

# A simple autonomous Jerk system with coexisting multiple attractors and its cryptographic application

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## ABSTRACT

In nonlinear dynamical systems, multiple attractors emerge, with the system converging to different attractors depending on the initial conditions, leading to complex dynamic behavior. This paper introduces a simple autonomous jerk system, which combines hyperbolic sine and absolute value functions. By adjusting the system's parameters, we explore its dynamic behavior, uncovering various complex phenomena such as period-doubling bifurcations, antimonotonicity, transient chaos, periodic windows, and multiple attractors, which collectively illustrate the system's rich dynamics. Furthermore, we examine the basins of attraction for each attractor, providing insights into the system's behavior under various initial conditions. Through this analysis of the basins, we deepen our understanding of the system's diversity and complexity, highlighting its sensitivity to initial conditions. This comprehensive analysis provides essential clues for future investigations. Additionally, the system is implemented on a field-programmable gate array platform, demonstrating its practicality for real-world use. Finally, the chaotic sequences generated by the system are applied to image encryption, and the influence of multistability on encryption performance is assessed. The results show that chaotic-state encryption outperforms periodic-state encryption in terms of cryptographic security. These findings emphasize the considerable impact of multiple attractors on the system's dynamic behavior and encryption performance, offering new theoretical insights and approaches for further research in this area.

**Keywords:** Multistability; Hopf bifurcation; Antimonotonicity; Hardware implementation; Image encryption



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## 1. Introduction

In recent decades, chaos has been widely studied for its applications in engineering fields such as secure communications, neural networks, image encryption, and others.<sup>1-8</sup> Multiple coexisting attractors in chaotic systems have also been increasingly investigated. Multiple coexisting attractors indicate multistability and hysteretic dynamics, suggesting that the system may transition between different attractors. Each attractor is associated with a specific set of initial conditions, and the trajectories converge to one of the attractors depending on these conditions.<sup>9</sup> Correspondingly, the basin of attraction consists of all initial points whose trajectories converge to the given attractor. Since the eventual state of motion is determined by the choice of initial conditions, it is essential to investigate whether multiple attractors exist across different sets of initial conditions, thereby enabling selection among desired states. Such a system has great application potential and efficient forecast methods; control methods should be studied to put the system in the desired state.<sup>10</sup> Owing to the lack of systematic methods for selecting initial values, such coexistence is investigated through tedious trial-and-error and extensive numerical searches. This issue merits further attention.

In recent years, numerous nonlinear systems with multiple coexisting attractors have drawn growing interest for their complexity and rich dynamics.<sup>11-15</sup> Yu *et al.*<sup>16</sup> developed a memristive hyperchaotic system in which coexisting attractors were observed. Xue *et al.*<sup>15</sup> then modified the canonical Chua's circuit by incorporating a memristive diode bridge. Soon afterward, Chen *et al.*<sup>17</sup> proposed a Hopfield neural network model with multiple coexisting attractors and chaos synchronization, and expounded its dynamic mechanism. Liang *et al.*<sup>18</sup> reported a three-dimensional (3D) chaotic system with multiple attractors. Additionally, Lai *et al.*<sup>19</sup> investigated how coexisting attractors can be constructed from a chaotic system. In many of these systems, symmetrical structures are more conducive to the formation of multiple attractors. Also, symmetry can give rise to various complex dynamical behaviors, including chaos crisis, coexisting bifurcations, and hysteresis.

A key research focus in chaotic systems with coexisting attractors is the development of simple, robust chaotic oscillators from both circuit and mathematical perspectives. Sprott<sup>20</sup> conducted an extensive search for autonomous three-dimensional chaotic systems with a prescribed explicit form  $\dot{x} = J(\dot{x}, x, x)$ . In this context, the nonlinear term  $J(x)$  is referred to as the "jerk" function, as it represents the third derivative of  $x$  with respect to time, corresponding to the first derivative of acceleration in mechanical systems.<sup>21,22</sup> Eichhorn *et al.*<sup>23</sup> showed that transforming certain functionally distinct three-dimensional dynamical systems can result in identical jerk-like dynamics. They also looked into the multistability of simple jerk systems. Kengne *et al.*<sup>24</sup> explored the dynamics of a jerk equation that included a cubic nonlinear term. By applying a linear

transformation to Model MO5, they used both numerical simulations and experimental methods to identify multiple attractors in the system. Fascinatingly, the characteristic of coexisting multiple attractors persisted even when the cubic nonlinearity was substituted with a hyperbolic tangent or when utilizing the inherent nonlinearity of a voltage-controlled memristor. The same authors later reported the existence of distinct attractors in a jerk system with a smooth piecewise-quadratic nonlinearity.<sup>25</sup> Additionally, a parameter range was identified where this jerk system exhibits magnetization. More strikingly, the symmetry of the nonlinear terms—such as memristors, hyperbolic tangent functions, and piecewise quadratic nonlinearities—renders this jerk system highly symmetric, potentially enabling multistable behavior.

