06, 0.72, 0.61) of the prey-intermediate predator-top predator model exists and is locally asymptotically stable. This result indicates that the prey, intermediate predator, and top predator can coexist permanently in the ecosystem at the equilibrium state (19.06, 0.72, 0.61). The local asymptotic stability of the interior equilibrium point implies that small disturbances in the population densities will gradually diminish over time, and the system will return to its stable coexistence state. Hence, the ecological interactions among the three species maintain a balanced and sustainable population structure.

The bifurcation analysis of model (1) reveals that the intrinsic growth rate r of the prey population plays a significant role in determining the dynamical behaviour of the ecosystem in **Figure 2**. The system undergoes a Hopf bifurcation as r increases, indicating a transition from oscillatory dynamics to stable coexistence. For lower values of r , i.e., $1 \leq r < 1.462$, the prey and intermediate predator populations exhibit persistent oscillations, while the top predator becomes extinct when $1 \leq r < 1.28$. As r further increases within $1.28 < r < 1.462$, the top predator also survives but shows oscillatory behaviour along with the other species. However, for higher values of r , i.e., $1.462 < r \leq 2.4$, all three species attain stable steady state coexistence. Biologically, this suggests that a sufficiently high intrinsic growth rate of the prey species promotes stability and persistence of the entire food chain.

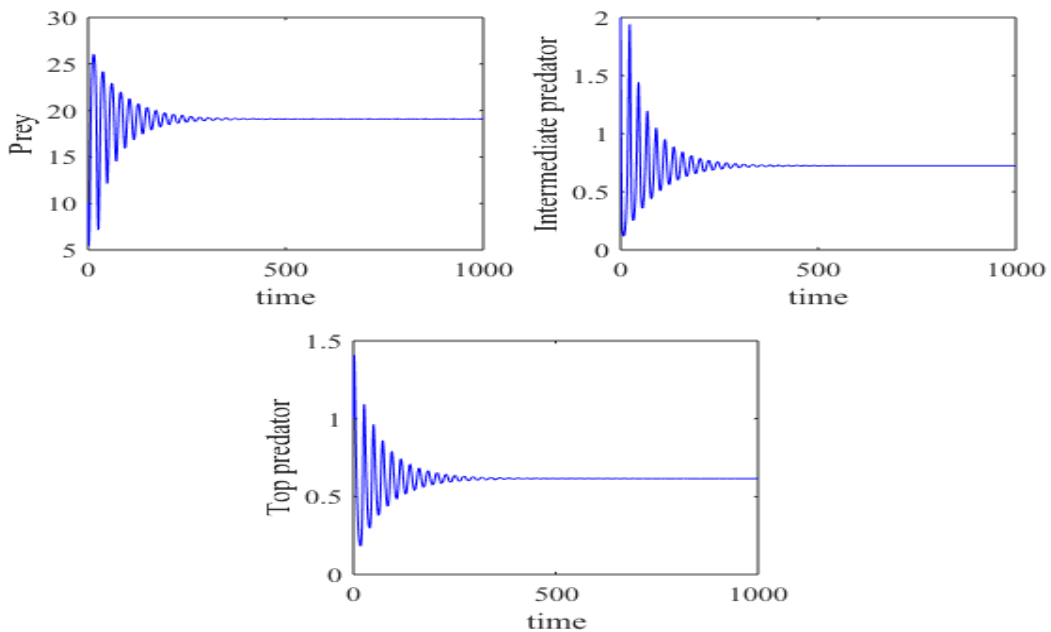


Figure 1. Local stability of model (1) around the interior equilibrium point for the set of parametric values $r = 1.5, K = 30, k = 4.5, \xi = 1.35, a = 0.18, e = 0.1, \zeta = 0.5, \rho = 0.5, c = 0.58, d_1 = 0.32, d_2 = 0.21, \sigma = 0.1$

