

ARTICLE

Sensitivity issue for surface seismic full-waveform inversion in acoustic vertical transverse isotropic media

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Abstract

In multi-parameter anisotropic full-waveform inversion (FWI) for surface seismic data, both the far-offset and near-offset data are required to obtain accurate estimations of anisotropic properties. However, in acoustic transversely isotropic media with a vertical symmetry axis (vertical transverse isotropy [VTI]), the near-offset data are insensitive to the high-resolution features of the two anisotropy parameters ϵ and δ in the typical parameterization (v_p, ϵ, δ) , which results in the corresponding low-resolution reconstruction in acoustic anisotropic FWI. To improve performance in surface seismic FWI, we propose a novel parameterization for acoustic VTI media using the Thomsen parameters ϵ and δ , along with a relevant parameter C_s (associated with the combination of vertical velocity and the Thomsen parameters). Using the Born approximation, we derive analytical expressions for the radiation patterns under this new parameterization and compare them with those under different parameterizations. The radiation patterns are imperative for indicating the angular influence of the perturbation on the parameters. This indicates that introducing the new parameter C_s is crucial for improving the contributions of ϵ and δ to near-offset data. We also derive the corresponding three-dimensional gradients and adjoint wave equations for parameterization (C_s, ϵ, δ) in acoustic VTI media. Furthermore, we develop multi-parameter point spread functions to numerically evaluate sensitivity during inversion. As a result, we can perform acoustic VTI FWI using the parameterization (C_s, ϵ, δ) , which is optimal for inverting both the low- and high-wavenumber components of the parameters $(C_s, \epsilon, \text{ and } \delta)$. Finally, we apply the synthetic data to test the accuracy and effectiveness of our multi-parameter acoustic VTI FWI algorithm. The results show that the proposed parameterization has the potential to invert near-offset records to high-resolution anisotropy parameters in an active-source seismic experiment.

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1. Introduction

Full-waveform inversion (FWI) produces high-resolution subsurface models by minimizing the misfit between seismic records and predicted data.^{1,2} However, the traditional isotropic assumption often fails to match near- and far-offset data simultaneously in wide-aperture seismic inversion.³ Seismic anisotropy describes the dependence of seismic velocity on propagation angle and is widely observed in exploration areas.^{4,5} Acoustic vertical transverse isotropy (VTI) FWI yields better results and is expected to produce anisotropic models.⁶⁻¹² FWI easily falls into local minima because the misfit function and model parameters are nonlinearly related.¹³ Consequently, an accurate initial model and low-frequency record are needed to avoid cycle-skipping. To reduce FWI's dependence on low frequencies, several approaches have guided the inversion toward a global minimum, such as multi-scale inversion,^{14,15} envelope misfit functions,¹⁶⁻¹⁸ and Laplace-domain inversion.¹⁹

Conventional surface seismic FWI cannot invert for all medium parameters because of their limited contributions to surface seismic records and parameter trade-offs. Thus, the main challenge in anisotropic parameter inversion is calculating the effective parameter gradient, which depends strongly on parameter sensitivities. Research has focused on acoustic VTI FWI based on sensitivity analysis. Radiation patterns provide meaningful descriptions of data sensitivities to each parameter, with variable scattering angles and the feasibility of multi-parameter inversion.²⁰ Furthermore, the multi-parameter point spread function (MPSF) can directly illustrate parameter sensitivities and solvability in a complex model.²¹ The sensitivities depend on the parameterization that is adopted to characterize the underground media. Commonly used parameterizations include the combination of normal moveout velocity v_n and anisotropy parameters δ and η , and the combination of horizontal velocity v_h and anisotropic parameters δ and ϵ , both of which are considered reliable for seismic data inversion.^{7,13,22,23} Based on sensitivity analysis, multi-parameter inversion in acoustic VTI media requires an appropriate parameterization and inversion strategy.^{8-10,12,13,23,24} Most current methods reduce parameter trade-offs by choosing a parameterization with less overlap in radiation patterns.^{7,13,22,25-28} However, inversion strategies that decompose parameter updates by scattering angle always introduce angular dependence, which limits the resolvability of different wavelength components in the model space and the capability for deep-layer inversion. Meanwhile, another study shows that one can use a parameterization with a full-angle-sensitive parameter, despite cross-talk during inversion.²⁹ Despite all these

contributions, a desirable parameterization that enables simultaneous retrieval of the low- and high-wavenumber components of the anisotropic parameter models remains an unsolved problem in FWI. Thus, our objective is to improve the sensitivity of anisotropic parameters in a surface observation system.

To obtain accurate multi-parameter inversion results for acoustic VTI media, we propose a new parameterization that can significantly improve the sensitivity of anisotropic parameters at near-offset and small scattering angles. It enables high-resolution estimation of anisotropic parameters from surface data. We derive the gradient formulas for multiple parameters in the proposed parameterization using the adjoint-state method. We derive radiation patterns with respect to the scattering angle for acoustic VTI media using the Born approximation to intuitively show parameter sensitivities. The MPSFs are displayed to show the effectiveness of the new parameterization in calculating gradients during FWI. Applications on synthetic data show that the proposed parameterization improves inversion results in multi-parameter FWI for acoustic VTI media compared with conventional parameterizations.

2. Full-waveform inversion for (C_s, ϵ, δ) parameterization

Using the acoustic VTI approximation, the density-normalized elastic stiffness moduli can be described in terms of the vertical velocity and the Thomsen parameters δ and ϵ .³⁰⁻³² The stiffness moduli C_{11} , C_{13} , and C_{33} are related to these quantities by (Equation 1):

$$\begin{aligned} C_{11} &= (1 + 2\epsilon)v_p^2 \\ C_{33} &= v_p^2 \\ C_{13} &= \sqrt{1 + 2\delta}v_p^2 \end{aligned} \quad (1)$$

The parameters C_{11} and C_{13} are associated with the properties of horizontal propagation, and C_{33} is the medium parameter governing the qP-wave propagation in the vertical direction. To use the scattered wavefield at small and large scattering angles simultaneously, we proposed a new parameterization defined by (C_s, ϵ, δ) to characterize acoustic VTI media. The parameter C_s , treated as an intermediate variable during inversion, captures wave-propagation properties in different directions. This new parameter can be defined by Equation 2:

$$C_s = \frac{C_{33}}{C_{11}} \frac{C_{33}}{C_{13}} C_{33} = \frac{v_p^2}{(1 + 2\epsilon)\sqrt{1 + 2\delta}} \quad (2)$$

Radiation pattern analysis helps us better understand the sensitivity of the scattered wavefield to the scattering

angle. We can compute the new radiation patterns for parameterization (C_s , ε , δ) in two ways. The first is a finite-difference approach that computes the partial derivative of the pressure wavefield with respect to each parameter.⁸ The other is an analytical approach based on the scattering integral (SI) method under the Born approximation. The SI provides a more compact representation of the sensitivity of the scattered data to parameter perturbations as a function of scattering angle.^{7,8} We derived the radiation pattern using the SI method.

Based on the first-order acoustic VTI wave equations,³³ the pressure field p and the anisotropic perturbation of the wavefield q can be described by the second-order acoustic VTI wave equation in terms of the parameters C_s , ε , and δ (Equation 3).

$$\begin{cases} -\frac{1}{C_s} \omega^2 p - (1+2\delta)^{\frac{3}{2}} (1+2\varepsilon) \partial_{xx} (p+q) \\ - (1+2\delta)^{\frac{3}{2}} (1+2\varepsilon) \partial_{yy} (p+q) \partial_{zz} \frac{1}{\sqrt{1+2\delta}} p = s \\ - (1+2\varepsilon)(1+2\delta) \\ -\frac{1}{C_s} \omega^2 q - 2(\varepsilon-\delta) \sqrt{1+2\delta} (1+2\varepsilon) (\partial_{xx} + \partial_{yy}) (p+q) = 0 \end{cases} \quad (3)$$

where ω is the angular frequency, and s is an isotropic source term. To obtain the Born single integral representation, we defined the perturbations r_{C_s} , r_ε , and r_δ using Equation 4:

$$\begin{aligned} C_s &= C_{s,0} (1 + r_{C_s}) \\ \varepsilon &= \varepsilon_0 + r_\varepsilon \quad \delta = \delta_0 + r_\delta \end{aligned} \quad (4)$$

The parameters ($C_{s,0}$, ε_0 , δ_0) represent the background medium. This definition is similar to the one shown in a previous work.²³ Assuming weak anisotropy, we perturbed the acoustic VTI medium parameters from an isotropic background (i.e., $\varepsilon_0 = 0$, $\delta_0 = 0$). In this sense, we obtained the background pressure equations for p_0 and q_0 by setting the spatial derivatives of the parameters to zero (Equation 5):

$$\begin{cases} -\frac{1}{C_s} \omega^2 p_0 - (\partial_{xx} p_0 + \partial_{yy} p_0 + \partial_{zz} p_0) = s \\ q_0 = 0 \end{cases} \quad (5)$$

According to the Born approximation, the scattered fields p_1 and q_1 satisfy a linear dependence on the parameters, and the scattered q_1 can be obtained directly from the background wavefield p_0 (Equation 6):

$$-\frac{1}{C_s} \omega^2 q_1 = 2r_\varepsilon (\partial_{xx} p_0 + \partial_{yy} p_0) - 2r_\delta (\partial_{xx} p_0 + \partial_{yy} p_0) \quad (6)$$