In earlier studies, Vaidyanathan *et al.*<sup>26</sup> proposed a 3D jerk chaotic system incorporating two hyperbolic sine nonlinearities. Subsequently, Kengne *et al.*<sup>27</sup> investigated a simple autonomous jerk circuit by replacing the cubic nonlinearity with Model MO5. This circuit can be regarded as an analog implementation of Model MO15 originally introduced by Sprott.<sup>20</sup> However, Model MO5 requires the cubic term to be implemented using two analog multipliers. In contrast, Model MO15 overcomes this drawback, as its hyperbolic sinusoidal nonlinearity can be conveniently realized using a pair of anti-parallel semiconductor diodes. To validate its applicability, Volos *et al.*<sup>28</sup> reported an extremely simple autonomous chaotic circuit with a hyperbolic-sine nonlinearity. Based on this system, an effective encryption scheme was further developed using a random number generator. Motivated by these results, a nonlinear function is introduced to realize the jerk circuit, which is composed of the hyperbolic sine and the absolute value function. Similar properties (e.g., period-doubling bifurcation, antimonotonicity) also appear in the new system. A window is found in the parameter space where four distinct non-static coexisting attractors are observed. This study addresses three aspects: (i) a systematic analysis of the proposed system to clarify the mechanism of chaos generation; (ii) identification of parameter regions where multiple coexisting attractors occur, along with an investigation of antimonotonicity and transient chaos; and (iii) evaluation of the influence of a key parameter on cryptographic performance under multistability.

Then Section 2 presents the mathematical model of the system. We first determine the stability of the equilibrium points and then derive the conditions for Hopf bifurcations. Section 3 conducts numerical experiments to explore the system's multistability and bifurcation structures. In Section 4, antimonotonicity and transient chaos behaviors are investigated in detail. Section 5 subsequently implements the jerk system on a field-programmable gate array (FPGA) platform. In Section 6, the image encryption combined S-box with different attractors obtained from the proposed system is discussed and compared. The encryption scheme's security is then evaluated. Finally, Section 7 concludes the paper.

## 2. Description and analysis of the model

### 2.1. The model

Building on Model MO15 introduced in the previous study, the Jerk system in this study is represented as **Equation 1**.<sup>27</sup>

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \sigma x_3, \\ \dot{x}_3 = x_1 - \gamma x_2 - \mu x_3 - \sinh(x_1) |x_1| \end{cases} \quad (1)$$

where, the overdot denotes the time derivative, with  $\sigma$ ,  $\gamma$ , and  $\mu$  being positive parameters that control the system. Notably, the right-hand side of the system consists of six terms, with only  $\sinh(x_1)|x_1|$  introducing nonlinearity. Furthermore, the nonlinearity is linked to the single state variable  $x_1$ , which is responsible for the system's intricate dynamical behavior. **Equation 1** can also be written in an equivalent form as  $\ddot{x} = -\mu\ddot{x} - \sigma\gamma\dot{x} + \sigma x - \sigma \sinh(x)|x|$ .

### 2.2. Symmetry and dissipation

**Equation 1** remains invariant under the coordinate transformation  $(x_1, x_2, x_3) \rightarrow (-x_1, -x_2, -x_3)$ , which implies point symmetry about the origin  $O = (0, 0, 0)$ . For a fixed set of parameters, if  $(x_1, x_2, x_3)$  is a solution of **Equation 1**, then  $(-x_1, -x_2, -x_3)$  is also a solution under the same conditions. In fact, the nonlinearity involves only the state variable  $x_1$ , and this feature leads to the system's complex dynamical behavior, including the coexistence of multiple attractors. The divergence of **Equation 1** can be written as **Equation 2**:

$$\nabla V = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} = -\mu \quad (2)$$

Clearly, for any state  $x = (x_1, x_2, x_3)$  in the phase space,  $\nabla V < 0$  holds whenever  $\mu > 0$ . As time approaches infinity, volumes associated with **Equation 1** contract to 0 at an exponential rate.<sup>29</sup> The system is therefore dissipative.

### 2.3. Fixed point analysis

The equilibrium points of **Equation 1** are obtained by solving **Equation 3**:

$$\begin{cases} x_2 = 0 \\ \sigma x_3 = 0 \\ x_1 - \gamma x_2 - \mu x_3 - \sinh(x_1) |x_1| = 0 \end{cases} \quad (3)$$

Evidently, from the first and second equations of **Equation 3**, we can obtain  $x_2 = x_3 = 0$ . And from the third equation, it is easy to get that  $\sinh(x_1)|x_1| = x_1$ . By calculating the hyperbolic function, it can be seen that there are three fixed points:  $x_0^E = F_0(0, 0, 0)$ ,  $x_{1,2}^E = F_{1,2}(\pm\vartheta, 0, 0)$ , where  $\vartheta \approx 0.8814$ . It can be observed that  $F_1$  and  $F_2$  are symmetric about the origin. The Jacobian matrix of **Equation 1** at an arbitrary equilibrium point is obtained by direct computation from **Equation 4**:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \sigma \\ \Gamma & -\gamma & -\mu \end{bmatrix} \quad (4)$$

where (**Equation 5**):

$$\begin{aligned} \Gamma &= 1 - \sinh(x_1) \cdot \text{sign}(x_1) - \cosh(x_1) |x_1| \\ &= \begin{cases} 1 & (x_0^E = F_0(0, 0, 0)) \\ -1.2465 & (x_{1,2}^E = F_{1,2}(\pm\vartheta, 0, 0)) \end{cases} \end{aligned} \quad (5)$$