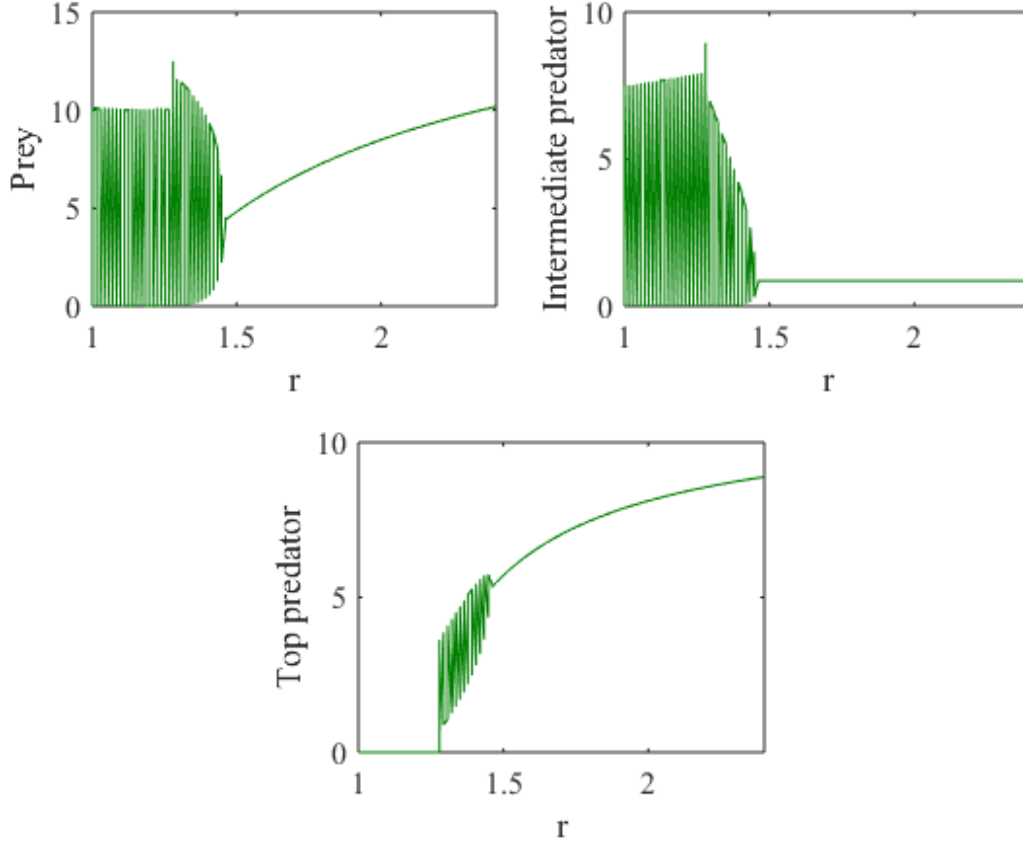


Figure 2. Bifurcation diagram of model (1) with respect to r for the set of parametric values $K = 18, k = 4.5, \xi = 1.35, a = 0.18, e = 0.62, \zeta = 0.055, \rho = 0.42, c = 0.58, d_1 = 0.32, d_2 = 0.21, \sigma = 0.5$

The bifurcation analysis of model (1) with respect to the attack rate of prey (ξ) in **Figure 3** demonstrates that this parameter has a crucial influence on the stability of the ecosystem. The system undergoes a Hopf bifurcation as (ξ) varies from 0.3 to 1.5, indicating a transition from oscillatory dynamics to stable coexistence. For lower values of the attack rate ($0.3 \leq \xi < 0.564$), the prey, intermediate predator, and top predator populations exhibit sustained oscillations, reflecting instability in the ecological interactions. However, when the attack rate exceeds the critical threshold ($0.564 < \xi \leq 1.5$), all species attain stable steady state behaviour. This result suggests that an increased attack efficiency of the prey-related interaction can stabilize the food chain and support long-term coexistence of all species.