Combining Equations 3 and 5, the field p_1 satisfies the equation with parameter perturbations as follows

(Equation 7):

$$\begin{aligned} &-\frac{1}{C_{s,0}} \omega^2 p_1 - (\partial_{xx} p_1 + \partial_{yy} p_1 + \partial_{zz} p_1) = \\ &-\omega^2 \frac{1}{C_{s,0}} r_{C_s} - \frac{C_{s,0}}{\omega^2} [\partial_{xx} 2(r_\varepsilon - r_\delta) (\partial_{xx} p_0 + \partial_{yy} p_0) \\ &+ \partial_{yy} 2(r_\varepsilon - r_\delta) (\partial_{xx} p_0 + \partial_{yy} p_0)] \\ &+ (2r_\delta \partial_{zz} p_0 + 2r_\varepsilon \partial_{zz} p_0 - \partial_{zz} r_\delta p_0) \\ &+ (2r_\varepsilon \partial_{xx} p_0 + 3r_\delta \partial_{xx} p_0 + 2r_\varepsilon \partial_{yy} p_0 + 3r_\delta \partial_{yy} p_0) \end{aligned} \quad (7)$$

The Green's function G was defined by Equation 8:

$$\left[-\frac{1}{C_{s,0}} \omega^2 - (\partial_{xx} + \partial_{yy} + \partial_{zz}) \right] G(\mathbf{x}, \mathbf{x}', \omega) = \delta(\mathbf{x} - \mathbf{x}') \quad (8)$$

With the Green's function, p_0 and p_1 can be expressed as in Equation 9:

$$\begin{cases} p_0(\mathbf{x}_s, \mathbf{x}, \omega) = s(\omega) G(\mathbf{x}_s, \mathbf{x}, \omega) \\ p_1(\mathbf{x}_s, \mathbf{x}, \omega) = -s(\omega) \int d\mathbf{x} \omega^2 \frac{1}{C_{s,0}} r_{C_s} G(\mathbf{x}_s, \mathbf{x}, \omega) G(\mathbf{x}_r, \mathbf{x}, \omega) \\ - s(\omega) \int d\mathbf{x} \frac{C_{s,0}}{\omega^2} (2r_\varepsilon - 2r_\delta) \\ \Delta_h G(\mathbf{x}_s, \mathbf{x}, \omega) \Delta_h G(\mathbf{x}_r, \mathbf{x}, \omega) \\ + s(\omega) \int d\mathbf{x} 3r_\delta G(\mathbf{x}_s, \mathbf{x}, \omega) \Delta_h G(\mathbf{x}_r, \mathbf{x}, \omega) \\ + 2r_\varepsilon G(\mathbf{x}_s, \mathbf{x}, \omega) \Delta_h G(\mathbf{x}_r, \mathbf{x}, \omega) \\ + s(\omega) \int d\mathbf{x} r_\delta \Delta_v G(\mathbf{x}_s, \mathbf{x}, \omega) G(\mathbf{x}_r, \mathbf{x}, \omega) \\ - s(\omega) \int d\mathbf{x} (2r_\varepsilon + 2r_\delta) G(\mathbf{x}_s, \mathbf{x}, \omega) \Delta_v G(\mathbf{x}_r, \mathbf{x}, \omega) \end{cases} \quad (9)$$

where $\Delta_h = \partial_{xx} + \partial_{yy}$, $\Delta_v = \partial_{zz}$.

To analyze the spatial features of scattered fields, we defined the Green's function in terms of Equation 10:

$$G(\mathbf{x}, \mathbf{y}, \omega) = A(\mathbf{x}, \mathbf{y}) \exp \left(i \frac{\omega}{\sqrt{C_{s,0}}} \mathbf{p} \cdot \mathbf{x} \right) \quad (10)$$

where $\mathbf{p}$ is the unitary vector related to the wavenumber vector. Equation 10 shows that the scattered pressure field p_1 can be further rewritten as Equation 11:

$$p_1(\mathbf{x}_s, \mathbf{x}_r, \omega) = -\omega^2 s(\omega) \int d\mathbf{x} A(\mathbf{x}_s, \mathbf{x}_r, \omega) \mathbf{a}(\mathbf{x}) \cdot \mathbf{r}(\mathbf{x}) \quad (11)$$

where

$$A(\mathbf{x}_s, \mathbf{x}, \mathbf{x}_r, \omega) = \frac{G(\mathbf{x}_s, \mathbf{x}, \omega) G(\mathbf{x}_r, \mathbf{x}, \omega)}{C_{s,0}} \quad (12)$$

and

$$\mathbf{r}_1 = \begin{pmatrix} r_{C_s} \\ r_\varepsilon \\ r_\delta \end{pmatrix}; \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ 2p_{sh}^2 p_{rh}^2 + 2p_{rz}^2 - 2p_{rh}^2 \\ -2p_{sh}^2 p_{rh}^2 + 3p_{rh}^2 + p_{sz}^2 - 2p_{rz}^2 \end{pmatrix} \quad (13)$$

Here, $A(x_s, x, x_r, \omega)$ represents terms related to Green's function, $\mathbf{r}_1$ is the perturbation matrix, $\mathbf{a}_1$ denotes the radiation patterns of parameters (C_s , ε , and δ), p_{sh} and p_{rz} denote the horizontal and vertical components of the source incident vector, and p_{rh} and p_{rz} denote the horizontal and vertical components of the reflector vector. To better understand the angular dependence of the radiation patterns, we introduce $\mathbf{P}$ as the unitary wave-propagation vector. $\mathbf{P}_s$ and $\mathbf{P}_r$ represent the source incident vector and reflected vector, respectively (Equation 14):

$$\begin{cases} \mathbf{P}_s = \begin{pmatrix} p_{sh} \\ p_{rz} \end{pmatrix} = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \\ \mathbf{P}_r = \begin{pmatrix} p_{rh} \\ p_{rz} \end{pmatrix} = \begin{pmatrix} -\sin(\theta + 2\alpha) \\ \cos(\theta + 2\alpha) \end{pmatrix} \end{cases} \quad (14)$$

where θ and α denote the source incident angle and the reflector dip angle, respectively. Similar definitions appear in the previous sensitivity analysis.^{10,13,23}

Assuming a horizontal Earth layer, Figure 1 shows the radiation patterns for the proposed parameterization. As observed, C_s is sensitive across all scattering angles and is mainly associated with vertical velocity. More importantly, the ε and δ parameters are both sensitive at small and large scattering angles. This suggests the possibility of simultaneously inverting the short- and long-wavelength parts of the ε and δ model.

To further emphasize the advantages of the parameterization (C_s , ε , δ), we compared the radiation patterns in Equation 13 with the conventional radiation patterns for the parameterization (v_p , ε , δ) and v_n , η , and δ . By assuming the linear relation between the parameter perturbations, the radiation patterns for the parameterization (v_p , ε , δ) can be written as Equation 15:

$$\mathbf{r}_2 = \begin{pmatrix} r_{v_p} \\ r_{\varepsilon} \\ r_{\delta} \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 2p_{sh}^2 \cdot p_{rh}^2 \\ 2 - 2p_{sh}^2 \cdot p_{rh}^2 - p_{sz}^2 - p_{rz}^2 \end{pmatrix} \quad (15)$$

where $\mathbf{r}_2$ and $\mathbf{a}_2$ represent the radiation patterns in parameterization (v_p , ε , δ), respectively. Similarly, the radiation patterns for the parameterization (v_n , η , δ)²³ can be given as Equation 16:

$$\mathbf{r}_3 = \begin{pmatrix} r_{v_n} \\ r_{\eta} \\ r_{\delta} \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} 2 \\ 2p_{sh}^2 p_{rh}^2 \\ -(p_{sh}^2 + p_{rz}^2) \end{pmatrix} \quad (16)$$

For the parameterization (v_p , ε , δ), we plotted the corresponding radiation patterns for a horizontal reflector in Figure 2. We can clearly see that v_p is sensitive to all scattering angles. However, both Thomsen's parameters ε and δ are insensitive to small scattering angles, and ε is sensitive to large scattering angles and mainly affects the horizontally traveling waves. From this observation, we conclude that angular dependence during inversion could impede high-resolution estimates of ε and δ . According to the radiation patterns for the conventional parameterization (v_p , ε , δ), in conventional anisotropic FWI based on surface receiver data, we consider near-offset data as seismic data that mainly come from small-scattering-angle (less than 60°) propagation and represent the influence of v_p perturbation. Far-offset data are mainly influenced by large-scattering-angle (more than 120°) propagation and represent the influence of anisotropy in subsurface media.

Figure 3 shows the radiation patterns of v_n , η , and δ on a horizontal reflector. In parameterization (v_n , η , δ), v_n shows a significant isotropic radiation pattern, which indicates a desirable resolvability in FWI applications on surface data.