According to the Jacobian matrix  $J$ , the characteristic equation is given by **Equation 6**:

$$\lambda^3 + \mu\lambda^2 + \sigma\gamma\lambda - \sigma\Gamma = 0 \quad (6)$$

For  $x_0^E = F_0(0, 0, 0)$ , **Equation 6** can be rewritten as **Equation 7**:

$$\lambda^3 + \mu\lambda^2 + \sigma\gamma\lambda - \sigma = 0 \quad (7)$$

Because the coefficients of the associated characteristic equation are not of the same sign, the equilibrium point  $F_0(0, 0, 0)$  is always unstable. For  $x_{1,2}^E = F_{1,2}(\pm\vartheta, 0, 0)$ , **Equation 7** can be rewritten as **Equation 8**:

$$\lambda^3 + \mu\lambda^2 + \sigma\gamma\lambda + 1.2465\sigma = 0 \quad (8)$$

In contrast, the stability of  $F_{1,2}(\pm\vartheta, 0, 0)$  is governed by the parameters  $\mu$  and  $\gamma$ . According to the stability criterion of Routh-Hurwitz, it can be inferred that  $F_{1,2}(\pm\vartheta, 0, 0)$  is stable only for **Equation 9**:

$$\begin{cases} \mu > 0, \\ \mu\sigma\gamma + \sigma\Gamma > 0, \\ -\sigma\Gamma(\mu\sigma\gamma + \sigma\Gamma) > 0 \end{cases} \quad (9)$$

After simplification, the stable condition is derived as **Equation 10**:

$$\mu > -\Gamma / \gamma \quad (10)$$

However, this state changes once the control parameter decreases past a critical threshold. Treating  $\mu$  as the Hopf bifurcation parameter and setting, assume that **Equation 6** admits a pair of purely imaginary conjugate eigenvalues  $\lambda_{1,2} = \pm j\omega$  ( $\omega \in R^+$ ), Under these conditions, one obtains **Equation 11**:

$$-j\omega^3 - \mu\omega^2 + j\sigma\gamma\omega - \sigma\Gamma = 0 \quad (11)$$

If  $\Gamma < 0$  is satisfied, then (**Equation 12**):

$$\lambda_1 = j\omega = j\sqrt{\sigma\gamma}, \lambda_2 = -j\omega = -j\sqrt{\sigma\gamma}, \lambda_3 = -\mu \quad (12)$$

Under this condition, the transversality condition (**Equation 13**):

$$\text{Re} \left( \frac{d\lambda}{d\mu} \Big|_{\lambda=j\omega} \right) = \frac{\omega^2(-3\omega^2 + \sigma\gamma)}{(-3\omega^2 + \sigma\gamma)^2 + (2\mu\omega)^2} \Big|_{\lambda=j\omega} = \frac{-\sigma\gamma}{2\sigma\gamma + 2\mu^2} \neq 0 \quad (13)$$

is also satisfied. Thus, the conditions for the Hopf bifurcation can be obtained as **Equation 14**:

$$\mu = \mu_c = -\Gamma / \gamma, \omega_H = \sqrt{\sigma\gamma} \quad (14)$$

**Equation 14** gives the oscillation frequency of the stable state, along with the critical value  $\mu_c$  at which **Equation 1** undergoes a Hopf bifurcation.<sup>30</sup>

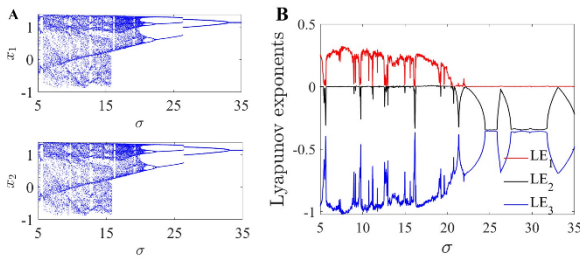
### 3. Numerical study

To analyze the chaotic dynamics of **Equation 1**, this section employs bifurcation diagrams, Lyapunov exponents, phase portraits, time-domain signals, and basins of attraction. Numerical simulations are conducted in MATLAB (R2019b, MathWorks, USA) with an initial time of 500, a step size  $\Delta t = 0.01$ , and a total duration of 5,000.

#### 3.1. Bifurcation and chaos

A bifurcation diagram is an effective way to depict the various pathways to chaos as the control parameter changes.<sup>31</sup> To investigate the complex dynamical behaviors demonstrated by the proposed jerk system, we numerically solve **Equation 1** utilizing the classical fourth-order Runge–Kutta technique. Setting  $\mu = 0.7$  and  $\gamma = 0.55$ , we assess the system's responsiveness to the control parameter  $\sigma$ . The bifurcation diagram is generated by plotting the local maxima of a state variable as  $\sigma$  is incrementally varied. In the periodic regime, the count of unique maxima aligns with the limit cycle's period, whereas the presence of chaos is suggested by a continuous array of points. Accordingly, Figure 1A displays a representative bifurcation diagram as  $\sigma$  changes. For each  $\sigma$  value, the system is initialized at (0.5, 0, 0), and data collection occurs after the transient effects have dissipated and the dynamics are stable.