The bifurcation analysis of model (1) with respect to the level of cognitive amplification (σ) in **Figure 4** indicates that this parameter significantly affects the dynamical stability of the ecosystem. The system undergoes a Hopf bifurcation as (σ) varies from 0.5 to 4, showing a transition from oscillatory dynamics to stable coexistence. For lower values of cognitive amplification ($0.5 \leq \sigma < 2.0$), the prey, intermediate predator,

and top predator populations exhibit sustained oscillations, indicating instability in the population interactions. However, for higher values of σ i.e., ($2 < \sigma \leq 4$), all species attain stable steady state behaviour. Biologically, this suggests that an increased level of cognitive amplification enhances the stabilizing effect in the ecosystem and promotes long-term coexistence among all interacting species.

The bifurcation analysis of model (1) with respect to the loss of prey due to the attack of the top predator (ζ) in **Figure 5** shows that this parameter strongly influences the stability of the ecosystem. The system undergoes a Hopf bifurcation as (ζ) varies from 0.9 to 2.0. It is observed that all three species attain stable steady state coexistence for lower values of ζ i.e., ($0.9 \leq \zeta < 1.087$). However, when ζ exceeds the critical threshold ($1.087 < \zeta \leq 2$), the prey, intermediate predator, and top predator populations exhibit oscillatory behaviour. This result suggests that an increased predation pressure of the top predator on the prey population may destabilize the ecosystem and lead to persistent population fluctuations.

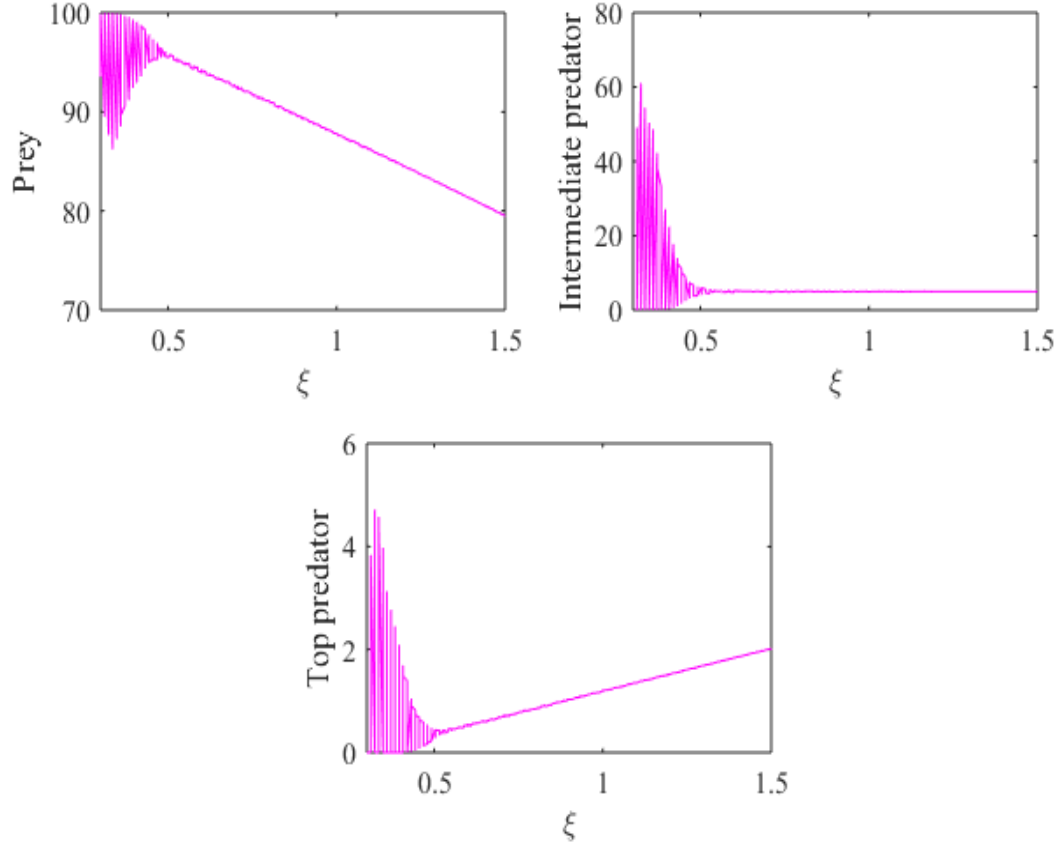


Figure 3. Bifurcation diagram of model (1) with respect to ξ for the set of parametric values $r = 1.5, K = 100, k = 100, \sigma = 0.3, a = 1.0, \zeta = 0.1, e = 0.3, \rho = 0.2, d_1 = 0.1, c = 0.1, d_2 = 0.1$.