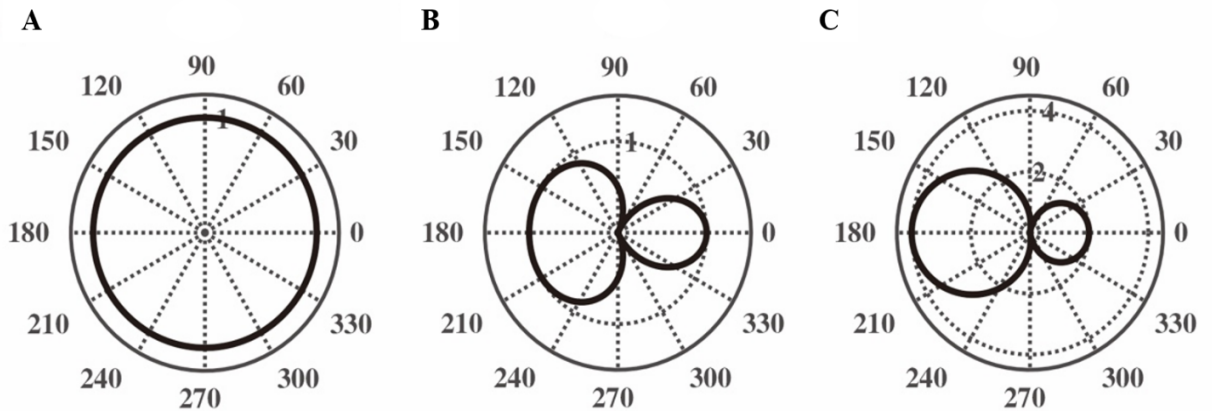


Figure 1. Radiation patterns for the parameterization (C_s , ε , δ) on a horizontal reflector. The radiation patterns here and throughout are shown as an absolute value. (A) C_s , (B) δ , and (C) ε .

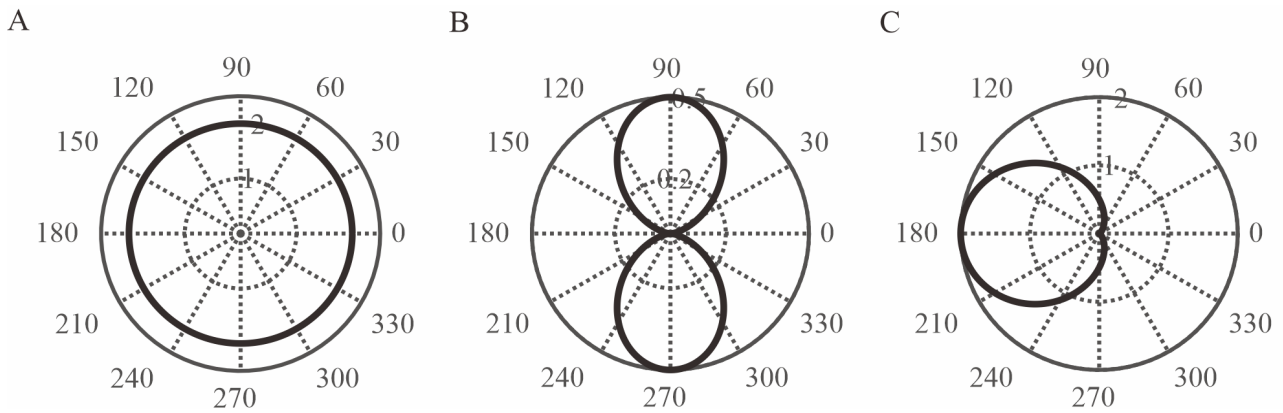


Figure 2. Radiation patterns for the parameterization (v_p , ϵ , δ) on a horizontal reflector. (A) v_p , (B) δ , and (C) ϵ .

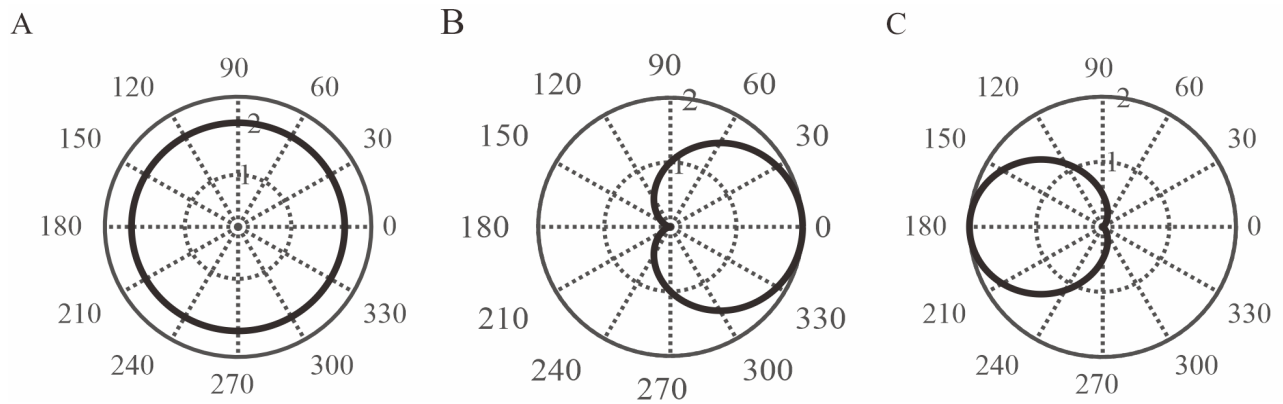


Figure 3. Radiation patterns for the parameterization (v_n , η , δ) on a horizontal reflector. (A) v_n , (B) δ , and (C) η .

The parameterization (v_n , η , δ) can effectively enhance the sensitivity of δ at small scattering angles, which is more relevant to reflection waves. However, η is still insensitive at small scattering angles. Depending on the large scattering angles, only the long-wavelength components of η can be inverted in FWI. The absence of scattered wavefield at large angles could limit the inversion of long-wavelength components of δ .

Comparing radiation patterns across parameterizations gives insight into FWI behavior. The parameterization (v_p , δ , ϵ) is reasonable if we only invert v_p with good initial anisotropic models. In parameterization (v_n , η , δ), the multi-parameter FWI can retrieve the short-wavelength parts of δ , the long-wavelength parts of η , and an accurate inversion result of v_n . Introducing C_s produces stronger changes in the radiation patterns and optimizes the sensitivities of ϵ and δ at small and large scattering angles. Thus, we consider the parameterization (C_s , ϵ , δ) optimal, since we intend to simultaneously recover short- and long-wavelength parts of all parameters by FWI. In VTI media, the parameterization affects both the model-parameter

scattering characteristics and the corresponding gradient formulations. The proposed parameterization alters the small-angle scattering behavior, thereby improving the inversion accuracy of surface-data FWI. This motivates the new parameterization to improve parameter sensitivity during inversion.

The scattering derivation adopted in this study assumes an isotropic background medium. According to previous studies,^{34,35} under an anisotropic background assumption, background anisotropic parameters have only a minor influence on the scattering wavefield, except under extremely strong anisotropy. Therefore, this assumption applies to most geological settings.

After deducing the C_s parameter by its definition, we can use the FWI method with the proposed parameterization and derive an efficient way to simultaneously invert for all three parameters in acoustic VTI media. Implementing FWI requires calculating wavefields and the gradients of an objective function (defined by **Equation A3** in the [Appendix](#)). The three-dimensional acoustic VTI wave in **Equation A1**, deduced from Hooke's law,³³ was used

to describe the propagation of quasi-P waves. Based on parameterization (C_s , ε , δ), the gradient formulas for FWI in acoustic VTI media derived from the adjoint-state method (**Appendix**) are given as follows (**Equations 17–19**):

$$\frac{\partial E}{\partial C_s} = \frac{1}{\rho C_s^2} \left(\frac{\partial \sigma_H}{\partial t} \sigma'_H + \frac{\partial \sigma_V}{\partial t} \sigma'_V \right) \quad (17)$$

$$\begin{aligned} \frac{\partial E}{\partial \varepsilon} = & -\frac{(1+2\varepsilon)^2}{\sqrt{1+2\delta}} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \sigma'_H - 2(1+2\varepsilon) \frac{\partial V_z}{\partial z} \sigma'_H \\ & - 2(1+2\varepsilon) \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \sigma'_V - \frac{1+2\varepsilon}{\sqrt{1+2\delta}} \frac{\partial V_z}{\partial z} \sigma'_V \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial E}{\partial \delta} = & -2(1+2\varepsilon) \sqrt{1+2\delta} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \sigma'_H - 2(1+2\delta) \frac{\partial V_z}{\partial z} \sigma'_H \\ & - 2(1+2\delta) \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \sigma'_V - 2\sqrt{1+2\delta} \frac{\partial V_z}{\partial z} \sigma'_V \end{aligned} \quad (19)$$

Here, the horizontal and vertical stress components of the wavefields are defined as σ_H and σ_V , respectively; V_x and V_z denote the corresponding particle velocities of the horizontal and vertical components, respectively; σ'_H , σ'_V , V'_x , and V'_z represent the corresponding adjoint wavefields, which can be obtained by calculating the following adjoint wave equations (**Equation 20**):

$$\begin{cases} \rho \frac{\partial V'_x}{\partial t} - (1+2\varepsilon)^2 \sqrt{1+2\delta} \frac{\partial \sigma'_H}{\partial x} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial x} = 0 \\ \rho \frac{\partial V'_y}{\partial t} - (1+2\varepsilon)^2 \sqrt{1+2\delta} \frac{\partial \sigma'_H}{\partial y} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial y} = 0 \\ \rho \frac{\partial V'_z}{\partial t} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_H}{\partial z} - \sqrt{1+2\delta}(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial z} = 0 \end{cases} \quad (20)$$

$$\begin{aligned} \frac{1}{\rho C_s} \frac{\partial \sigma'_H}{\partial t} &= \frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \\ \frac{1}{\rho C_s} \frac{\partial \sigma'_V}{\partial t} &= \frac{\partial V'_z}{\partial z} \end{aligned}$$

The synthetic seismic record was obtained by solving the acoustic VTI wave equations with parameterization (C_s , ε , δ), and the data residual is the difference between observed and synthetic data. To minimize the misfit in **Equation A3**, we used gradient-based FWI optimization.⁶ We used the steepest descent method and computed the gradients using **Equations 17–19**. Finally, we obtained the inversion results for parameters (C_s , ε , δ) after multiple iterations. Since v_p is commonly used to describe acoustic VTI media, we convert our inversion result for C_s to v_p .