As  $\sigma$  varies gradually in **Figure 1A**, the system transitions through multiple dynamical regimes—from chaos initially, into a wide periodic window, then back to chaos. Further variation eventually drives the system from chaos to periodic motion via an inverse period-doubling cascade, signaling a chaos crisis. Interspersed within the chaotic regions are several small periodic windows. **Figure 1B** shows the corresponding Lyapunov exponent spectrum computed using Wolf's algorithm.<sup>32</sup> The bifurcation diagram and the Lyapunov spectrum are in close correspondence throughout.



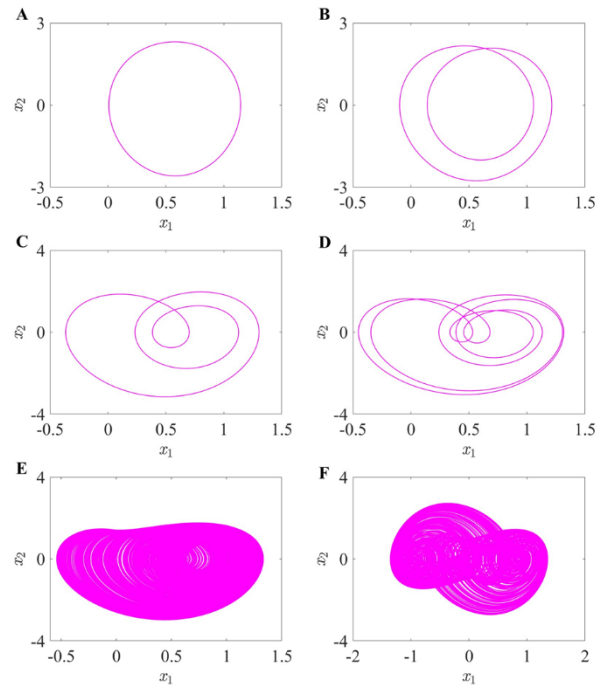
**Figure 1.** Dynamics of **Equation 1**. (A) Bifurcation diagrams showing  $x_1$  and  $x_2$  as functions of  $\sigma$ , with  $\mu = 0.7$ ,  $\gamma = 0.55$ , and initial condition (0.5, 0, 0). (B) Corresponding Lyapunov exponent spectrum.

Under the same parameter settings as in **Figure 1A**, the system's phase portraits shown in **Figure 2** give a direct visualization of the transition to chaos. **Figure 2A–F** shows the attractors of the system for  $\sigma = 34, 30, 25, 21, 18$ , and  $12$ , respectively, corresponding to period-1, period-2, period-3,

and period-6 limit cycles, as well as single-band and double-band chaotic attractors. Under the same parameter settings, the Lyapunov exponents are evaluated and reported in **Table 1**. For periodic orbits,  $\lambda_1 = 0, \lambda_2, \lambda_3 < 0$ , whereas chaotic attractors are characterized by  $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$ . These results agree with the bifurcation diagram.

**Table 1.** Lyapunov exponents for various  $\sigma$

| $\sigma$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | Behavior | Figure 2 |
|----------|-------------|-------------|-------------|----------|----------|
| 34       | 0           | −0.808      | −0.610      | Periodic | A        |
| 30       | 0           | −0.344      | −0.357      | Periodic | B        |
| 25       | 0           | −0.349      | −0.352      | Periodic | C        |
| 21       | 0           | −0.044      | −0.660      | Periodic | D        |
| 18       | 0.196       | 0.000       | −0.896      | Chaotic  | E        |
| 12       | 0.273       | 0.000       | −0.972      | Chaotic  | F        |



**Figure 2.** Phase portraits for selected values of  $\sigma$ . (A) Period-1 at  $\sigma = 34$ . (B) Period-2 at  $\sigma = 30$ . (C) Period-3 at  $\sigma = 25$ . (D) Period-6 at  $\sigma = 21$ . (E) Single-band chaos at  $\sigma = 18$ . (F) Double-band chaos at  $\sigma = 12$ .

The Lyapunov dimension of the chaotic attractor is further estimated using the Kaplan–Yorke formula<sup>33</sup>:

$$D_L = k + \frac{1}{|L_{k+1}|} \sum_{i=1}^k L_i \quad (15)$$

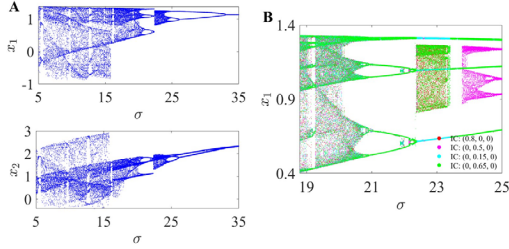
where  $k$  satisfies  $\sum_{i=1}^k L_i \geq 0$  and  $\sum_{i=1}^{k+1} L_i \leq 0$ . Therefore,  $D_L = 2.22$  for  $\sigma = 18$ , and  $D_L = 2.28$  for  $\sigma = 12$ , indicating that the new system has a fractal dimension.