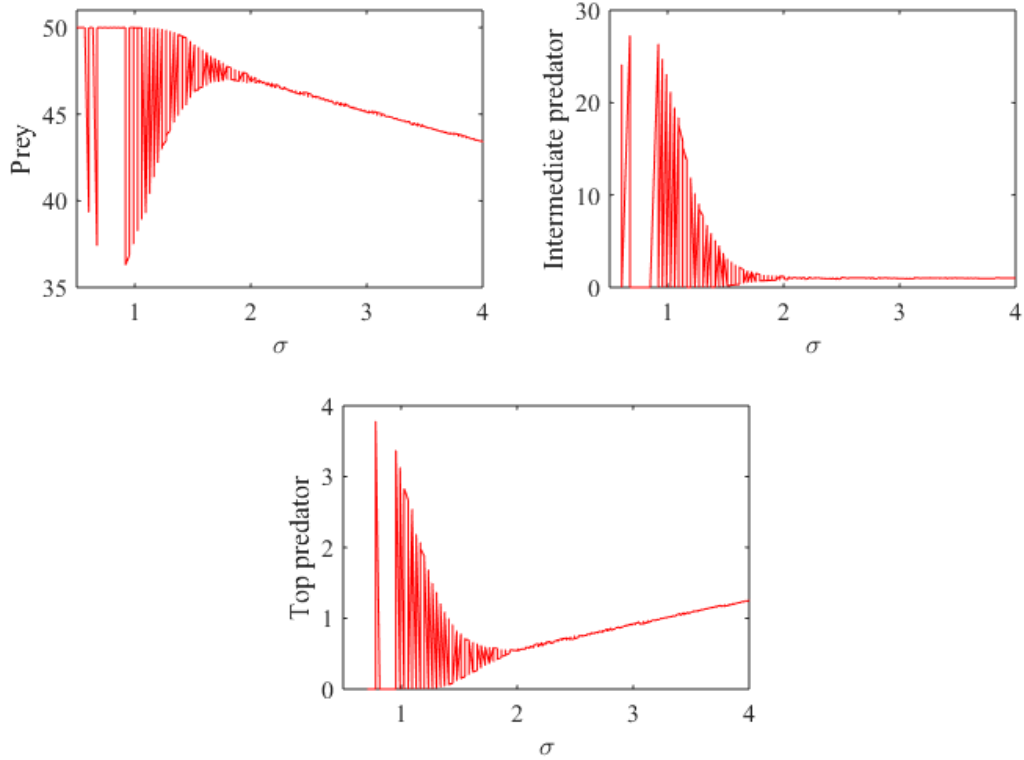


Figure 4. Bifurcation diagram of model (1) with respect to σ for the set of parametric values $r = 2.0, K = 50, k = 50, \xi = 0.3, a = 1.5, \zeta = 0.2, e = 0.4, \rho = 0.1, d_1 = 0.1, c = 0.1, d_2 = 0.01$.