3. Multi-parameter point spread function for full-waveform inversion

The point spread function (PSF) helps illustrate sensitivity, resolution, and illumination in a specific observed system and model space. To investigate parameter sensitivities during inversion with complex subsurface structure, we calculated the MPSF as follows. A modified Marmousi model was adopted to produce the synthetic data. The true models of the parameters in terms of v_p , ε , and δ are given in **Figure 4**. Additionally, we set the density to a constant value of 1,500 kg/m³. The model size was 298 by 150 grids with a grid interval of 10 m. We considered 18 shots with a source spacing of 150 m, starting at 50 m and ending at 2,750 m. We distributed 298 geophones along the surface at an equidistant 10 m spacing from 0 m to 2,980 m. A Ricker wavelet with a center frequency of 20 Hz was used as the source wavelet. The initial models shown in **Figure 5** are smoothed versions of the true models.

We defined model perturbations of 20 m/s, 0.01, and -0.01 for v_p , ε , and δ , respectively. By applying the perturbation of v_p at $x = 1,500$ m and $z = 1,000$ m, we obtained the PSF for v_p based on the synthetic perturbation data. We used the same method to calculate the PSF for ε and δ . The perturbations for ε and δ were $x = 750$ m, $z = 500$ m, and $x = 2,250$ m, $z = 1,000$ m, respectively. **Figure 6** displays the MPSF for parameterization (v_p , ε , δ), which shows that, using surface-observed receivers, FWI with conventional parameterization can barely detect a gradient when the anisotropic parameter changes. A similar perturbation for (C_s , ε , δ) parameterization was applied to calculate the MPSFs: $\Delta C_s = 30,429.0$; $\Delta \varepsilon = 0.01$; $\Delta \delta = -0.01$. **Figure 7** shows that the new parameterization can effectively enhance the sensitivities of ε and δ by using surface seismic data.

To better compare the MPSFs, we further analyzed the single-trace gradient PSFs of the three parameters under different parameterizations. We introduced perturbation points for ε , v_p , and δ at ($x = 750$ m, $z = 500$ m), ($x = 1,500$ m, $z = 750$ m), and ($x = 2,250$ m, $z = 1,000$ m), respectively. The blue curves represent the single-trace PSFs obtained using the conventional v_p parameterization, while the red curves represent the PSFs calculated with the cs parameterization.

As shown in **Figures 8–13**, both parameterizations can extract effective PSFs at the perturbation locations. However, away from the perturbation points, the conventional method produced incorrect amplitudes for the single-trace PSFs of anisotropic parameters. This further indicates that the conventional parameterization

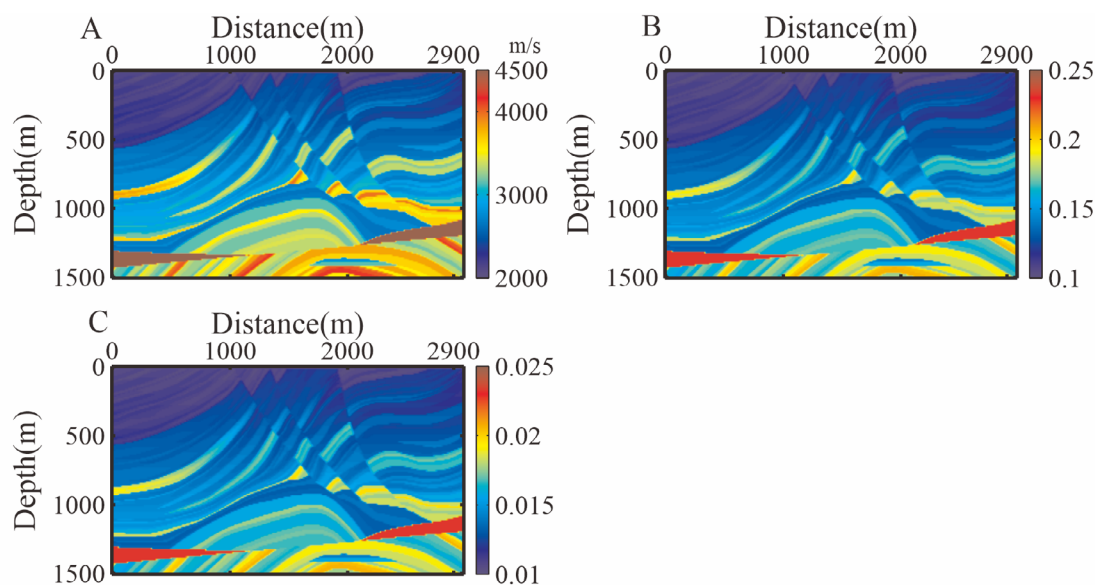


Figure 4. The true parameters of the Marmousi model. (A) Vertical qP-wave velocity v_p , (B) ϵ , and (C) δ .

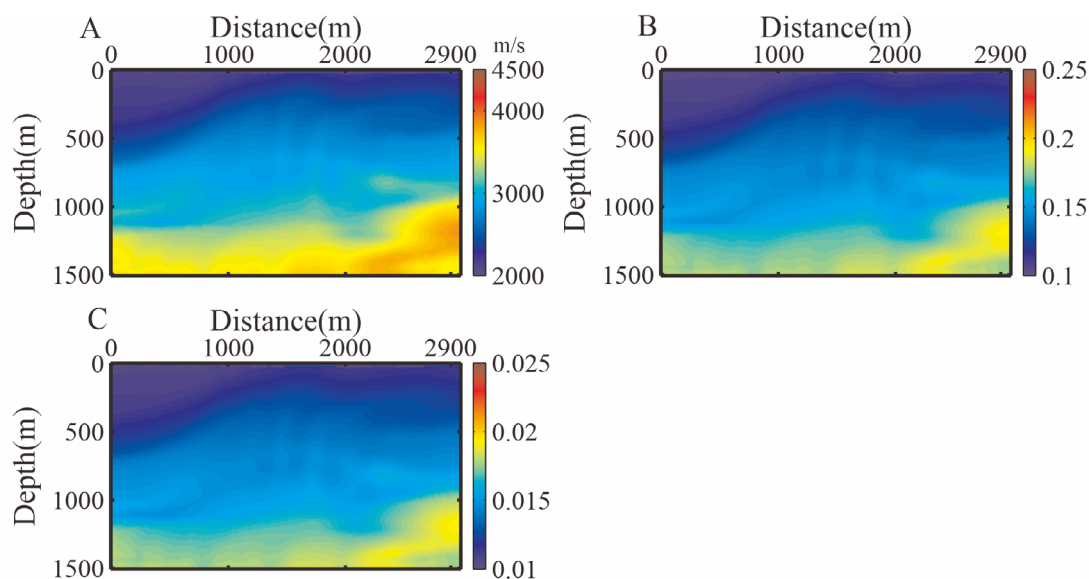


Figure 5. Initial full-waveform inversion models: (A) v_p , (B) ϵ , and (C) δ

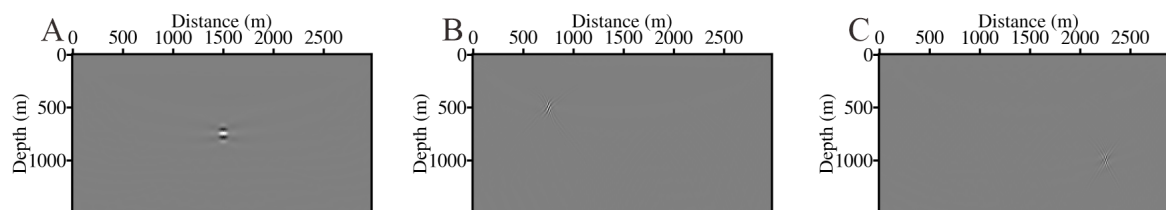


Figure 6. The multi-parameter point spread functions (PSFs) for full-waveform inversion in parameterization (v_p , ϵ , δ): (A) PSF for gradient of v_p , (B) PSF for gradient of ϵ , and (C) PSF for gradient of δ .