#### 3.2. Coexisting bifurcation and multiple attractors

Coexisting attractors describe a situation in which multiple attractors arise under identical parameter settings from different initial conditions, exhibiting nonlinear hysteretic behavior. Similar behavior has been observed in different

systems. The influence of initial conditions on the behavior of Equation 1 is also examined in the numerical analysis.

To this end, with the parameters  $\gamma = 0.55$  and  $u = 0.7$ , the bifurcation diagram for the coexisting attractor is shown in **Figure 3A**, obtained by plotting the local peaks of the system variable as a function of the control parameter varied over  $5 < \sigma < 35$ . As the control parameter varies, a random number is used as the initial condition, which can assume various values. Thanks to the system's symmetry and the randomness of the initial conditions, the numerical method can identify isolated bifurcation branches that may coexist in the system's response.

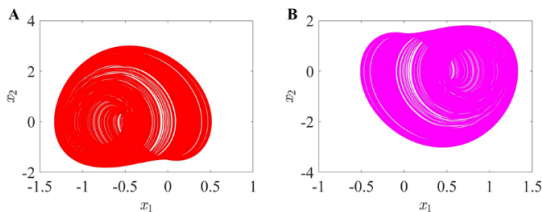


**Figure 3.** Coexisting bifurcation and multiple attractors. (A) Bifurcation diagrams showing  $x_1$  and  $x_2$  versus  $\sigma$  for  $\mu = 0.7$ ,  $\gamma = 0.55$ , and initial conditions are random. (B) Hysteresis window for  $18.8 < \sigma < 25$ ; curves in red, magenta, cyan, and green correspond to initial conditions  $(0.8, 0, 0)$ ,  $(0, 0.65, 0)$ ,  $(0, 0.5, 0)$ , and  $(0, 0.15, 0)$ , respectively.

As shown in **Figure 3A**, Equation 1 enters a chaotic state through an inverse period-doubling bifurcation, while simultaneously exhibiting multiple coexisting attractors. Obviously, coexisting attractors can take many forms, including single-band chaos, double-band chaos, and period attractors. By selecting different parameter values and initial conditions, different coexisting attractors can be obtained. **Figure 3A** shows the hysteretic dynamics window for  $\sigma$  in the range 18.8–25. This phenomenon can be further understood by referring to the bifurcation diagram in **Figure 1A** and the corresponding zoom-in shown in **Figure 3B**. In contrast, the red curve is obtained for a different initial condition  $(0.8, 0, 0)$ ; the graphs in magenta, cyan, and green correspond to initial values  $(0, 0.65, 0)$ ,  $(0, 0.5, 0)$ , and  $(0, 0.15, 0)$ , respectively. Up to four different attractors can be obtained within this range.

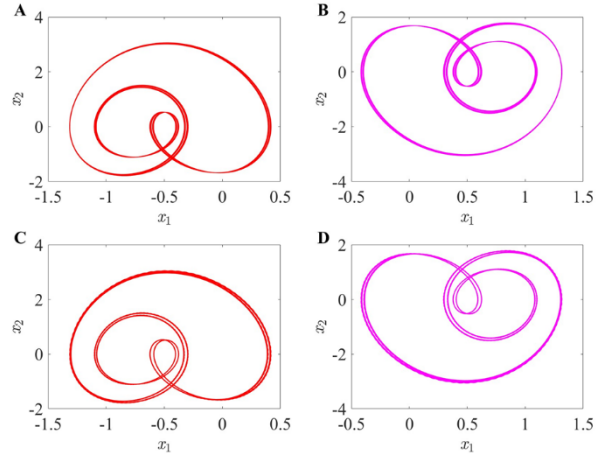
At  $\sigma = 19$ , two chaotic attractors coexist under the initial conditions  $(0, \pm 0.65, 0)$ , as shown in **Figure 4**.

At  $\sigma = 21.9$ , four periodic attractors coexist under the initial conditions  $(0, \pm 0.15, 0)$  and  $(\pm 0.8, 0, 0)$ , as shown in **Figure 5**.

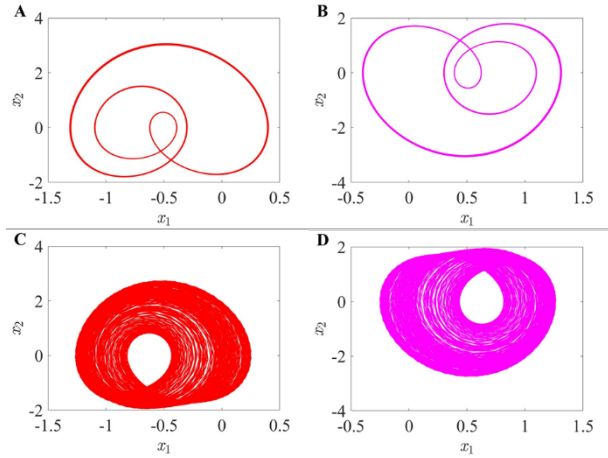


**Figure 4.** Coexistence of two distinct attractors at  $\sigma = 19$  for different initial conditions. (A)  $(0, -0.65, 0)$ . (B)  $(0, 0.65, 0)$ .