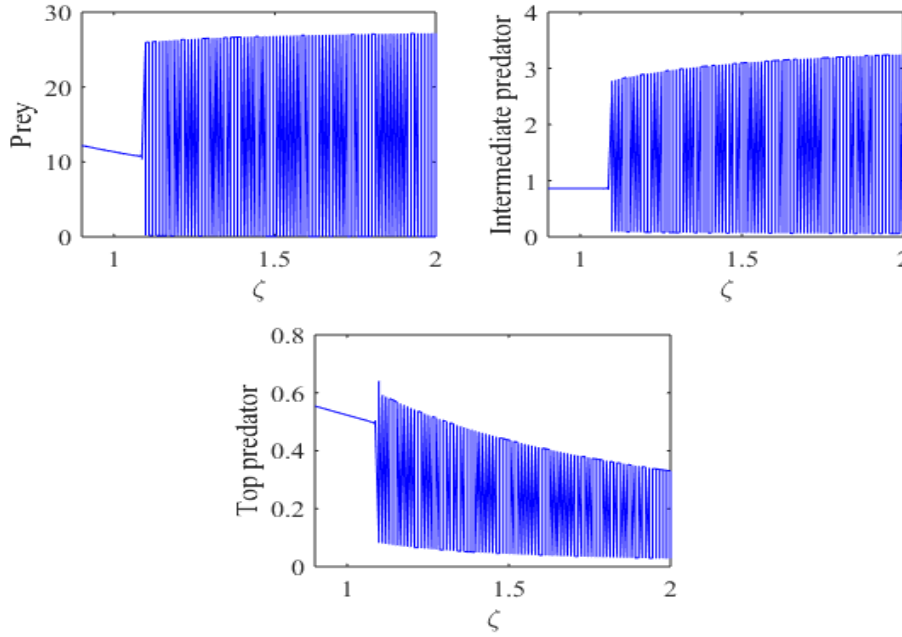


Figure 5. Bifurcation diagram of model (1) with respect to ζ for the set of parametric values $r = 1.5, K = 30, k = 4.5, \xi = 1.35, a = 0.18, e = 0.1, \rho = 0.42, c = 0.58, d_1 = 0.32, d_2 = 0.21, \sigma = 0.1$

The above bifurcation analysis indicates that the conversion efficiency of prey into predator biomass e , plays a crucial role in determining the long-term dynamics of the ecosystem in **Figure 6**. For lower values of e , the system tends to attain a stable steady state where the prey population persists, while the intermediate and top predators may become extinct due to insufficient energy transfer from prey consumption. As the value of e increases beyond certain threshold levels, both predator populations are able to survive, but the system loses stability through Hopf bifurcation and exhibits persistent oscillations. Biologically, this implies that a higher conversion rate enhances predator growth and strengthens predator-prey interactions, which destabilizes the ecosystem and generates cyclic fluctuations in all species populations. Thus, the parameter e acts as a key regulatory factor controlling species persistence, extinction, and the transition from stable coexistence to oscillatory dynamics.

The bifurcation analysis with respect to the predation rate of the top predator on the intermediate predator, In **Figure 7**, (ρ) reveals that this parameter significantly influences the stability and persistence of the ecosystem. For lower values of ρ , the prey and intermediate predator populations exhibit sustained oscillatory dynamics, while the top predator becomes extinct due to insufficient predation efficiency. As ρ crosses the critical threshold value ($\rho = 0.356$), the system undergoes a Hopf bifurcation, leading to the disappearance of oscillations and the emergence

of a stable steady state. Beyond this threshold, all three species coexist stably in the ecosystem. Biologically, this result suggests that an increased consumption rate of the intermediate predator by the top predator strengthens trophic regulation, suppresses population fluctuations, and promotes stable coexistence among species. Thus, the parameter (ρ) acts as an important stabilizing factor in the food chain dynamics.

In **Figure 8**, The bifurcation analysis with respect to the conversion efficiency of the intermediate predator c , demonstrates that this parameter has a strong influence on the stability and persistence of the ecological system. For lower values of c , the prey and intermediate predator populations exhibit oscillatory or unstable dynamics, while the top predator becomes extinct due to inadequate energy conversion and insufficient population growth. As the value of c exceeds the critical threshold near $c = 0.415$, the system undergoes a Hopf bifurcation, after which the oscillations disappear and the populations attain stable steady state behaviour. In this region, all three species coexist persistently in the ecosystem. Biologically, this indicates that a higher conversion efficiency of the intermediate predator improves energy transfer within the food chain, enhances the survival of the top predator, and stabilizes the overall ecosystem dynamics. Hence, the parameter c acts as an important stabilizing factor that governs species coexistence and the transition from unstable oscillations to stable equilibrium behaviour.