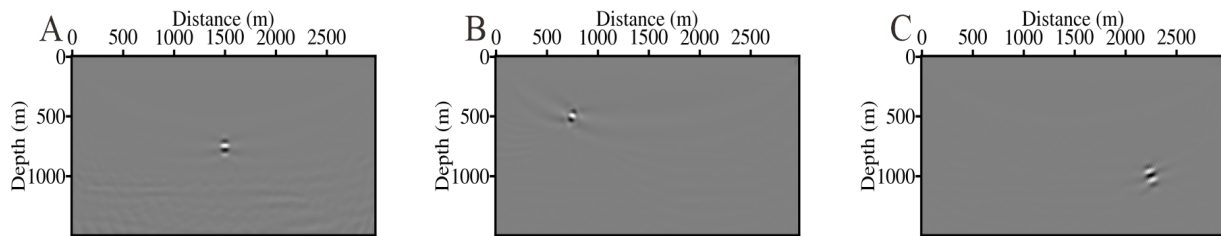


Figure 7. The multi-parameter point spread functions (PSFs) for full-waveform inversion parameterization (C_s , ϵ , δ): (A) PSF for gradient of C_s , (B) PSF for gradient of ϵ , and (C) PSF for gradient of δ .

cannot accurately calculate the gradients of weakly sensitive parameters.

4. Synthetic examples

We applied the multi-parameter FWI to the modified Marmousi model (Figure 4) to demonstrate the performance of the C_s -based parameterization. To reduce cycle-skipping during FWI, we used a multi-scale inversion strategy in all synthetic examples. We performed multi-scale inversion by low-pass filtering, obtaining different scales by filtering the data and source with filters with cutoff frequencies of 5 Hz, 10 Hz, and 15 Hz, respectively. For each inversion frequency band, we performed FWI with 10 iterations. The step length was determined using a quadratic function in all iterations. To ensure a fair comparison, all inversion strategies used the same settings except for the parameterization.

Figure 14 displays the simultaneous inversion results of multi-parameter FWI for parameterization (v_p , ϵ , δ). Only v_p was recovered well, and ϵ and δ cannot be accurately recovered in FWI based on parameterization (v_p , ϵ , δ). The details of v_p , such as the stratification, the thrust fault, and the caprock at the bottom of the model, were well retrieved. However, the ϵ and δ parameters can only be partially updated in the shallow part.

We conducted FWI with the parameterization (v_n , δ , η) for multi-parameter inversion (Figure 15). Compared with the results in Figure 14, a more accurate δ model can be inverted in parameterization (v_n , δ , η) and the high-resolution features of δ can be recovered well. Accompanied by the accurate inversion of δ , the anisotropic FWI for parameterization (v_n , δ , η) provides more details than the previous results (Figure 14). Nevertheless, an effective inversion result of the η parameter still cannot be obtained, and the updates of η only work on shallow parts (Figure 15B). Thus, the converted ϵ is inaccurate and needs further improvement to address the angular dependency.

For comparison, we further conducted FWI for parameterization (C_s , ϵ , δ). According to the definition, we calculated the initial C_s model. We then calculated the gradients of C_s , ϵ , and δ using Equations 17–19 and obtained the final inversion results after multiple iterations (Figure 16). The C_s parameterization significantly affected the accuracy and resolution of multi-parameter FWI. The inversion recovered both anisotropy parameters ϵ and δ of the acoustic VTI media in greater detail in the shallow and deep layers of the model. Furthermore, the v_p converted from the inversion results was more accurate in the deep part of the model than in previous results, indicating the effectiveness of the proposed parameterization.

We also compared the vertical profiles of the inversion results with different parameterizations at 1.2 km (Figures 17–19). The vertical profile of δ results (Figure 17) indicates that the inverted δ from the parameterizations (v_n , η , δ) (green solid line) and (C_s , ϵ , δ) (red solid line) fits the true δ (the black solid line) well. Figure 18 displays the inverted ϵ by FWIs for different parameterizations. We can only obtain the shallow part (depth less than 0.8 km) of η by FWI for parameterization (v_n , η , δ) because of the limited offset. The converted ϵ was consistent with η and required few updates in the deep part (green line in Figure 18). The inverted η for parameterization (v_n , η , δ) showed strong dependence on the long-offset data and could invert only the low-wavenumber part. In FWI for parameterization (C_s , ϵ , δ), ϵ was sensitive at the short offset and could be effectively inverted in both shallow and deep parts. The inversion result (red line in Figure 18) fits the true model (black line, Figure 18) fairly well. The data misfits over iterations show that the FWI with the parameterization (C_s , ϵ , δ) converges better compared with the others during long-wavelength and short-wavelength components inversion (Figure 20).

Comparing the inversion results of FWI for parameterizations (v_p , ϵ , δ), (v_n , η , δ), and (C_s , ϵ , δ), it is evident that the choice of parameterization significantly

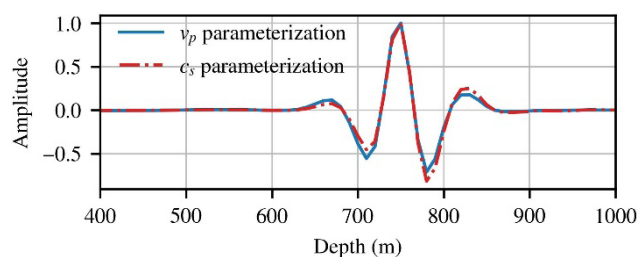


Figure 8. The gradient point spread function for the v_p parameter at $x = 1,500$ m

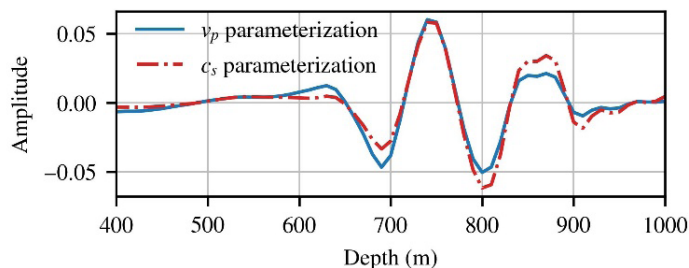


Figure 9. The gradient point spread function for the v_p parameter at $x = 1,400$ m

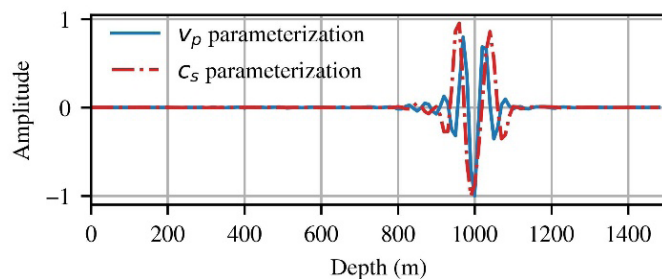


Figure 10. The gradient point spread function for the δ parameter at $x = 2,250$ m

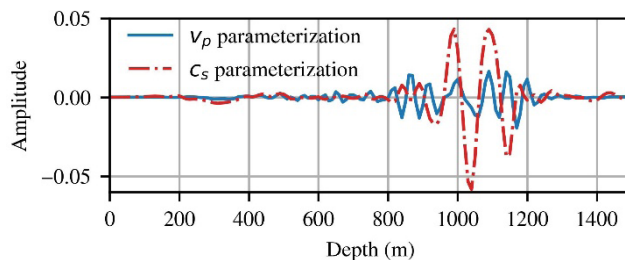


Figure 11. The gradient point spread function for the δ parameter at $x = 2,150$ m.

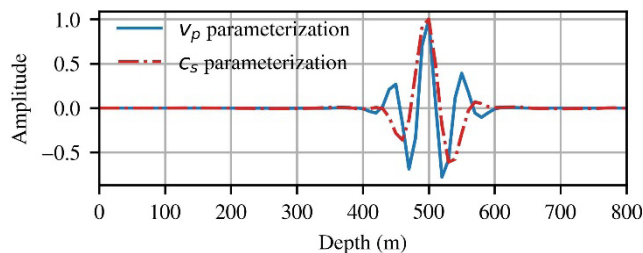


Figure 12. The gradient point spread function for the ϵ parameter at $x = 750$ m.