**Figure 6** illustrates a case at  $\sigma = 22.5$  where two pairs of attractors coexist. Specifically, one pair corresponds to period-3 behavior, while the other consists of single-band chaotic attractors arising from the initial conditions  $(0, \pm 0.5, 0)$  and  $(\pm 0.8, 0, 0)$ .



**Figure 5.** Coexistence of two distinct attractors at  $\sigma = 21.9$  for different initial conditions. (A)  $(0, -0.15, 0)$ . (B)  $(0, 0.15, 0)$ . (C)  $(-0.8, 0, 0)$ . (D)  $(0.8, 0, 0)$ .



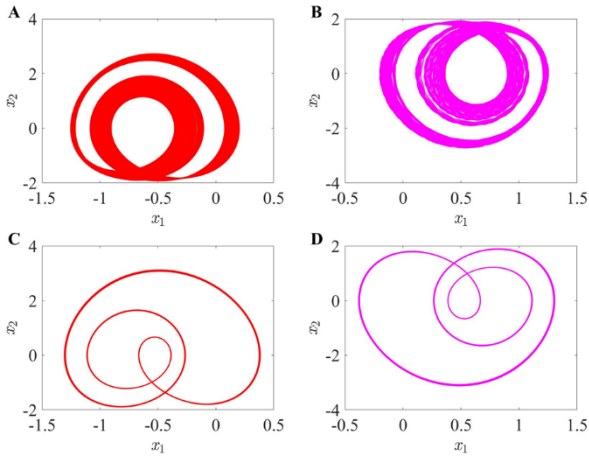
**Figure 6.** At  $\sigma = 22.5$ , four coexisting attractors appear for different initial conditions. (A)  $(0, -0.5, 0)$ . (B)  $(0, 0.5, 0)$ . (C)  $(-0.8, 0, 0)$ . (D)  $(0.8, 0, 0)$ .

**Figure 7** presents another coexistence scenario at  $\sigma = 24$ , where one pair corresponds to period-3 attractors, while the other consists of torus attractors associated with the initial conditions  $(\pm 0.5, 0, 0)$  and  $(\pm 0.15, 0, 0)$ .

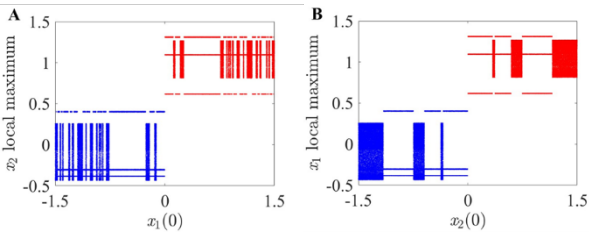
Therefore, both asymmetric and symmetric coexisting attractors can be observed in this system. When multiple attractors are present, the state space is partitioned into distinct basins of attraction, each associated with a specific attractor.

**Figure 8** shows the basins associated with the coexisting attractors in **Figure 6**. Under the same parameter settings, a scan over initial conditions is performed to identify the regions leading to each attractor.



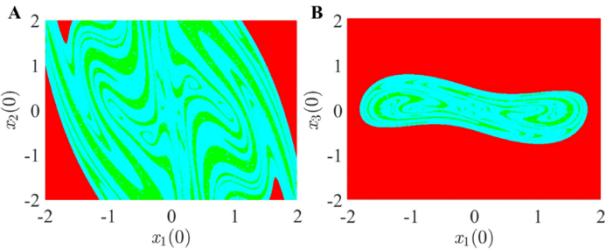


**Figure 7.** At  $\sigma = 24$ , four coexisting attractors appear for different initial conditions. (A)  $(0, -0.5, 0)$ . (B)  $(0, 0.5, 0)$ . (C)  $(0, -0.15, 0)$ . (D)  $(0, 0.15, 0)$ .



**Figure 8.** Bifurcation diagrams of local maxima at  $\sigma = 22.5$ . (A)  $x_1$  versus the initial value  $x_1(0)$  with  $x_2(0) = x_3(0) = 0$ . (B)  $x_2$  versus initial value  $x_2(0)$  with  $x_1(0) = x_3(0) = 0$ .

**Figure 9** shows cross sections of the basins of attraction for  $x_3(0) = 0$  and  $x_2(0) = 0$ . The coexistence of chaotic attractors (green) and symmetric period-3 cycles (cyan) reflects the complexity of the basin structure, while the red regions indicate unbounded solutions.