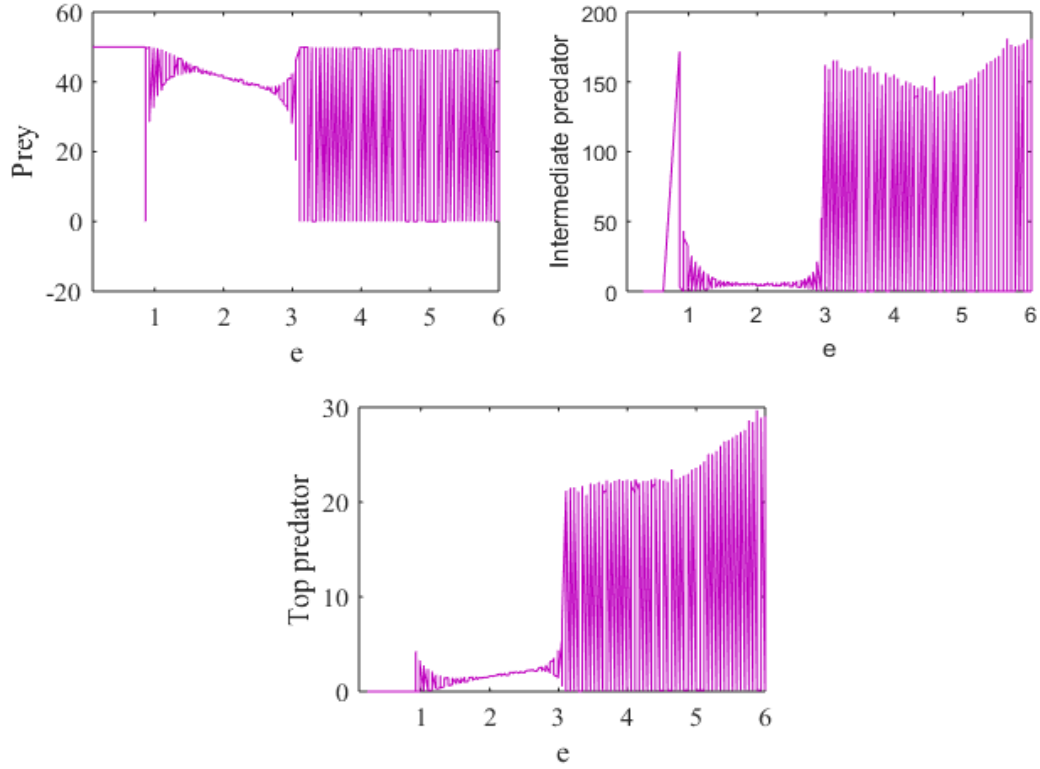


Figure 6. Bifurcation diagram of model (1) with respect to e for the set of parametric values $r = 2.0, K = 50, k = 50, \xi = 0.1, \sigma = 0.2, a = 0.5, \zeta = 0.2, \rho = 0.2, d_1 = 0.1, c = 0.1, d_2 = 0.1$

