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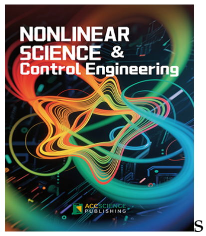

On inverse problems in mathematical neuroscience: Nonlinear neurodynamic processes described by second-order bilinear evolution equations with delay

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ABSTRACT
The quest to mathematically realize the complex dynamics of neuromorphic processes has led to the development of differential models that approximate neural signal behavior within controlled dynamical systems. Based on the maximum entropy principle and the tensor product of Hilbert spaces, we examine the solvability of the problem concerning the existence of a differential realization of nonlinear neuromorphic dynamic processes in a class of bilinear nonstationary ordinary differential equations of the second order (with and without delay) in a separable Hilbert space. Additionally, we analyze the metric conditions of continuity of the projectivization of the entropy Rayleigh–Ritz operator, and compute the fundamental group of its compact image. These problems belong to the class of nonstationary operator inverse problems for evolutionary equations in an infinite-dimensional Hilbert space. They provide new insights into the development of the theory of nonlinear inverse problems for higher-order multilinear non-autonomous differential equations with a "delay factor," which cannot be solved with respect to the higher-order derivatives. The results obtained in this study may be applicable to the precision modeling of differential equations of nonlinear neurodynamics, providing a meta-basis for the analysis of the cognitive activity of local neuropopulations under investigation.
Keywords: Entropy Rayleigh–Ritz operator; Inverse problems of nonlinear neurodynamics; Maximum principle for entropy; Second-order bilinear differential realization with delay; Tensor analysis

1. Introduction

The physical world is so complex that researchers of nature are satisfied if they manage to perceive and understand at least some of its inherent laws, even the simplest ones. To try to understand the world, researchers introduce simplified and idealized mathematical models (freed, from

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their point of view, from unimportant and complicating details), which they hope reflect the most essential properties of the physical objects under consideration. The set of such problems includes the challenging idea of constructing physical laws expressed by differential models (D-models; we consider the terms "D-model," "D-realization," and "D-equation" as synonyms), which describe the behavior of neurodynamic processes—that is, vector functions of neural signals, integrally represented by an $aposteriori$ protomodel of the "black box" in the form of dynamic processes of the multichannel interface-platform "brain–machine." This approach seeks to answer the question: to what extent (and by what characteristic criteria) may non-autonomous bilinear D-models, which take into account the delay factor of neurosignals, be classified among them? It should be noted that, as a rule, in the course of such investigations the situation is as follows: (i) at the first stage, the decision on choosing the model class is made; (ii) at the second stage, deductive constructions or conclusions are applied on the basis of the proposed model class (in the case of neurodynamic models, see Brzychczy and Poznanski[1]). In this context, the qualitative theory of differential realization (QTDR) occupies a special place in the methodology of step (i).

Notably, Kalman et al.,[2] the founders of QTDR, while assuming that the problem of differential implementation plays a central role in the theory of dynamical systems, formulated the following me thodological approach: consider the problem of realization as an attempt to infer the D-equations of motion of the dynamical system from the behavior of its input and output signals, or as the problem of constructing a physical model that adequately explains experimental data of the "input–output" type, which are represented by the "black box" protomodel (p. 286).

The current period of intensive development of QTDR in the infinite-dimensional formulation is largely associated with the creation of a new mathematical language for the entropy theory of extensions of M_2-operators.[3] This theory has substantially rebuilt and strengthened the theoretical foundations of QTDR and provided a harmonious connection between the functional-analytical ideas of M_2-continuance and the methods of geometric modeling bilinear D-equations,[4] focusing on the analysis of the existence of models in infinite-dimensional spaces, and emphasizing the "entropy approach"[5] rather than pursuing the maximum generality of the presentation of QTDR results.

This circumstance inevitably influences the content of the present study. In many problems of mathematical neurophysiology, models of differential representation of neuromorphic processes of the type "response reaction (the state of the synapses) to external influence (the state of the receptors)" in local populations of neurons necessitate consideration of nonlinear nonstationary interactions. These interactions arise from the reaction itself, from its rate, and from the external influence. Accordingly, our principal attention below is directed toward investigating the D-realization, which depends on five bilinear nonstationary structures.

The first of them (a vector function of time) depends on the reaction; the second bilinear operator depends on this reaction and its rate; the third bilinear operator depends only on the rate of this reaction, while the other two take into account these variables in connection with the influence of neurosignals from the receptors (external influence). At the same time, each bilinear operator has its own neurodynamic parameter of "delay," which reflects the factor of time delay in the course of physicochemical interactions occurring in the activated synapses of the neuropopulation.

Furthermore, QTDR—as a research field framed within the infinite-dimensional formulation of inverse problems in nonlinear mathematical neurophysiology—is more complex, more interesting, deeper in terms of applications, and highly important for understanding the basic properties of D-models. Its geometric constructions may serve as new starting points for the contemporary development of the general (axiomatic) theory of controlled dynamical systems (see, e.g., Chapter 10 in Kalman et al.[2] and Chapter 3 in Mesarovic and Takahara[6]). Such constructions may also establish QTDR as a powerful mathematical tool for the precision $aposteriori$ modeling of complex infinite-dimensional dynamical systems with or without delay, including nonlinear non-autonomous neuromorphic D-models.[1]

2. Problem statement: Bilinear differential realization modeling with delay in the bilinear operators

Let $(X, \|\cdot\|_X)$, $(Y, \|.\|_Y)$ denote real separable Hilbert spaces—normed spaces that satisfy "the requirement of parallelogram" (see Kantorovich and Akilov,[7] p. 162). In this case, we will use the linear isometry (which preserves the norm) $E : Y \to X$ of spaces Y and X (Kantorovich and Akilov,[7] p. 176). As usual, $L(B', B_{\rm J})$ is the Banach space (equipped with the operator norm) of all continuous linear operators between the two Banach spaces B' and B''; $L(X^2, X)$ is the space of all continuous bilinear maps from the Cartesian square $X \times X$ into space X. Therefore, we actively use the property of linear isometry between spaces $L(X^2, X)$ and $L(X, L(X, X))$ (see Kantorovich and Akilov,[7] p. 650). The notation $i = \overline{1, n}$ indicates that $i = 1,...,n$.

Let us denote by T a segment of the numerical line R ($\bar{R} = R \cup \{-\infty, \infty\}$) equipped with the Lebesgue measure μ, and by $\wp_\mu$ the σ-algebra of all μ-measurable subsets of T. The notation $S \underset{\mathrm{mod}\,\mu}{\subseteq} Q$ (S,$Q \in \wp_\mu$) indicates that $\mu(S \backslash Q) = 0$. Furthermore, let us assume that $AC^1(T, X)$ is the set of all the functions $g : T \to X$, whose first derivative dg/dt is absolutely continuous with respect to the measure μ on the interval T.