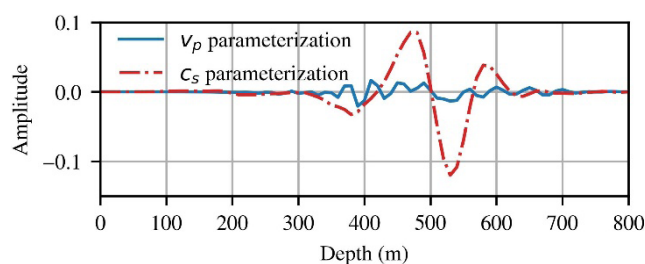


Figure 13. The gradient point spread function for the ϵ parameter at $x = 650$ m

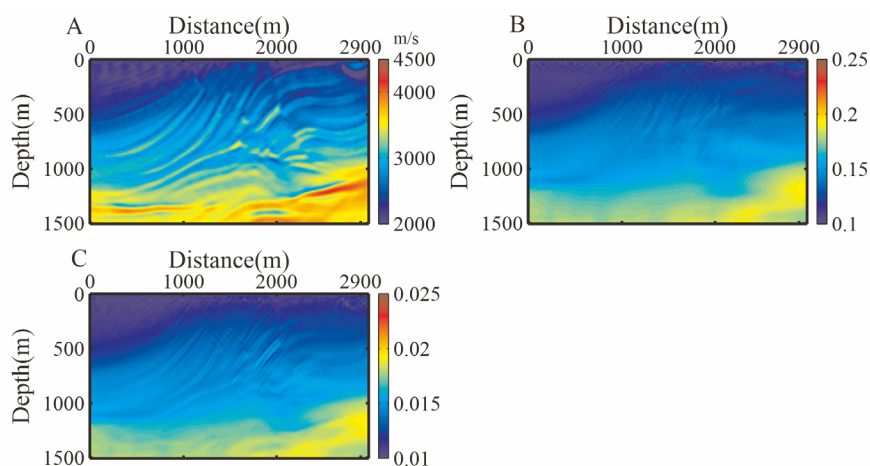


Figure 14. The full-waveform inversion results for parameterization (v_p, ϵ, δ) : (a) v_p , (b) ϵ , (c) δ

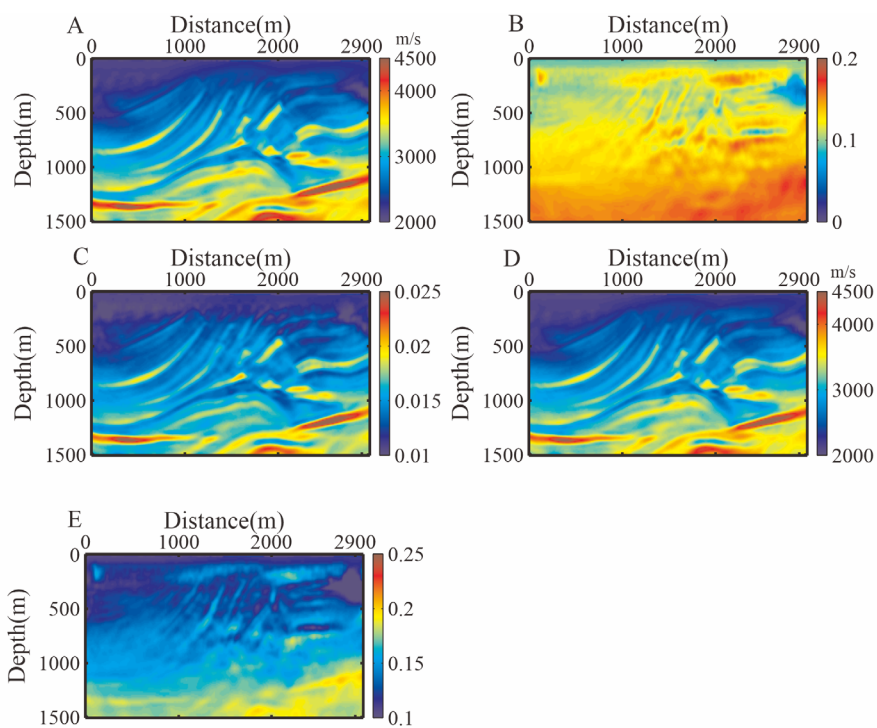


Figure 15. The full-waveform inversion results for parameterization (v_n, η, δ) : (a) v_n , (b) η , and (c) δ . The results of (d) v_p and (e) ϵ are converted from the inversion results according to their definitions.

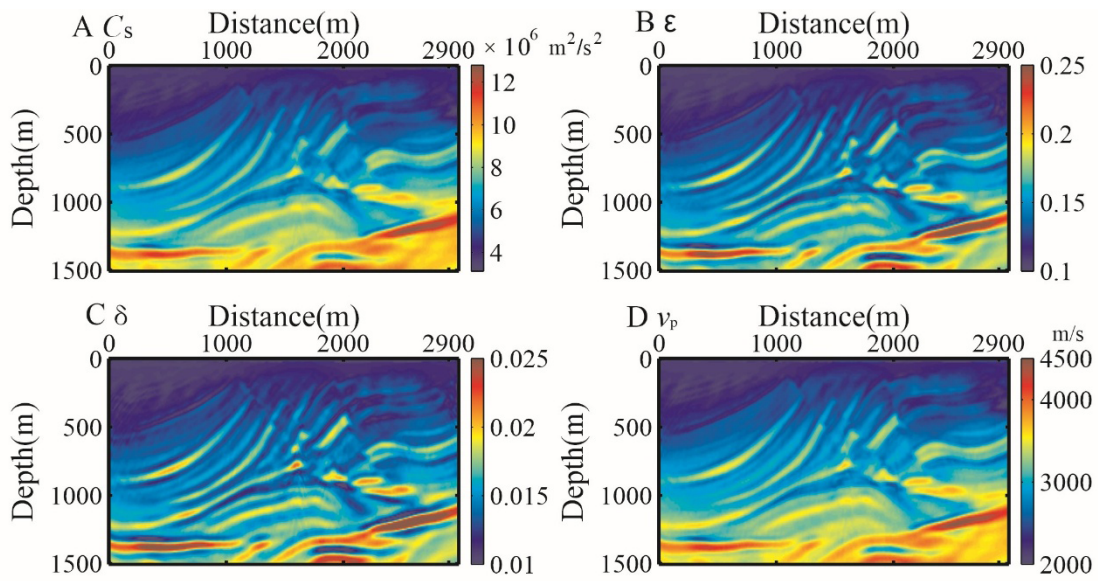


Figure 16. The full-waveform inversion results for parameterization (C_s , ϵ , δ): (a) C_s , (b) ϵ and (c) δ . The result in (d) v_p is converted from the inversion results according to its definition.

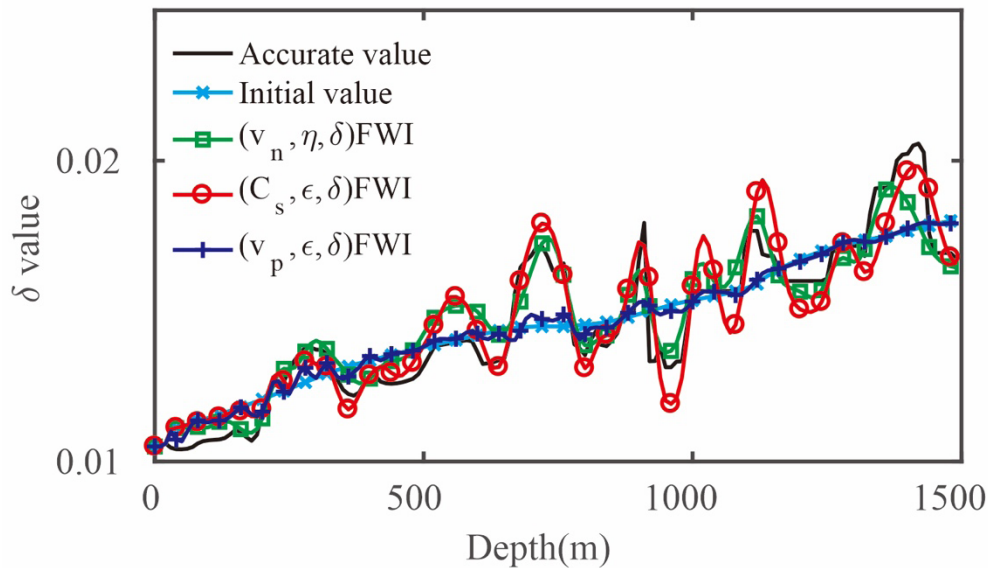


Figure 17. Comparison of the vertical profiles of the inverted δ models at $x = 1.2$ km with different parameterizations. The black and cyan lines indicate true δ and initial δ , respectively; the green, red, and blue lines indicate the converted δ from results of full-waveform inversion (FWI) for parameterizations (v_n, η, δ) , (C_s, ϵ, δ) , and (v_p, ϵ, δ) , respectively.

affects the inversion results of multi-parameter FWI by changing the scattering patterns. In deep layers, the inversion of anisotropy parameters for conventional parameterizations is affected by angular dependence and the limited illumination of surface acquisition. With the proposed choice of parameterization (C_s , ϵ , δ), our method

yields more accurate ϵ and δ models. These inversion results agree with the previous sensitivity analysis of radiation patterns. Parameterization (C_s , ϵ , δ) improves the sensitivity of anisotropic parameters (ϵ and δ) in near offset and also maintains high sensitivity in far offset. Hence, it can contribute to high-resolution results in multi-scale

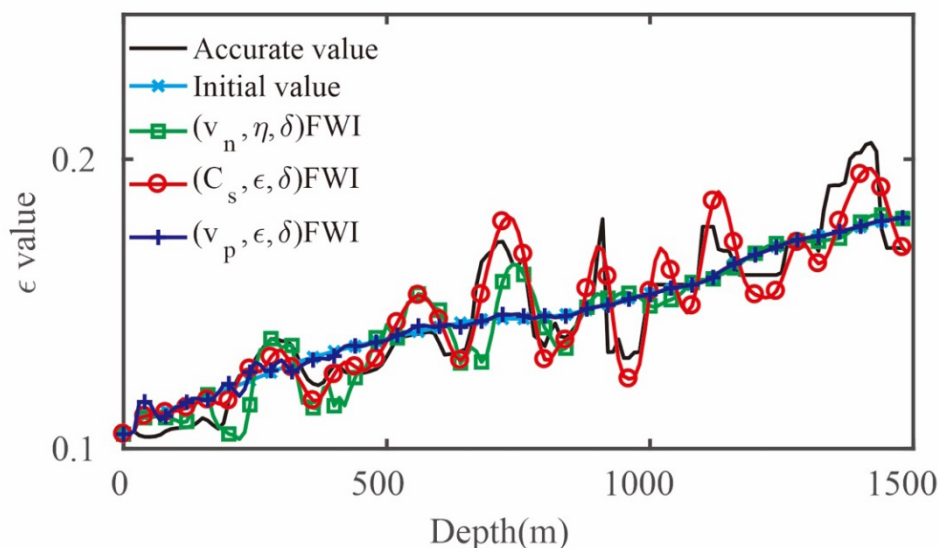


Figure 18. Comparison of the vertical profiles of the inverted ϵ models at $x = 1.2$ km with different parameterizations. The black and cyan lines indicate true ϵ and initial ϵ , respectively; the green, red, and blue lines indicate the converted ϵ from results of full-waveform inversion (FWI) for parameterizations (v_n, η, δ) , (C_s, ϵ, δ) , and (v_p, ϵ, δ) , respectively.