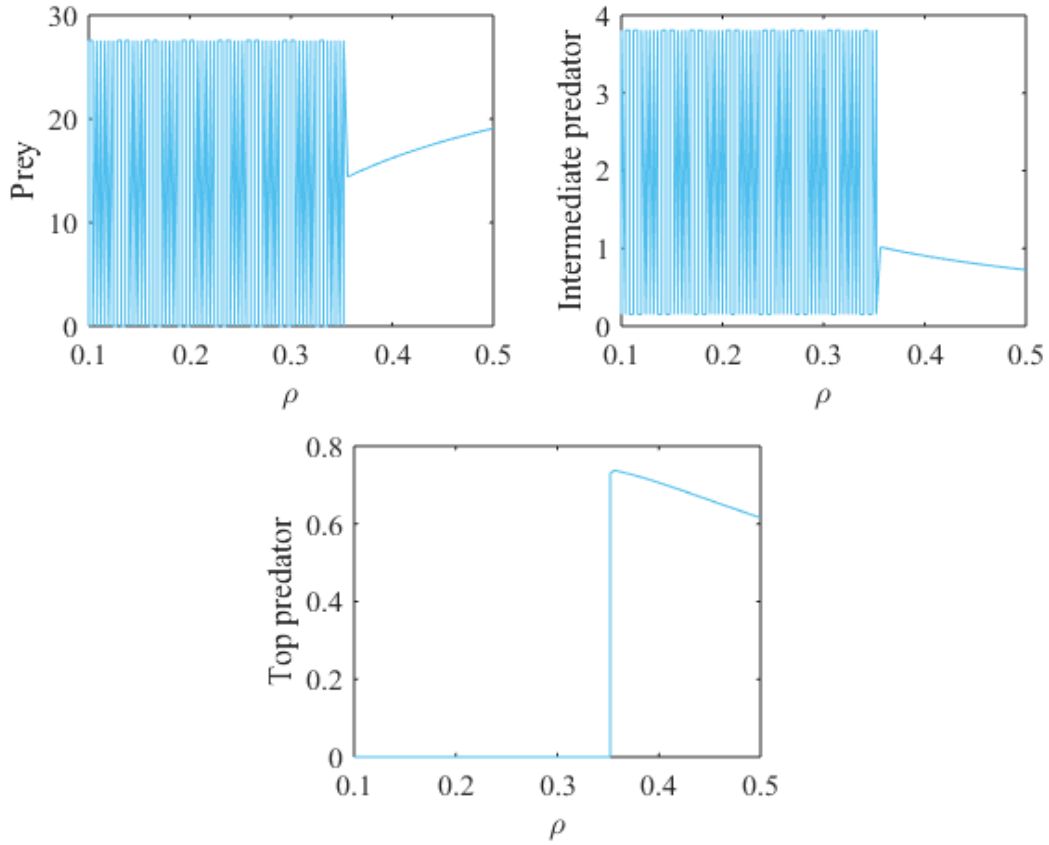


Figure 7. Bifurcation diagram of model (1) with respect to ρ for the set of parametric values $r = 1.5, K = 30, k = 4.5, \xi = 1.35, a = 0.18, e = 0.1, \zeta = 0.5, c = 0.58, d_1 = 0.32, d_2 = 0.21, sigma = 0.1$