If below $(\mathbf{B}, \|\cdot\|_B)$ is a definite Banach space, then by $\mathrm{L}_p(T, \mathbf{B})$, $p \in [1, \infty)$ we denote the Banach space of all classes of μ-equivalent Bochner integrable (Yosida[8], p. 189) maps f: $T \to \mathbf{B}$ with norm $(\int_T ||f(\tau)||^p_{\mathbf{B}} \mu(d\tau))^{1/p} < \infty$, and, respectively, by $\mathrm{L}_\infty(T, \mathbf{B})$, we denote the Banach space of the given classes with norm ess $\sup_T \|f\|_{\mathbf{B}} < \infty$. In this context, let us agree that:

$$\mathbf{II} = \mathrm{L}_2(T, Y) \times \mathrm{AC}^1(T, X) \tag{1}$$

$$\begin{aligned}\mathbf{L_2} = {} & \mathrm{L}_2\big(T, L(X, X)\big) \times \mathrm{L}_2\big(T, L(X, X)\big) \times \mathrm{L}_2\big(T, L(X, X)\big) \\ & \times \mathrm{L}_2\big(T, L(X^2, X)\big) \times \mathrm{L}_2\big(T, L(X^2, X)\big) \times \mathrm{L}_2\big(T, L(X^2, X)\big) \\ & \times \mathrm{L}_2\big(T, L(X^2, X)\big) \times \mathrm{L}_2\big(T, L(X^2, X)\big)\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{L}^* = {} & L(X, X) \times L(X, X) \times L(Y, X) \times L(X^2, X) \\ & \times L(X^2, X) \times L(X^2, X) \times L(X^2, X) \times L(X^2, X)\end{aligned} \tag{3}$$

Next, we assume that the time interval T is fixed. $A\,posteriori$, the behavior of the examined (simulated) local neuropopulation, represented in the form of a nonlinear beam N of observed neural processes in the state space X, is given by tuples (pairs) of vector functions of the type "external influence on the receptors, response of the synapses." This representation corresponds to th+e behavioral system[4] in the form of a "black box" model (see, for instance, Definition 1.8 in Kalman et al.,[2] p. 20 and Definition 1.2 in Mesarovic and Takahara,[6] p. 21), or, equivalently, to a nonlinear bundle of tuples (u,x) of the "input–output" type, formally:

$$N \subset \mathbf{\Pi},\ \text{Card}\ N \leq \exp\ \aleph_0,$$

where $(u,x) \in N$ is the pair "control (effect on the receptors), trajectory (state of the synapses)" [2]; $\aleph_0$is aleph-null; $\exp\ \aleph_0$ is the continuum. The term "nonlinear beam N" indicates that the superposition principle (Kalman et al.,[2] p. 18) is not assumed $a\,priori$ for the trajectories of the given beam, even when the dependence of x-trajectories on u-controls is linear (see Definition 1.1 in Mesarovic and Takahara,[6] p. 84).

In addition, we assume that the differential realization of the N-system must be of the second order, since second-order D-models allow modeling, as an endogenous factor, variations in the rate of neuroimpulses. In other words, they contain the term $\hat{A}d^2x/dt^2$, where the operator-function $\hat{A} : T \to L(X,X)$ has the following properties:

$$\hat{A} \in \mathrm{L}_\infty(T,L(X,X)), \quad \mu\{t \in T :\ \text{Ker}\ \hat{A}(t) \neq \{0\} \subset X\} \geq 0 \tag{4}$$

Let us call the variant $\mu\{t \in T :\ \text{Ker}\ \hat{A}(t) \neq \{0\} \subset X\} = 0$ the "general reducibility" to the normal D-model. To make this description more precise, we again require information about the pair $(N, \hat{A})$:

(i) Definition 1: The pair $(N, \hat{A})$ is called non-defective (or defective when ... $\underset{\mathrm{mod}\,\mu}{\neq}$ T), if:

$$N \subset \mathbf{\Pi},\ \text{Card}\ N \leq \exp\ \aleph_0, \tag{5}$$

(ii) Consider the following problem: For a non-defective pair $(N, \hat{A})$, determine the necessary and sufficient conditions under which the bilinear differential realization (BDR) of the second order with delays $\hat{\tau}_i = \text{const} \geq 0, i = \overline{1,5}$ is feasible (realizable) and has an analytical representation of the form:

$$\begin{aligned}
&\exists\ (A_1, A_0, B,\ D_1,D_2, D_3,D_4,D_5) \in \mathbf{L_2}\ :\\
&\hat{A}\, d^2x/dt^2 + A_1 dx/dt + A_0 x =\\
&= B\,u + D_1(x_1,x_1) + D_2(x_2,dx_2/dt) + D_3(dx_3/dt,dx_3/dt)+\\
&+ D_4(E(u), x_4) + D_5(E(u), dx_5/dt)\ \ \forall\,(u,x) \in N\,,
\end{aligned} \tag{6}$$

$$t \mapsto x_i(t) = \begin{cases} x\big(t - \hat{\tau}_i\big), & \text{if } t_0 + \hat{\tau}_i \leq t \leq t_1; \\ 0 \in X, & \text{if } t_0 \leq t < t_0 + \hat{\tau}_i, \quad i = \overline{1,5}. \end{cases}$$

The equality in Equation (6) is considered an identity in a functional space $\mathrm{L}_1(T, X)$. If the modeled operators of the BDR Equation (6) are presupposed to be sought in the class of stationary ones, they will instead be constructed in the class of continuous ones. In this case, we may write $(A_1, A_0, B,\ D_1,D_2, D_3,D_4,D_5) \in \mathbf{L}^*$.

In connection with this mathematical formulation, we note that each field of mathematics, as a rule, contains leading problems that are so difficult that their complete solution is not even expected. Nevertheless, such problems stimulate a constant flow of research and serve as the main milestones on the path toward progress in that field.

In the case of QTDR investigations, such a problem is the classification of continuous behavioral systems that precisely coincide with the solutions of idealized D-models, including those of higher orders. In the strongest analytical form, the solution requires classification of the scrutinized dynamical systems with precision up to the corresponding class of D-realization models, including the class of nonstationary BDR-models Equation (1) with delay. Such an approach is justified by Theorems 1 and 2, which allow us to substantially approach an ideal combination of functional transparency[3] and geometric clarity[4,9] in this model classification.

3. Entropy indicator of the delayed bilinear differential realization model: Equivalent formulations

Contemporary theoretical and applied works in the field of $a\,posteriori$ differential modeling are usually initiated with the phrase: "Let the object of mathematical modeling be described with a D-equation of the form..." However, we begin the construction of the qualitative theory of inverse problems of nonlinear non-autonomous neurodynamics by demonstrating the "fact of existence" of bilinear D-equations with delay for a local neuropopulation, represented through certain analytical properties of its behavior and, moreover, characterized by a functional set N.

Let $Z = X \otimes X$ be the tensor product (Kirillov,[10] p. 54) of Hilbert spaces X and X with the cross-norm $\|\cdot\|_Z$ defined by the scalar product in X. In addition, let us introduce new notations, which will be used extensively below:

$$U = X \times X \times Y \times Z \times Z \times Z \times Z \times Z, \tag{7}$$

$$\begin{aligned}\|(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)\|_U = (&\|\cdot\|_X^2 + \|\cdot\|_X^2 + \|\cdot\|_Y^2 + \|\cdot\|_Z^2\\ &+ \|\cdot\|_Z^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2)^{1/2};\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{L}_{2,U} = &\mathrm{L}_2(T, L(X,X)) \times \mathrm{L}_2(T, L(X,X)) \times \mathrm{L}_2(T, L(Y,X))\\ &\times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X))\\ &\times \mathrm{L}_2(T, L(Z,X)) \times \mathrm{L}_2(T, L(Z,X));\end{aligned} \tag{9}$$

The product space $\mathbf{L}_{2,U}$ (with the product topology) is linearly homeomorphic to the Banach functional space $\mathrm{L}_2(T,L(U,X))$.