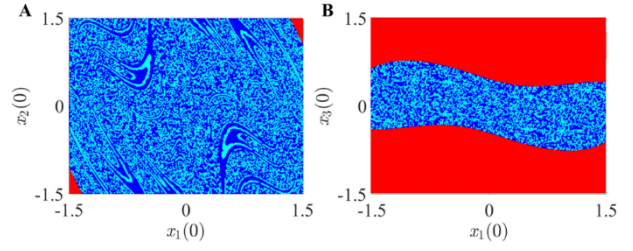
**Figure 9.** Basins of attraction at  $\sigma = 22.5$ ,  $\gamma = 0.55$ ,  $\mu = 0.7$ . (A) Cross sections for  $x_3(0) = 0$ . (B) Cross sections for  $x_2(0) = 0$ .

Similarly, cross sections of the basins of attraction for  $\sigma = 21.9$ ,  $\gamma = 0.55$ ,  $\mu = 0.7$  are presented in **Figure 10**, where the blue, cyan, and red indicate the period-6 cycle attractor, period-3 cycle attractor, and unbounded solutions, respectively.

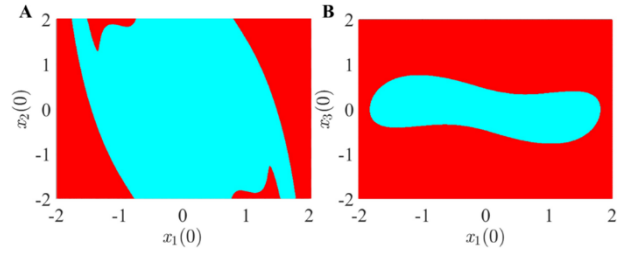
Set  $\gamma = 0.55$ ,  $\mu = 0.7$ , cross sections of the attraction basin for  $\sigma = 24$  and  $\sigma = 25$  are presented in **Figure 11**, where the green, cyan, and red regions denote a chaotic attractor, a period-3 limit cycle, and unbounded solutions, respectively.

Each attractor is associated with a distinct basin of attraction. The coexistence of multiple attractors increases

system complexity and may be exploited in chaos-based signal encryption and secure communication.



**Figure 10.** Basins of attraction at  $\sigma = 21.9$ ,  $\gamma = 0.55$ ,  $\mu = 0.7$ . (A) Cross sections for  $x_3(0) = 0$ . (B) Cross sections for  $x_2(0) = 0$ .

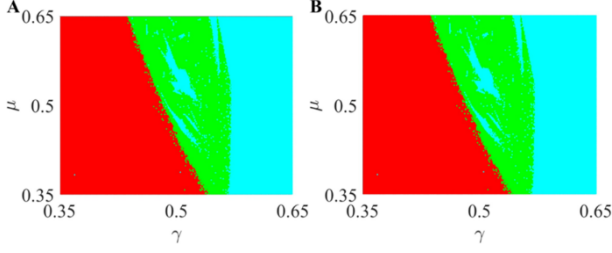


**Figure 11.** Basins of attraction at  $\sigma = 24$  and  $\sigma = 25$ . (A) Cross sections at  $\sigma = 24$  with  $x_3(0) = 0$ . (B) Cross sections at  $\sigma = 25$  with  $x_2(0) = 0$ .

### 3.3. Two-parameter dynamical map

As discussed above, **Equation 1** exhibits rich dynamical behavior as the control parameters vary. Dynamical maps are commonly used to analyze chaotic systems.<sup>34</sup> To further explore the complex dynamics of **Equation 1**, a two-parameter dynamical diagram is presented in **Figure 12A**, illustrating the chaotic region in the  $(\gamma, \mu)$  plane for  $\gamma, \mu \in [0.35, 0.65]$ . The remaining parameter is fixed at  $\sigma = 22.5$  with initial conditions  $(x_1(0), x_2(0), x_3(0)) = (0.5, 0, 0)$ . As shown in **Figure 12A**, periodic behavior is mainly distributed on the right side of the diagram. Three distinct regions can be identified: the green and cyan regions correspond to chaotic and periodic states, respectively, while the red region denotes unbounded solutions. For small values of  $\gamma$ , the system exhibits chaotic or unbounded behavior, whereas increasing  $\gamma$  leads to a transition from chaos to periodic motion. In particular, for  $0.48 < \gamma < 0.54$  and  $0.42 < \mu < 0.5$ , cyan regions appear in the green area, indicating frequent transitions between chaotic and periodic states. This suggests that **Equation 1** is highly sensitive in these regions and exhibits complex state-switching behavior. Additionally, the colored regions remain relatively concentrated and form well-defined green (chaotic) and cyan (periodic) areas, indicating strong robustness and structural stability of **Equation 1**. Furthermore, under  $\gamma = 0.55$  and initial conditions  $(0.5, 0, 0)$ , **Figure 12B** presents a two-parameter dynamical map in the  $(\sigma, \mu)$  plane, where  $\sigma \in [15, 24]$  and  $\mu \in [0.35, 0.65]$ . Obviously, chaos is the principal part in the region of the graph. Varying  $\sigma$  in the range  $16 < \sigma < 20$  and  $\mu$  in the range  $0.42 < \mu < 0.46$ , finely punctate red regions are scattered within the chaotic domain, indicating the presence of multiple periodic windows. Overall, the dynamical map provides useful guidance for parameter

selection in engineering applications of the system.