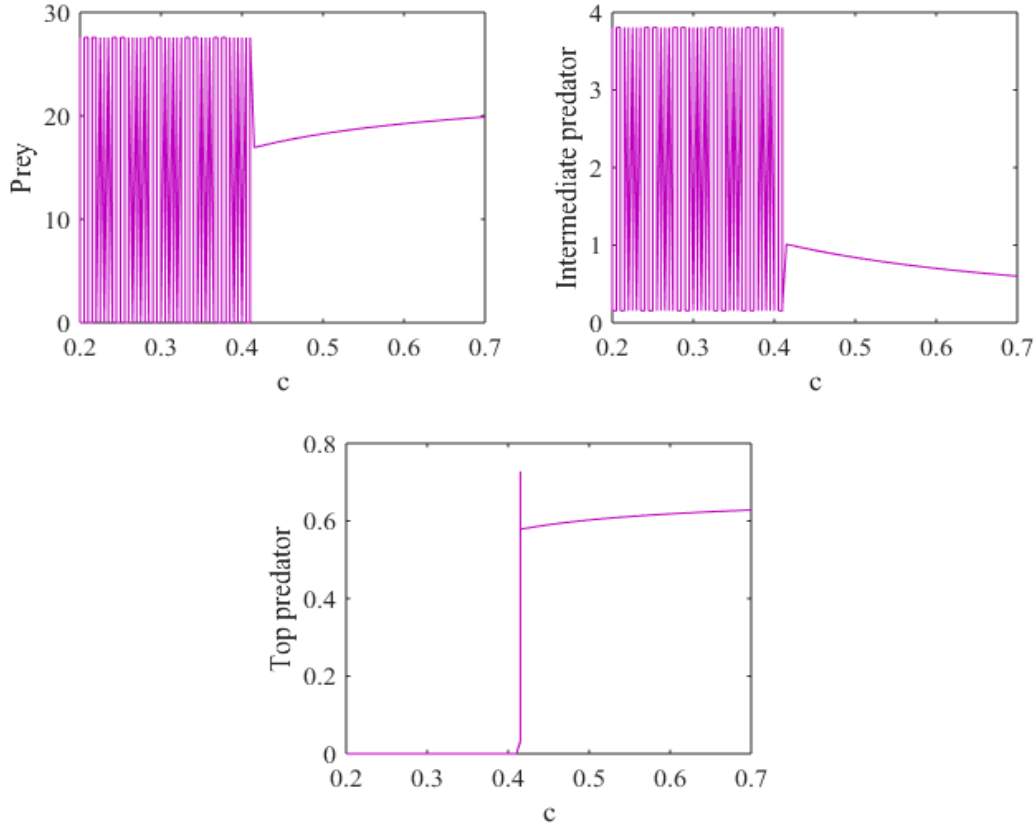


Figure 8. Bifurcation diagram of model (1) with respect to c for the set of parametric values $r = 1.5, K = 30, k = 4.5, \xi = 1.35, a = 0.18, e = 0.1, \zeta = 0.5, \rho = 0.5, d_1 = 0.32, d_2 = 0.21, \sigma = 0.1$

10. Conclusion and future work

In the current study, a new mathematical model describing a three species predator–prey system with instantaneous eco-cognitive modulation in the predation process was developed and analyzed. Unlike traditional trophic systems in which the interaction rate is determined only by the density of the interacting populations, in the current system, predator efficiency is dynamically affected by the degree of cognitive activation triggered by the prey. This is achieved by considering a nonlinear bounded cognitive function in the functional response, thus enabling the interactions to be modulated by adaptive behaviors.

The current model incorporates logistic prey growth, nonlinear trophic interaction, and eco-cognitive effect into a common dynamical system. Basic mathematical properties of the system such as positivity, boundedness, existence and stability of equilibria, sensitivity behavior, and conditions for Hopf bifurcation were thoroughly analyzed. It is shown that the cognitive amplification parameter plays a crucial role in the dynamics of the

system in question. Namely, low cognitive amplification leads to stable coexistence of species, while high cognitive amplification may destabilize the equilibrium point leading to Hopf bifurcation.

From the obtained results, it is clear that cognition can play an important role in modifying trophic balance through increasing the nonlinearity of the interactions between species. From the sensitivity analysis, it was evident that the model was very sensitive to changes in the values of the cognitive amplification and saturation parameters, stressing the role of behavioral adjustment in ecological dynamics. Also, numerical investigations of parameter regimes have revealed the presence of a variety of dynamical phenomena, such as stable equilibrium points and periodic oscillations.

In conclusion, this work has succeeded in creating a mathematical link between nonlinear ecological dynamics and cognitive ecology.

10.1. Future work

Some possible extensions of the current model

that could be addressed in future research include the following. First, the effect of time delay could be included in the modeling of the cognitive response to obtain a more accurate representation of memory and delayed behavioral change. Second, stochasticity could be introduced into the system to assess the impact of environmental fluctuations and random disturbances on eco-cognitive dynamics. Third, spatial diffusion and reaction–diffusion models could be considered to analyze patterns in behaviorally structured ecosystems. In addition, the introduction of prey refuge, fear, harvesting, or stage structure could enhance the biological realism of the current model while giving rise to more complicated dynamics. Mathematically, future research directions might involve chaos detection, global bifurcations, fractional-order modeling, and optimal control related to eco-cognitive ecological systems. Finally, calibration of the model based on ecological or behavioral data would certainly enhance its practical value.

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Conflict of interest

The authors declare that they have no competing interests.

Availability of data

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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