Let us denote by π a universal bilinear map $\pi : X \times X \to X \otimes X$. In the language of categories, a morphism π defines (algebraically) a tensor product as a universal mapping object (Kirillov,[10] p. 40). The universality of the bilinear map π also implies the fact that:

$$\pi : X \times X \to X \otimes X, \tag{10}$$

$$(h_1, h_2) \mapsto \pi(h_1, h_2) = h_1 \otimes h_2,\ \ \|h_1 \otimes h_2\|_Z = \|h_1\|_X \|h_2\|_X\,; \tag{11}$$

These relations are important for defining the construction of the nonlinear Rayleigh–Ritz functional operator in terms of specifying the norm $\|\cdot\|_U$ (see Definition 2).

Next, we assume that the Cartesian square is endowed with the norm $(\|\cdot\|_X^2 + \|\cdot\|_X^2)^{1/2}$. Under this problem statement, it is obvious that $\pi \in L(X^2,Z)$, and, according to Theorem 2 (see Kantorovich and Akilov,[7] p. 245), for any continuous bilinear map $D \in L(X^2,X)$, there always exists a linear continuous operator $D \in L(Z,X)$, such that $D = D \circ \pi$, and, for any pair $(u,x) \in N$, the following inclusions will be satisfied:

$$\begin{aligned} &\pi(x,x), \pi(x,dx/dt), \pi(dx/dt,dx/dt) \in \mathrm{L}_\infty(T,Z), \\ &\qquad \pi(E(u),x), \pi(E(u),dx/dt) \in \mathrm{L}_2(T,Z)\,; \end{aligned} \tag{12}$$

It is clear that these inclusions are not violated by the effect of delays $\hat{\tau}_i = \text{const} \geq 0$, $i = \overline{1,5}$.

The preliminary constructions introduced above may be summed up in the following statement:

(a) Lemma 1

For any set $(A_1, A_0, B,\ D_1,D_2,D_3,D_4,D_5) \in \mathbf{L}_2$ and map

$$\begin{aligned} &F:\ \mathrm{L}_2(T,X) \times \mathrm{L}_2(T,X) \times \mathrm{L}_2(T,Y) \times \\ &\times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \mathrm{L}_2(T,X^2) \times \\ &\mathrm{L}_2(T,X^2) \to \mathrm{L}_1(T,X), \end{aligned} \tag{13}$$

$$\begin{aligned} &(y_1,...,y_8) \mapsto F(y_1,...,y_8) := A_1 y_1 + A_0 y_2 + B\, y_3 + D_1 y_4 \\ &+ D_2 y_5 + D_3 y_6 + D_4 y_7 + D_5 y_8 \end{aligned} \tag{14}$$

There exists a unique cortege of operator functions

$$(D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8) \in \mathbf{L}_{2,U} \tag{15}$$

and, correspondingly, a unique linear map

$$M{:}\mathrm{L}_2(T,U) \to \mathrm{L}_1(T,X), \tag{16}$$

which has an analytical representation of the form:

$$(z_1,...,z_8) \mapsto M(z_1,...,z_8) = D_1 z_1 + ... + D_8 z_8, \tag{17}$$

such that the following functional equality holds:

$$\begin{aligned} &(y_1,...,y_8) \mapsto F(y_1,...,y_8) = M\,(y_1, y_2, y_3, \pi(y_4), \\ &\pi(y_5), \pi(y_6), \pi(y_7), \pi(y_8)), \end{aligned} \tag{18}$$

which, in turn, induces the following operator relations for the operator functions arising from the map constructions F and M:

$$A_1 = D_1,\ \ A_0 = D_2,\ \ B = D_3, \tag{19}$$

$$D_i = D_{i+3} \circ \pi,\ \ i = \overline{1,5}. \tag{20}$$

Everywhere below (in the context of geometric design of the BDR-problem Equation (6), we assume:

$$\begin{aligned} &V_N = \mathrm{Span}\{(dx/dt, x, u, \pi(x_1,x_1), \pi(x_2, dx_2/dt), \\ &\pi(dx_3/dt, dx_3/dt), \end{aligned} \tag{21}$$

$$\pi(E(u), x_4), \pi(E(u), dx_5/dt)) \in \mathrm{L}_2(T,U) : (u,x) \in N\}. \tag{22}$$

The following Lemma generalizes the behavioral condition Equation (7).[11]

(b) Lemma 2

Let $g = (v_1,...,v_8) \in V_N$ and $S_g = \{t \in T : g(t) = 0 \in U\}$, $Q_g = \{t \in T : dv_1(t)/dt = 0 \in X\}$, and it is clear that $(S_g, Q_g) \in \wp_\mu \times \wp_\mu$. Hence, $S_g \underset{\mathrm{mod}\,\mu}{\subseteq} Q_g$.

Next, let $(\mathrm{L}_+(T,R),\leq_\mathrm{L})$ be a positive cone[12] of μ-equivalence classes of all real, nonnegative, μ-measurable, and almost everywhere finite functions given on T with a quasi-ordering $\leq_\mathrm{L}$ for which $q_1 \leq_\mathrm{L} q_2$, if $q_1(t) \leq q_2(t)$, if almost everywhere in T. In this context, below $\inf_\mathrm{L}$ is the largest lower $\leq_\mathrm{L}$-edge and $\sup_\mathrm{L}$ is the smallest upper $\leq_\mathrm{L}$-edge.

Let subset $W \subset \mathrm{L}_+(T,R)$ has $\sup_\mathrm{L} W$; in particular, for $q_1,q_2 \in \mathrm{L}_+(T,R)$ we have

$$\sup\nolimits_\mathrm{L}\{q_1, q_2\} = q_1 \vee q_2 = 2^{-1}(q_1 + q_2 + |q_1 - q_2|). \tag{23}$$

In the present problem statement, we consider $R(W) = \{q \in L_+(T,R) : q \leq_L \sup_L W\}$. Hence, $(\mathcal{R}(W),\leq_\mathrm{L})$ is a lattice[12] with the smallest element $\inf_\mathrm{L} W = \chi_\varnothing \in \mathrm{L}_+(T,R)$ and the largest element $\sup_\mathrm{L} W \in \mathrm{L}_+(T,R)$. From now on, $\chi_\varnothing$ is the "zero-function" of the cone $\mathrm{L}_+(T,R)$. In the context of Theorem 17 (Kantorovich and Akilov,[7] p. 68) and Corollary 1 (Kantorovich and Akilov,[7] p. 69), the generalization for the lattice can be formulated.

(c) Lemma 3

The lattice $\mathcal{R}(W)$ is complete, that is, for any $V \subsetneq \mathcal{R}(W)$, there exist $\inf_\mathrm{L} V,\ \sup_\mathrm{L} V \in \mathcal{R}(W)$.