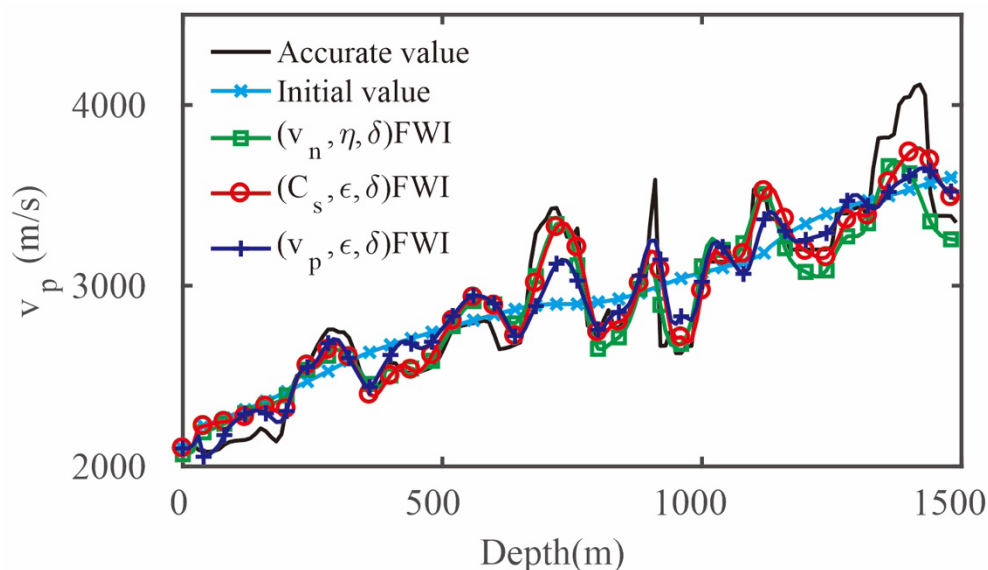


Figure 19. Comparison of the vertical profiles of the inverted v_p models at $x = 1.2$ km with different parameterizations. The black and cyan lines indicate true v_p and initial v_p , respectively; the green, red, and blue lines indicate the converted v_p from results of FWI for parameterizations (v_n, η, δ) , (C_s, ϵ, δ) , and (v_p, ϵ, δ) , respectively.

FWI with surface data.

From an objective-function minimization perspective, steepest descent can directly reduce the objective function to a relatively low value. Considering the high computational cost of FWI, we performed 10 iterations for each inversion frequency band. Therefore, the steepest descent method is better suited to this study. Theoretically, conjugate gradient

and limited-memory Broyden–Fletcher–Goldfarb–Shannon methods may achieve better inversion results. However, this study mainly focuses on addressing weak parameter sensitivity. The optimization of gradient-based update strategies can be further investigated in future work.

To further evaluate the accuracy of the inversion

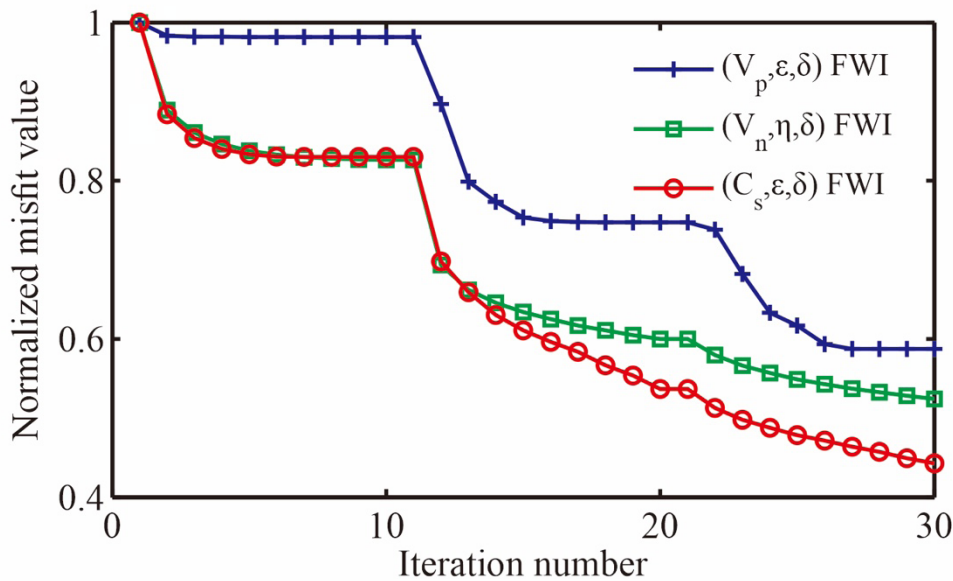


Figure 20. The data misfits for different parameterizations. All values of the objective functions are normalized by the value of the first iteration. The black, blue, and red lines indicate the data misfit of FWI for (v_p, ϵ, δ) , (v_n, η, δ) , and (C_s, ϵ, δ) , respectively.

results, we calculated the global normalized root-mean-square error (NRMSE) for a quantitative comparison. The calculation formula of NRMSE is given as **Equation 21**:

$$\text{NRMSE} = \frac{\sqrt{\frac{1}{n} \sum_{i=1}^n (y_{\text{pred}} - y_{\text{true}})^2}}{\bar{y}} \quad (21)$$

We calculated the NRMSE values using **Equation 21** (Table 1). The comparison indicates that the proposed parameterization scheme yielded inversion results closer to the true model values.

Table 1. Normalized root-mean-square error (NRMSE) for full-waveform inversion in different parameterizations

Parameterization	NRMSE- v_p	NRMSE- ϵ	NRMSE- δ
(v_p, ϵ, δ)	0.099677	0.108137	0.110033
(v_n, η, δ)	0.078546	0.120672	0.084278
(C_s, ϵ, δ)	0.074638	0.095031	0.085153

5. Discussion

The primary issue with FWI is calculating gradients that contain detailed information about the parameters. The

proposed parameterization provides a potential way to obtain a higher-accuracy multi-parameter model simultaneously by considering both small- and large-angle seismic data during the inversion. However, similar to other existing parameterization methods, this proposed parameterization (C_s, ϵ, δ) still suffers from the cross-talk effect, especially when the subsurface layers or/and anomalies are spatially uncorrelated. To address the cross-talk problem economically, previous research prefers to decompose updates for parameters with scattering angles and use a hierarchical inversion method.^{8-10,13,23,24} Based on the PSF results, we calculated effective gradients for anisotropic parameters and then studied parameter decomposition during FWI in future work.

After introducing the new parameterization, the computational cost of the main procedures, including forward modeling and adjoint wavefield propagation, did not increase significantly compared with the conventional method. In addition, the conversion from the new parameters to velocity parameters depends theoretically on the accuracy of the anisotropic parameters. The accuracy of anisotropic parameter inversion directly affects the converted velocity results. Our numerical experiments show that the proposed parameterization improved anisotropic-parameter inversion accuracy. Therefore, the converted velocity models remain reliable and accurate.

6. Conclusion

The FWI for parameterization (C_s, ϵ, δ) proposed in this

study provides a new method to solve the sensitivity issue and angular dependency in acoustic VTI inversion. It differs from existing methods by using a new parameter, C_s . Using the parameterization (C_s , ε , δ), we derived the new gradient formula and adjoint equation in FWI using the adjoint-state method. To obtain sensitivity analysis results, we used the Born approximation to derive the radiation patterns for the new parameterization in VTI pseudo-acoustic wave equations. MPSFs for the Marmousi model verify the effectiveness of our new parameterization in calculating the gradients. This approach can invert for high-resolution components of anisotropic parameters in acoustic VTI media using surface data. In addition, because anisotropic parameters are sensitive to both small and large scattering angles, we can accurately invert for the long- and short-wavelength components of the model.

We tested our FWI method for parameterization (C_s , ε , δ) using synthetic seismic data. Inversion results show that the proposed parameterization helps invert for desirable ε and δ models. The numerical example indicates that this new parameterization can not only enhance the sensitivity of anisotropic parameters on surface data but also improve the accuracy of the inversion result.