Before proceeding further, it is useful to introduce additional terminology related to the formalization of the mathematical constructions of the entropy Rayleigh–Ritz operator.[3,4]

(d) Definition 2 Let $\Psi{:}V_N \to \mathrm{L}_+(T,R)$ be a modification of the Rayleigh–Ritz operator of the form:

$$\begin{aligned} &t \mapsto \Psi(\varphi)(t) : \\ &= \begin{cases} \left\| \widehat{A}(t) dv_1(t)/dt \right\|_X / \|\varphi(t)\|_U \,, if \varphi(t) \neq 0 \in U; \\ \qquad 0 \in R, if \varphi(t) = 0 \in U; \end{cases} \end{aligned} \tag{24}$$

$$\begin{aligned} &||\varphi(t)||_U = ||(v_1(t), ..., v_8(t))||_U = \\ &= (\|v_1(t)\|_X^2 + \|v_2(t)\|_X^2 + \|v_3(t)\|_Y^2 + \|v_4(t)\|_Z^2 \\ &+ \|v_5(t)\|_Z^2 + \|v_6(t)\|_Z^2 + \|v_7(t)\|_Z^2 + + \|v_8(t)\|_Z^2)^{1/2} \end{aligned} \tag{25}$$

where $\varphi = (v_1,...,v_8) \in V_N$. Furthermore, the functions

$$(v_1, v_2, v_3) \mapsto \eta_L(v_1, v_2, v_3) = ||v_1, v_2, v_3, 0, ..., 0||_U^2, \tag{26}$$

$$\begin{aligned} &(v_4, v_5, v_6, v_7, v_8) \mapsto \eta_B(v_4, v_5, v_6, v_7, v_8) \\ &= ||0, 0, 0, v_4, v_5, v_6, v_7, v_8||_U^2 \end{aligned} \tag{27}$$

may be referred to, respectively, as the linear and bilinear characteristics of the Rayleigh–Ritz operator.

Due to Lemma 2, the following equality holds over the time interval T:

$$\mathrm{supp}\,\Psi(\varphi) \underset{\mathrm{mod}\,\mu}{=} \mathrm{supp}\left\| \hat{A}\, dv_1/dt \right\|_X, (\varphi = (v_1,...,v_8) \in V_N \tag{28}$$

In defining the support construction of the function carrier, we follow the definition provided by (Kantorovich and Akilov,[7] p. 137), where the carrier is defined up to a set of measure zero, which differs from the definition of the function carrier in Yoshida[8] and Edwards.[12]

The entropy operator Ψ satisfies the following simple yet important relations:

$$\chi_{\varnothing} \leq_L \Psi(\varphi) = \Psi(r\varphi), r \in R^* = R\backslash\{0\}, \varphi \in V_N; \quad (29)$$

In the notation, we will distinguish between the image of a point $\Psi(\varphi)$ and the image of a set $\Psi[\{\varphi\}]$.

(e) Definition 3 The map $P\Psi : P_N \to \mathrm{L}_+(T, R)$ is called the projectivization of the Rayleigh–Ritz operator if:

$$P\Psi(\gamma) = \Psi[\gamma], \gamma \in P_N(\gamma \subset V_N), \quad (30)$$

where P_N denotes the real projective space associated with V_N, equipped with the topology induced from $L_2(T, U)$. Here, P_N is a set of orbits of the multiplicative group R^* acting on $V_N\backslash\{0\}$.

In the given interpretation of the Rayleigh–Ritz operator, the key point is the topological structure of space P_N, $\dim P_N < \aleph_0$. First, consider its "compactness." In particular, if $\dim V_N = 3$, then the compact 2-manifold P_N is arranged as a Mobius sheet, to which a circle is attached along its boundary (Prasolov[13], p. 162). Additionally, the geometric structure of the CW-complex may be introduced on the manifold P_N (Prasolov[13], p. 140), which, in turn, substantially simplifies (see Theorem 9.7 in Prasolov[13], p. 149) the analysis of the issue of the geometric interpretation of the manifold P_N.

There is a characteristic of the N-system associated with the concept of entropy,[3,5] which requires particular attention. Below, exp$\mathbf{\Pi}$ denotes a family of all subsets of $\mathbf{\Pi}$.

(f) Definition 4

A functional Entrp$(\cdot,\cdot)$:exp$\mathbf{\Pi}$ $\times \wp_\mu \to \bar{R}$ represented in the form

$$(N, S) \mapsto \mathrm{Entrp}(N, S) = \left(\int_S (\sup_L P\Psi[P_N](\tau))^2 \mu(d\tau)\right)^{1/2} \quad (31)$$

will, by analogy with Rusanov et al.,[3] be referred to as the entropy of the behavioral N-system on the σ-algebra $\wp_\mu$.

(g) Proposition 1

The entropy functional Equation (31) possesses the following properties:

$$(i) \quad S, S_i \in \wp_\mu,\ i = \overline{1,n},\ S \subseteq \bigcup_{i=1}^{n} S_i \Rightarrow \mathrm{Entrp}(N, S) \leq \sum_{i=1}^{n} \mathrm{Entrp}(N, S_i) \quad (32)$$

$$(ii) \quad N \subseteq N^*, \mathrm{Entrp}(N^*, S) \neq \infty \Rightarrow \mathrm{Entrp}(N, S) \leq \mathrm{Entrp}(N^*, S) \quad (33)$$

The first property is called "the semiadditivity property of entropy," while the second property expresses the increase of entropy with the expansion of the N-system. In the general case, the QTDR role of entropy Equation (31) becomes evident from Theorem 1 below.

(h) Theorem 1

Theorem 1 serves as the entropy indicator and a characteristic of the BDR model. Each of the first three conditions implies the other two:

(i) The BDR problem Equation (6) is solved with respect to $(A_1, A_0, B,\ D_1, D_2, D_3, D_4, D_5) \in \mathbf{L}_2$;
(ii) There exists a function $\theta \in \mathrm{L}_2(T, R)$, such that for all $\varphi \in V_N$, one has $\Psi(\varphi) \leq_{\mathrm{L}} \theta$;
(iii) $\mathrm{Entrp}(N, T) < \infty$.
(iv) Satisfaction of condition $(A_1, A_0, B,\ D_1, D_2, D_3, D_4, D_5) \in \mathbf{L}^*$ necessitates satisfaction of condition $\mathcal{R}(P\Psi[P_N]) \subset \mathrm{L}_\infty(T, R)$.

(i) Remark 1

Obtaining Theorem 1 may be considered—within the framework of a set of inverse problems in neurodynamics—as an initial stage in the study of the problem, where it is necessary to compare a nonlinear bundle N from an "implicit" D-equation of the highest order (with delay) with an "explicit," nonstationary bilinear second order D-equation of (Equation 6) having the same bundle N, and a fixed operator-function $\hat{A}$. In particular, this problem statement is appropriate when the BDR problem, solvable for a tuple $(N,\ \hat{A}_1)$, needs to be reduced to a solvable BDR problem for a tuple $(N,\ \hat{A}_1)$, in the case where $\mu\{t \in T:\ \mathrm{Ker}\ \hat{A}_1(t) \neq 0 \in X\} \neq 0$, with $\hat{A}_2$ being a homothety operator with coefficient 1. The character of the respective computations is illustrated in Examples 1 and 2.

(j) Proof of Theorem 1

For the proof of Theorem 1, we adopt the approach presented in Rusanov et al.[3] Based on Definition 1,[3] we consider the construction M_2 of the operator M:$\mathrm{L}_2(T, U) \to \mathrm{L}_1(T, X)$ of the form

$$\exists\,(D_1, ..., D_8) \in \mathbf{L}_{2,U} : M(v_1, ..., v_8) = D_1 v_1 + ... + D_8 v_8\,, (v_1, ..., v_8) \in \mathrm{L}_2(T, U)\,. \quad (34)$$

The remaining details of the proof, with minor refinements and taking into account Lemmas 1–3 for the lattice, are presented in the continuation scheme of Theorem 3 in Rusanov et al.,[3] which incorporates the principle of maximum entropy.[5] The second and third conditions of Theorem 1 are connected by the following relation: $\mathrm{Entrp}(N, S) \leq \left(\int_S \theta^2(\tau)\mu(d\tau)\right)^{1/2}\ \forall S \in \wp_\mu$.

The necessary condition, corresponding to the fourth point, is obtained by a modification of the proof of Theorem 3 in the study by Rusanov et al.[11]

(k) Remark 2

In the case when $1 < \mathrm{Card}\, N < \aleph_0$, we have $\mathrm{Card}\, P_N = \exp \aleph_0$, which complicates the verification of the third condition of Theorem 1 due to Equation (31). However, it can be shown (see Theorem 17 in Kantorovich and Akilov[7], p. 68) that there exists a countable set such that, if the cone $\mathrm{L}_+(T, R)$ contains $\sup_{\mathrm{L}} P\Psi[P_N]$, then the function $\zeta = \sup_L P\Psi[P_N] = \bigvee_{\gamma \in G} P\Psi(\gamma)$, can be realized via the following sup-construction:

$$t \mapsto \zeta(t) = \sup\{P\Psi(\gamma)(t) \in R\text{: } \gamma \in G\}. \quad (35)$$

The second condition of Theorem 1 allows us to construct analogues of Theorem 2 in Lakeyev et al.,[14] which provides sufficient conditions for the existence of a bilinear second-order D-realization that realizes the given dynamic bundles $N_1,\ N_2$. Each bundle has its specific BDR model Equation (6). This also relates to the case where BDR-simulated operators of the D-equation Equation (6) are stationary (i.e., lie in space $\mathbf{L}^*$), particular with a minimum operator norm. One such QTDR criterion can be observed in the following particular case (when $\mathrm{Card}\, N < \aleph_0$):

(l) Corollary 1

The BDR problem Equation (6) is solvable for $\dim V_N = n$ when the following two conditions hold:

(i) Let $\{\xi_i\}_{i=\overline{1,n}}$ be an algebraic basis in $\dim V_N$, hence $\Psi[\{\xi_i\}_{i=\overline{1,n}}] \subset \mathrm{L}_2(T, R)$.
(ii) Let $p \in [1, \infty)$, hence $\Psi(\varphi_1 + \varphi_2) \leq_{\mathrm{L}} p\Psi(\varphi_1) + p\Psi(\varphi_2)$, $(\varphi_1, \varphi_2) \in V_N \times V_N$.

(m) Remark 3

The first condition of Corollary 1 indicates that any tuple $(u,x) \in N$ has a D-realization (Equation 6), possibly with an "individual" cortege for $(A_1, A_0, B,\ D_1, D_2, D_3, D_4, D_5) \in \mathbf{L}_2$ (uniqueness of the BDR solution). When $p = 1$, the second condition corresponds to the property of "sublinearity" [12] of the functional operator Ψ.

4. Continuity of the Rayleigh–Ritz operator in the analysis of the solvability of the bilinear differential realization problem with delay

In the case of compactness of a projective variety P_N (which is equivalent to $\dim\ P_N < \aleph_0$), it is natural to attempt to relate this property to the problem of constructing a lattice $\mathcal{R}(P\Psi[P_N])$ in the context of the continuity conditions of the projectivization of the Rayleigh–Ritz operator (see Lakeyev et al.[15]). Below, when choosing a metric structure in a cone $\mathrm{L}_+(T,R)$ in Theorem 2, we apply Theorems 15 (Kantorovich and Akilov,[7] p. 65) and 16 (Kantorovich and Akilov,[7] p. 67). In this problem statement $\mathrm{L}_+(T,R)$ is a complete separable metric space.

The principal QTDR results of this section follow directly from the following proposition:

(a) Proposition 2

Let $\dim\ P_N < \aleph_0$ and cone $\mathrm{L}_+(T,R)$ be endowed with the topology induced by convergence with respect to measure μ, or, equivalently, endowed with an invariant metric:

$$\rho_T(f_1,f_2) = \int_T |f_1(\tau) - f_2(\tau)|(1+|f_1(\tau) - f_2(\tau)|)^{-1}\mu(d\tau),$$
$$f_1, f_2 \in L_+(T,R). \tag{36}$$

Hence, operator $P\Psi : P_N \to \mathrm{L}_+(T,R)$ is continuous when bundle N satisfies:

$$\operatorname{supp} \|\varphi\|_U \underset{\operatorname{mod}\mu}{=} T,\ \ \varphi \in V_H\{0\}, \tag{37}$$

In particular, when:

$$\operatorname{supp} P\Psi(\gamma) \underset{\operatorname{mod}\mu}{=} T,\ \ \gamma \in P_H. \tag{38}$$

(b) Remark 4

The invariance of the metric requires the satisfaction of the equality:

$$\rho_T(f+q, g+q) = \rho_T(f,g), f, g, q \in \mathrm{L}_+(T,R). \tag{39}$$

A variant of a non-invariant, incomplete metric on $L+(T,R)$, which does not depend on Equations (37) and (38) but ensures the continuity of the Rayleigh–Ritz operator, is discussed in Lakeyev et al.[15]

Notably, Proposition 2 extends Corollary 12.1 in D'yachenko and Ul'yanov[16] (p. 54), highlighting its methodological importance in the a posteriori modeling of complex dynamical systems.[2–6] One of the homotopy-based applications of this result is reflected in the following statement.

(c) Corollary 2

If, when Equation (37) or Equation (38) is satisfied, operator $P\Psi$ is one-to-one, then $P\Psi$ is a homeomorphism, and the fundamental group of the metric space $(P\Psi[P_N], \rho_T)$ is isomorphic to the additive group of integers Z when $\dim\ \operatorname{Span} N = 2$, and to the group Z_2 of residues when $\dim\ \operatorname{Span}\ N \geq 3$. Furthermore, the space $(P\Psi[P_N], \rho_T)$ is orientable when the dimension of the linear shell Span N is even, and non-orientable when this dimension is odd.