The proposed parameterization (C_s , ε , δ) provides a practical FWI method with less angular dependency on surface seismic data, but still suffers from the cross-talk issue; further investigation is needed.

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Conflict of interest

The authors declare they have no competing interests.

Author contributions

Conceptualization: Shilei Sun, Weijian Mao

Formal analysis: Shilei Sun, Weijian Mao

Investigation: Shilei Sun

Methodology: Shilei Sun

Writing—original draft: Shilei Sun

Writing—review & editing: Weijian Mao, Lingli Gao

Availability of data

Data associated with this research are available and can be obtained by contacting the corresponding author.

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Appendix

Gradient formula and adjoint wave equations for full-waveform inversion using C_s parameterization

Based on the adjoint-state method, we derived the gradient formula and adjoint equation in C_s parameterization. Since full-waveform inversion (FWI) uses seismic amplitudes to update subsurface parameters, we selected the first-order velocity-stress formulation to preserve dynamic accuracy. The first-order acoustic VTI wave equations in C_s parameterization are:

$$\begin{cases} \rho \frac{\partial V_x}{\partial t} = \frac{\partial \sigma_H}{\partial x} \\ \rho \frac{\partial V_y}{\partial t} = \frac{\partial \sigma_H}{\partial y} \\ \rho \frac{\partial V_z}{\partial t} = \frac{\partial \sigma_V}{\partial z} \\ \frac{1}{\rho C_s} \frac{\partial \sigma_H}{\partial t} = (1+2\varepsilon)^2 \sqrt{1+2\delta} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) + (1+2\delta)(1+2\varepsilon) \frac{\partial V_z}{\partial z} \\ \frac{1}{\rho C_s} \frac{\partial \sigma_V}{\partial t} = (1+2\varepsilon)(1+2\delta) \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) + \sqrt{1+2\delta} (1+2\varepsilon) \frac{\partial V_z}{\partial z} \end{cases} \quad (A1)$$

The defined C_s parameter only considered the velocity and anisotropic parameters, while the density parameter was not included. Therefore, the density parameter was added to **Equation A1**. We expressed the wave **Equation A1** in matrix form:

$$M\mathbf{p} = \mathbf{s}, \text{ where the } \mathbf{s} \text{ is the source term and } M = \mathbf{A} \frac{\partial}{\partial x} + \mathbf{B} \frac{\partial}{\partial y} + \mathbf{C} \frac{\partial}{\partial z} + \mathbf{D} \frac{\partial}{\partial t},$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -(1+2\varepsilon)^2 \sqrt{1+2\delta} & 0 & 0 & 0 & 0 \\ -(1+2\varepsilon)(1+2\delta) & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(1+2\varepsilon)^2 \sqrt{1+2\delta} & 0 & 0 & 0 \\ 0 & -(1+2\varepsilon)(1+2\delta) & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -(1+2\delta)(1+2\varepsilon) & 0 & 0 \\ 0 & 0 & -\sqrt{1+2\delta}(1+2\varepsilon) & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho C_s} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho C_s} \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} V_x \\ V_y \\ V_z \\ \sigma_H \\ \sigma_V \end{bmatrix} \quad (A2)$$

A measure of the difference between observed and modeled data can be obtained by the least-squares function (**Equation A3**):

$$J = \frac{1}{2} \int_H dx dy dz \int_0^T dt \{ \mathbf{p}(x, y, z, t) - \mathbf{p}'(x, y, z, t) \}^2 \quad (A3)$$

where $\mathbf{p}'$ and $\mathbf{p}$ are the observed and modeled data, respectively. T represents the maximum record time, and H represents the calculation area. The gradient of the objective function (A3), constrained by the wave equation, can be obtained using the Lagrange multiplier technique. The Lagrangian E is given as follows (**Equation A4**):

$$E = J + \frac{1}{2} \int_H dx dy dz \int_0^T \boldsymbol{\lambda}^T (M\mathbf{p} - \mathbf{s}) dt \quad (A4)$$

where $\boldsymbol{\lambda}$ is the Lagrange function and represents the adjoint wavefield in the form of $\boldsymbol{\lambda}^T = [V'_x \ V'_y \ V'_z \ \sigma'_H \ \sigma'_V]^T$. When the model is close to the true model, there is the following approximation: $\frac{\partial E}{\partial \mathbf{p}} = 0$, and E satisfies the relationship:

$$\begin{aligned} \frac{\partial E}{\partial \mathbf{p}} &= \int_{(x,y,z)} dx dy dz \int_0^T (\mathbf{p} - \mathbf{p}') dt + \frac{\partial \int_{(x,y,z)} \int_0^T \boldsymbol{\lambda}^T (\mathbf{M}\mathbf{p} - \mathbf{s}) dt}{\partial \mathbf{p}} = \int_{(x,y,z)} dx dy dz \int_0^T (\mathbf{p} - \mathbf{p}') dt - \frac{\partial \int_{(x,y,z)} \mathbf{p}^T \mathbf{M}^T \boldsymbol{\lambda} dt}{\partial \mathbf{p}} \\ &= \int_{(x,y,z)} dx dy dz \int_0^T (\mathbf{p} - \mathbf{p}') dt - \int_{(x,y,z)} dx dy dz \int_0^T \mathbf{M}^T \boldsymbol{\lambda} dt = 0 \end{aligned} \quad (\text{A5})$$

Thus, in the approximation of the basic accurate model value, the adjoint wavefield can be represented as:

$$\mathbf{M}^T \boldsymbol{\lambda} = \mathbf{p} - \mathbf{p}' \quad (\text{A6})$$

Equation A6 is equal to:

$$\begin{cases} \rho \frac{\partial V'_x}{\partial t} - (1+2\varepsilon)^2 \sqrt{1+2\delta} \frac{\partial \sigma'_H}{\partial x} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial x} = 0 \\ \rho \frac{\partial V'_y}{\partial t} - (1+2\varepsilon)^2 \sqrt{1+2\delta} \frac{\partial \sigma'_H}{\partial y} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial y} = 0 \\ \rho \frac{\partial V'_z}{\partial t} - (1+2\delta)(1+2\varepsilon) \frac{\partial \sigma'_H}{\partial z} - \sqrt{1+2\delta}(1+2\varepsilon) \frac{\partial \sigma'_V}{\partial z} = 0 \\ \frac{1}{\rho C_s} \frac{\partial \sigma'_H}{\partial t} = \frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \\ \frac{1}{\rho C_s} \frac{\partial \sigma'_V}{\partial t} = \frac{\partial V'_z}{\partial z} \end{cases} \quad (\text{A7})$$

The adjoint equation in C_s parameterization is given in Equation A7. The gradient formula in the new parameterization was obtained by derivation of the objective function with respect to parameters in model space:

$$\frac{\partial E}{\partial C_s} = \boldsymbol{\lambda}^T \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\rho C_s^2} \frac{\partial \sigma'_H}{\partial t} \\ \frac{1}{\rho C_s^2} \frac{\partial \sigma'_V}{\partial t} \end{bmatrix} = \frac{1}{\rho C_s^2} \left(\frac{\partial \sigma'_H}{\partial t} \sigma'_H + \frac{\partial \sigma'_V}{\partial t} \sigma'_V \right) \quad (\text{A8})$$

$$\begin{aligned} \frac{\partial E}{\partial \delta} &= \boldsymbol{\lambda}^T \begin{bmatrix} 0 \\ 0 \\ -\frac{(1+2\varepsilon)^2}{\sqrt{1+2\delta}} \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) - 2(1+2\varepsilon) \frac{\partial V'_z}{\partial z} \\ -2(1+2\varepsilon) \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) - \frac{1+2\varepsilon}{\sqrt{1+2\delta}} \frac{\partial V'_z}{\partial z} \\ -2(1+2\varepsilon) \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) \sigma'_V - \frac{1+2\varepsilon}{\sqrt{1+2\delta}} \frac{\partial V'_z}{\partial z} \sigma'_V \end{bmatrix} \\ &= -\frac{(1+2\varepsilon)^2}{\sqrt{1+2\delta}} \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) \sigma'_H - 2(1+2\varepsilon) \frac{\partial V'_z}{\partial z} \sigma'_H \\ &\quad -2(1+2\varepsilon) \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) \sigma'_V - \frac{1+2\varepsilon}{\sqrt{1+2\delta}} \frac{\partial V'_z}{\partial z} \sigma'_V \end{aligned} \quad (\text{A9})$$

$$\frac{\partial E}{\partial \varepsilon} = -2(1+2\varepsilon) \sqrt{1+2\delta} \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) \sigma'_H - 2(1+2\delta) \frac{\partial V'_z}{\partial z} \sigma'_H - 2(1+2\delta) \left(\frac{\partial V'_x}{\partial x} + \frac{\partial V'_y}{\partial y} \right) \sigma'_V - 2\sqrt{1+2\delta} \frac{\partial V'_z}{\partial z} \sigma'_V \quad (\text{A10})$$

The gradient formulas of multi-parameter FWI in C_s parameterization are given as Equations A8–A10. This technique provides a sensitive solution for gradient calculation in multi-parameter FWI based on surface seismic data, expressed in the C_s parameterization.