Considering that a continuous real function on a compact space attains its largest and smallest values, we conclude that, when Equation (37) or Equation (38) is satisfied and the conditions in Theorem 5 (Kantorovich and Akilov[7], p. 28) are taken into account, for the case $1 \leq \dim\ P_N < \aleph_0\ \sup_{\mathrm{L}} P\Psi[P_N] \in \mathrm{L}_+(T,R)$, the points $\gamma', \gamma'' \in P_N$ can be identified such that the following estimated relations hold:

$$\begin{aligned}\rho_T(\,P\Psi(\gamma'), \chi_\varnothing) &= \sup\{\rho_T(\,P\Psi(\gamma), \chi_\varnothing) :\ \gamma \in P_N\} \\ &\leq \rho_T(\sup_{\mathrm{L}} P\Psi[P_N], \chi_\varnothing) < \mu(T),\end{aligned} \tag{40}$$

$$\begin{aligned}\rho_T(\,P\Psi(\gamma''),\ \sup_{\mathrm{L}} P\Psi[P_N]) = \inf\{&\rho_T(\,P\Psi(\gamma), \\ &\sup_{\mathrm{L}} P\Psi[P_N]) :\ \gamma \in P_N\} \geq 0\end{aligned} \tag{41}$$

Note that condition $P\Psi(\gamma') \in \mathrm{L}_2(T,R)$ does not guarantee that $R(P\Psi[P_N]) \subset \mathrm{L}_2(T,R)$ (see Example 1 in D'yachenko and Ul'yanov[16]). Notably (see Remark 3), $\dim\ P_N = 0$ leads to the condition:

$$\begin{aligned}&\sup_{\mathrm{L}} P\Psi[P_N] = P\Psi[P_N] = \left\|\hat{A}\, d^2x/dt^2\right\|_X / (\,\|\,dx/dt\|_X^2 \\ &+ \|x\|_X^2 + \|u\|_Y^2 + \|x_1\|_X^4 + \|x_2\|_X^2 \\ &\|\,dx_2/dt\|_X^2 + \|\,dx_3/dt\|_X^4 + \|E(u)\|_X^2\,\|x_4\|_X^2 \\ &+ \|E(u)\|_X^2\,\|\,dx_5/dt\|_X^2\,)^{1/2}.\end{aligned} \tag{42}$$

In the context of Theorem 1 and Proposition 2, it is possible to refine the analytical conditions for the existence of the lattice $R(P\Psi[P_N])$. To do this, we take an auxiliary construction as the starting point: for a natural number i, let us denote by W_i a finite i^{-1}-dense subset in the metric space $(P\Psi[P_N], \rho_T)$. The subset W_i may be found (when either condition in Equation (37) or Equation (38) is satisfied) due to Proposition 2. We agree that below $\operatorname{Lim}_{\rho_T}\{\xi_n\}$denotes the limit of the sequence $\{\xi_n\} \subset \mathrm{L}_+(T,R)$ in the topology induced by the metric ρ_T.

Based on the above statement, and in the context of the semi-additivity property of entropy for an N-system (by the Radon–Nicodemus Theorem [Reed and Simon,[17] p. 38]), the following statement can be easily proved as a QTDR result.

(d) Theorem 2

Let the condition in Equation (37) or Equation (38) be satisfied, and let

$$\{W_i\}_{i=\overline{1,n}},\ \ W_i = \{\zeta_1, \ldots, \zeta_{k_i}\} \subset P\Psi[P_N], \tag{43}$$

Additionally,

$$f_n := \xi_1 \vee \ldots \vee \xi_n, \xi_i = \zeta_1 \vee \ldots \vee \zeta_{k_i}, i = \overline{1,n}\,. \tag{44}$$

Hence, cone $\mathrm{L}_+(T,R)$ contains the lattice $R(P\Psi[P_N])$ if and only if

$$\rho_T(f_n, f_m) \to 0\ (n,m \to \infty),$$

Furthermore, the BDR solvability takes the following form: the non-defective tuple $(N, \hat{A})$ possesses the property of D-realization Equation (6) if and only if $\operatorname{Lim}_{\rho_T}\{f_n\} \in \mathrm{L}_2(T,R)$, which is equivalent to any one of the following conditions:

(i) $\mathcal{R}(P\Psi[P_N]) \subset \mathrm{L}_2(T,R)$;

(ii) $\mathrm{Entrp}^2(N,\cdot) : \wp_\mu \to R$ is an absolutely continuous measure (with respect to μ) on $\wp_\mu$.

In summary, let us consider examples that challenge the necessity of emphasizing the ideological aspect of each QTDR concept, which might otherwise lead to inadvertently neglecting its computational consideration. Hence, we assume that simulations are conducted each time with "zero delays" $\hat{\tau}_i = 0,\ i = \overline{1,5}$ (i.e., $x_i = x$).

(e) Example 1

Let $(T = [0, 10], Y = X,)$, $\hat{A}$ be the operator of homothety[12] with the coefficient of 1, and let

$$A_1 = 0 \in L(X,X), D_1 = D_3 = D_4 = D_5 = 0 \in L(X^2, X) \tag{45}$$

$$e \in X, ||e||_X = 1 \tag{46}$$

In addition, let us assume that the modeled dynamic process has the form

$$N = \{(u,x)\} \tag{47}$$

$$t \mapsto u(t) = 0 \in \mathrm{L}_2(T, X), t \mapsto x(t) = (t \sin t)e. \tag{48}$$

Then, according to Equation (42), the function

$$f = sup_L P\Psi(P_N) = ||d^2x/dt^2||_X(||x||_X^2 + ||x||_X^2\,||dx/dt||_X^2)^{-1/2} \tag{49}$$

does not belong to the space $\mathrm{L}_2(T, R)$ (Figure 1). In other words, $\mathrm{Entrp}(N, T) = \infty$ and therefore, according to Theorem 1, D-realization Equation (6) for the uncontrolled process $N = \{(u,x)\}$ does not exist.

(f) Example 2

Let us change the statement of Example 1 as follows:

$$t \mapsto u(t) = (t\sin^2 t + 2^{-1}t^2 \sin 2t + \cos t)e. \tag{50}$$

Then, according to Equation (42), we have (Figure 2):

$$\begin{aligned} f = sup_L P\Psi(P_N) = ||d^2x/dt^2||_X(||x||_X^2 \\ +||x||_X^2\,||dx/dt||_X^2 + ||u||_Y^2)^{-1/2} \in \mathrm{L}_2(T, R) \end{aligned} \tag{51}$$

where $\mathrm{Entrp}(N, T) < \infty$, and thus, the D-implementation (Equation [6]) for the controlled process $N= \{(u,x)\}$ exists. It can readily be verified that $d^2x/dt^2 + x = 2u - 2D_2(x, dx/dt)$, where $D_2 =< \cdot,\cdot>_X e$, $< \cdot,\cdot>_X$ is a scalar product in X.

Note that for more complex variants of assigning the tuple ($t \mapsto u(t)$,$t \mapsto x(t)$), symbolic computations of the function $t \mapsto f^2(t)$ (similar to those presented in Figures 1 and 2), as well as the computation of $\mathrm{Entrp}(N)$, may be conducted using computer algebra systems.[18] In this context, the scheme for analyzing the solvability of the BDR problem in Examples 1 and 2 may be adapted for the qualitative analysis of the reduction of exact multidimensional diffusion solutions with power nonlinearities to an initial-value problem for a countable system of ordinary D-equations with a polylinear structure.

The following example demonstrates that the nonstationarity property of the BDR model arises as an endogenous factor in the D-realization of the bundle N (see the fourth condition of Theorem 1).

(g) Example 3

Let $T = [0,\ 1]$ be an operator with homothety coefficient $\hat{A}$ = 5.

Furthermore,

$$N = \{(u,x)\}, \tag{52}$$

$$t \mapsto u(t) = \chi_T(t)e, t \mapsto x(t) = t^{1,6}e. \tag{53}$$

where χ_T is the characteristic function of interval T.

Hence, the function $f = \sup_L P\Psi(P_N)$, according to Equation (42), satisfies the inequalities:

$$0,6\,t^{-0,4} \le \sup\nolimits_{\mathrm{L}} P\Psi(P_N) \le 4,8\,t^{-0,4} \tag{54}$$

where the second inequality guarantees—due to Equation (42) and the third condition of Theorem 1—the existence of a BDR model realizing the process-bundle N. Furthermore, according to the fourth condition of Theorem 1, the first inequality indicates that this D-model cannot be stationary.

Consider an example that illustrates a situation in which the assumed linearity of the D-realization model fails to meet the requirements for precise construction of the D-equations governing the dynamics of the bundle. This precision is essential for accurately modeling the synergistic factor of the scrutinized neuromodulation.

(h) Example 4

Let $T = [1,\ 2]$, $\hat{A}$ be a homothety operator with the coefficient of -1 and $N = \{(u_i,x_i)\}_{i=\overline{1,3}}$:

$$t \mapsto u_1(t) = 2^{-1}\chi_T(t)te, t \mapsto x_1(t) = (t^2 + 2)e, \tag{55}$$

$$t \mapsto u_2(t) = \chi_T(t)te, t \mapsto x_2(t) = te, \tag{56}$$

$$t \mapsto u_3(t) = \chi_T(t)te, t \mapsto x_3(t) = (2 - 4\sqrt{2})te; \tag{57}$$

It is apparent that the tuple $(N,\ \hat{A})$ is not defective (in this case, the tuple $(\{(u_i,x_i)\}_{i=\overline{2,3}}, \hat{A})$ is defective).

First, we must demonstrate that a dynamic bundle N cannot admit a linear D-realization. Therefore, it is sufficient to establish that the linear characteristic η_{L} of the Rayleigh–Ritz operator induces the function $t \mapsto \eta_{\mathrm{L}}(g, w, v)(t)$ with a zero of order 2. The characteristic condition for this assertion is given by the following system of equations (with respect to the parameters α,β):

$$\begin{aligned} ||x_1(t) + \alpha x_2(t) + \beta x_3(t)||_X^2 = ((t^2 + 2) + \alpha t \\ + \beta(2 - 4\sqrt{2})t)^2 = 0, \end{aligned} \tag{58}$$

$$\begin{aligned} ||dx_1(t)/dt + \alpha dx_2(t)/dt + \beta dx_3(t)/dt||_X^2 = (2t + \alpha\chi_T(t) \\ + \beta(2 - 4\sqrt{2})\chi_T(t))^2 = 0, \end{aligned} \tag{59}$$

$$||u_1(t) + \alpha u_2(t) + \beta u_3(t)||_X^2 = (2^{-1}t + \alpha t + \beta t)^2 = 0, \tag{60}$$

Hence, it can readily be computed that $\alpha = -1, \beta = 0,5$, in this case, the zero point is $t = \sqrt{2}$.

Next, we must demonstrate that a BDR model Equation (6) for the dynamic bundle N exists. To do this, it is sufficient to show that the sum of the linear and bilinear characteristics (parametrized with respect to α,β) of the Rayleigh–Ritz operator for the bundle N is bounded from below by a definite function of the form $t \mapsto r\chi_T(t),\ r \in (0, \infty)$.

For any $t \in [1,\ 2]$, α,$\beta \in R$, it can be readily shown that:

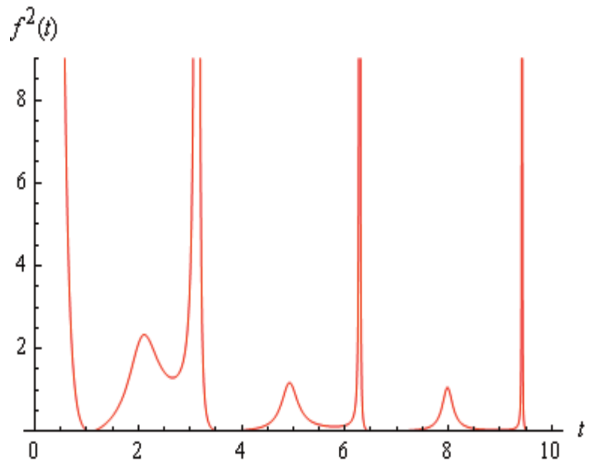

Figure 1. $f^2(t) = (2\cos t - t\sin t)^2 \times ((t\sin t)^2 + (t\sin t)^2(\sin t + t\cos t)^2)^{-1}$.

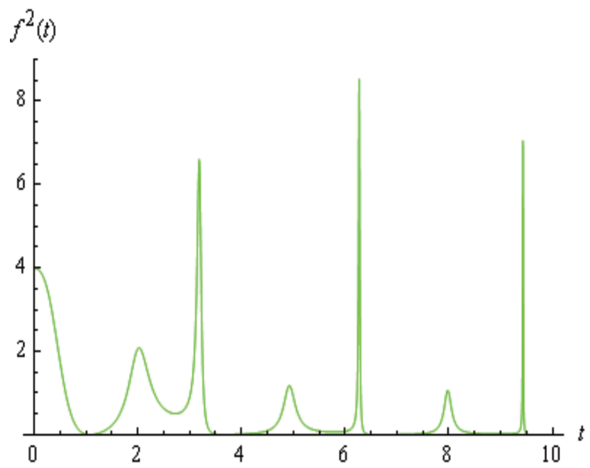

Figure 2. $f^2(t) = (2\cos\ t - t\sin t)^2 \times ((t\sin t)^2 + (t\sin t)^2(\sin t + t\cos t)^2 + (t\sin^2 t + 2^{-1}t^2 \sin 2t + \cos t)^2)^{-1}$.

$$
\begin{aligned}
&\eta_L(t,\alpha,\beta) + \eta_B(t,\alpha,\beta) \geq \\
&\geq ||dx_1(t)/dt + \alpha dx_2(t)/dt + \beta dx_3(t)/dt||_X^2 + \\
&+ || < u_1(t), dx_1(t)/dt >_X e + \alpha < u_2(t), dx_2(t)/dt >_X e \\
&+ \beta < u_3(t), dx_3(t)/dt >_X e||_X^2 = \\
&= (2t + \alpha + \beta(2 - 4\sqrt{2}))^2 + (t^2 + \alpha t + \beta(2 - 4\sqrt{2})t)^2 = \\
&= [(2t + \gamma(\alpha,\beta))^2 + t^2(t + \gamma(\alpha,\beta))^2]\,\Big|_{\gamma(\alpha,\beta) = \alpha + \beta(2\text{-}4\sqrt{2})} \geq \\
&\geq [(2t + \gamma(\alpha,\beta))^2 + (t + \gamma(\alpha,\beta))^2]\,\Big|_{\gamma(\alpha,\beta) = \alpha + \beta(2\text{-}4\sqrt{2})} = \\
&= [(t + \lambda(t,\gamma))^2 + \lambda^2(t,\gamma)]\,|_{\lambda(t,\gamma) = t + \gamma(\alpha,\beta)} \geq \\
&\geq [(t + \theta(t))^2 + \theta^2(t)]\,|_{\theta(t) = -t/2} = t^2/2 \geq 0,5_T(t).
\end{aligned}
\tag{61}
$$

It is possible to verify that the BDR model Equation (6) for the bundle has an analytical representation of:

$$
-d^2x/dt^2 + x = D_5(u, dx/dt) \tag{62}
$$

where $D_5(\cdot,\cdot) = < \cdot,\cdot >_X e$.

(i) Example 5

The construction in Example 5 (Reed and Simon,[17] p. 54) allows us to construct a general BDR (with the state vector $X \oplus X \oplus X$) for dynamic bundles described in Examples 2–4.

From Example 4, several important model-theoretic conclusions follow as its empirical generalization. Verification of these conclusions should essentially precede any analysis (including qualitative analysis) of the BDR modeling problem Equation (6). This approach helps reduce the gap between theory and practice by relying on conclusions grounded in experience and common sense.

The first conclusion states that a finite dynamic bundle N may have an interpolation representation on the interval T in the class of polynomial spline functions.[19] The second conclusion raises a natural question: "Under what conditions can the D-realization be unsatisfactory in the class of linear D-models?" In Proposition 3, we briefly address the solution to the above-mentioned problem of "causality" in the nonlinear formulation of D-modeling, which remains partially unresolved but constitutes an important methodological issue in QTDR.

(j) Proposition 3

Let the dynamic bundle N satisfy the condition:

$$
\begin{aligned}
&\exists\, (v_1, ..., v_8,) \in V_N : ||\hat{A}dg/dt||_X(\sqrt{\eta_L(v_1, v_2, v_3)} \\
&+ \chi_{S_{\dot{v}}})^{-1} \notin \mathrm{L}_2(T, R)
\end{aligned}
\tag{63}
$$

where $\chi_{S_{\dot{v}}}$ is the characteristic function of the set $S_{\dot{v}_1} = T \backslash \mathrm{supp}\ dv_1/dt$.

Hence, the differential realization of this bundle cannot have an analytical representation given by Equation (6), in which all bilinear operators are zero.

The condition in Equation (62) can be weakened, or, constructively, strengthened, by reducing it to the problem of searching, by means of computer algebra, for the zeros of function $\eta_L + \eta_B$ parametrized by the coefficients of the interpolation representation of the bundle N in the class of polynomial spline functions (see Proposition 2.6.1) and Proposition 2.6.4 in Laurent[19] (p. 69) on the arrangement of polynomial zeros.

5. Conclusion

Multidimensional neurodynamic models that describe the evolution of synapses or receptors, changes in the speed of neuroimpulses, and other neurophysiological factors in local neuron populations typically take the form of systems of integro-differential Equation (1). For such differential representations of neuromorphic processes, the problem of the solvability of these infinite systems is of particular relevance. Consequently, various approaches and techniques from contemporary functional analysis can be employed, including comparison theorems, monotone iterative methods, truncation techniques, and fixed-point topological methods. On the other hand, with respect to neurodynamic processes of the "input–output" type, defined $a\,posteriori$ in terms of the "black box" protomodel (see Definition 1.2 in Mesarovic and Takahara,[6] p. 21), the problem of their realization[2,6] can be formulated in the class of nonlinear, non-autonomous D-models (with or without delay). In this context, D-equations in infinite-dimensional Hilbert spaces prove highly effective for addressing integrative aspects of both theoretical and applied problems in mathematical neurobiology. At the same time, the factors of nonstationarity, multilinearity, and delay in such D-models capture the adaptive tuning of synergetic complexes of synapses in the neuron populations under study (simulation).

The QTDR results confirm that the permanently developed realization theory serves as a universal language for interdisciplinary exchange of scientific results within the framework of entropy analysis, since it is sufficiently general to avoid the introduction of arbitrary constraints into the mathematical description of the processes involved in a posteriori modeling of nonlinear infinite-dimensional dynamical systems. At the same time, because of its strength and generality, the theory eliminates the possibility of common misunderstandings in the interpretation of such a key concept in the contemporary theory of realization for multilinear dynamical systems as structural identifiability, preventing this concept from assuming subjective aspects and, instead, assigning it a completely formalized meaning.[6] Overall, it can be concluded that understanding the geometric nature of a posteriori modeling of polylinear D-models of complex dynamical systems, including neuromorphic ones, helps to clarify intuitive ideas about the equations of infinite-dimensional behavioral systems.

Additionally, the entropy QTDR apparatus proposed above provides at least an element of esthetic satisfaction. Mathematical models and laws are often postulated on the basis of Aristotle's philosophy and an esthetic perspective, and only later it becomes evident that these models and laws can also be derived—to some extent—from already existing (or supplemented) mathematical knowledge and observed facts of the physical world. For example, we may point to the history of the discovery of the law of universal gravitation, formulated by Hooke in one of his letters to Newton (see the historical commentary in Arnold,[20] p. 115). Hooke speculatively argued that the force of attraction of a body spreads evenly over the sphere surrounding the body (equidistantly), and is therefore inversely proportional to the square of the radius of that sphere. In fact, Hooke claimed that this law is a consequence of "the three-dimensional geometry of the surrounding physical space." Later, Einstein, developing this geometric model of gravity, added the factor of space curvature, determined by the mass of the body. It was this letter from Hooke—rather than the anecdote about the "fallen apple"—that prompted Newton to reconsider abandoning science and provided the impetus for writing the famous Principia,[21] which effectively marked the beginning of contemporary mathematical physics.

In order to proceed further, we can point to the system-theoretic direction in QTDR, which will form the algebraic basis (without excessive reliance on the technical apparatus of algebraic geometry) for the next stage of QTDR development, applicable to inverse problems of nonlinear neurodynamics with a delay factor. Such a direction will provide a foundation for the transition from the bilinear structure of nonlinear connections to polylinear interactions in higher-order D-models,[4,14] including models with periodic operators (Massera and Schaffer,[22] p. 387). Methodologicall y, this transition consists of the integration of the geometric language of tensor structures of Fock spaces[17] and projective representations (Kirillov[10], p. 238), in the context of employing the continuity property of the Rayleigh–Ritz entropy operators[15] through computer algebra.[18]

In the context of this remark, we note that polylinear D-equations occupy a special place in the qualitative theory of nonlinear dynamical systems because—according to one of the main ideas of nonlinear multidimensional analysis—any sufficiently smooth function (with respect to Fréchet derivatives) of many variables can be well approximated in the neighborhood of each of its points by a polylinear function, namely through the Taylor expansion for mappings in infinite-dimensional Banach spaces. The resulting operation of polylinearization leads to polylinear D-models as efficient precision approximations in the study of nonlinear D-equations in the vicinity of some of their solutions. This fact makes it possible to analyze the behavior of representative nonlinear neuromorphic D-models of higher order (with or without delay) in a multidimensional phase space on the basis of polylinear structures of the D-equations of the neuropopulation under study. Therefore, through differential modeling of neuroprocesses,[23–26] we claim to identify special (or typical) entropy-dynamic properties of such D-models.[5,27]

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Conflict of interest

The authors declare they have no competing interests.

Author contributions

Conceptualization: V. A. Rusanov
Formal analysis: V. A. Rusanov
Methodology: V. A. Rusanov
Software: R. A. Daneev
Writing–original draft: V. A. Rusanov
Writing–review & editing: R. A. Daneev

Availability of data

Data is available from the corresponding author upon reasonable request.

AI Tools Statement

The authors confirm that no artificial intelligence (AI) tools were used in the preparation of this manuscript.

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