**Figure 12.** Two-parameter dynamical maps of **Equation 1**. Green: chaos; cyan: periodic; red: unbounded. (A) In the  $(\gamma, \mu)$  plane with  $\sigma = 22.5$ . (B) In the  $(\sigma, \mu)$  plane with  $\gamma = 0.55$ .

#### 4. Antimonotonicity and transient chaos behavior

In this section, the same dynamical analysis method as in the previous section is used to analyze **Equation 1**. The simulation runs for 5,000 time units with  $\Delta t = 0.01$ .

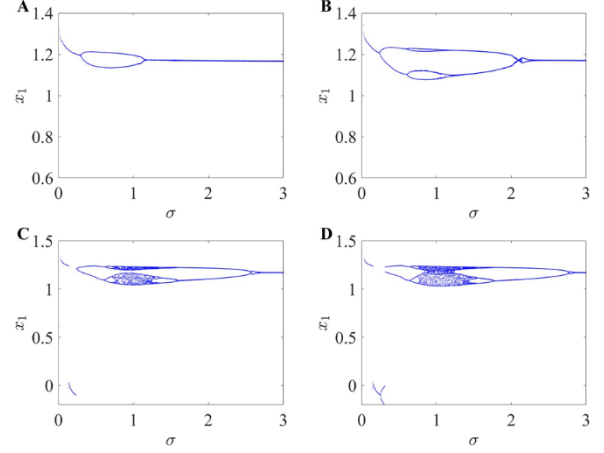
It is noteworthy that **Equation 1** can display the complete Feigenbaum tree when suitable parameters are chosen. As the control parameter is gradually adjusted, periodic orbits can emerge and disappear. This behavior has been analyzed in numerous nonlinear systems and is referred to as antimonotonicity. To validate this phenomenon within the current system, several bifurcation diagrams of the state variable  $x_1$  are presented as  $\sigma$  is systematically varied over  $0 < \sigma < 3$  for specific values of  $\gamma$ . The other parameter is maintained at  $\mu = 0.7$ , with initial conditions set to  $(0.6, 0, 0)$ . A selection of results is shown in **Figure 13**. In this figure, a period-2 bubble is evident at  $\gamma = 1.1$ , and the branch transitions into a stable period-4 bubble at  $\gamma = 1.03$ . As  $\gamma$  decreases, more bubbles appear, ultimately leading to an infinite bifurcation tree.

Transient chaos describes a dynamic behavior in which a system transitions from chaotic to periodic motion over time, according to numerous studies.<sup>35</sup> Transient chaos is rarely observed in real laboratory experiments due to its relatively short lifetime. **Equation 1** exhibits complex transient chaotic behavior under appropriate parameter settings and initial conditions. To investigate this nonlinear phenomenon, the parameters are set to  $\sigma = 21.9$ ,  $\mu = 0.7$ , and  $\gamma = 0.55$ , with the initial condition  $(0.7, 0, 0)$ . The time series, along with the related phase portraits, are shown in **Figure 14**. In **Figure 14C**, it is evident that **Equation 1** undergoes a transition from transient chaos to a stable period-6 state over time. **Figure 14D** shows that the Lyapunov exponent is initially positive and gradually converges to 0, indicating the presence of transient chaos followed by periodic dynamics with rich nonlinear behavior.

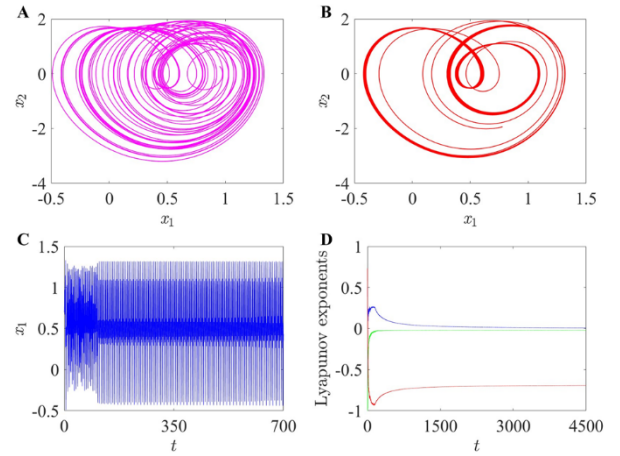
#### 5. Hardware implementation based on system-on-programmable-chip

To verify the previous theoretical results, we presented an appropriate hardware implementation approach based on a system-on-programmable-chip (SOPC) platform. SOPC is an embedded system, which can be regarded as an FPGA-based solution. This design offers convenience and

flexibility for algorithm upgrades. The implementation is conducted on the AC620 FPGA development board. The EP4CE10F17 chip is selected as the FPGA; it is part of the Intel CYCLONE IV E series. W9812g6KH-6 is selected as the SDRAM, with a size of 128 M. The digital-to-analog converter (DAC) chip is the high-speed dual-channel 14-bit AD9767 ASTZ. Hardware customization is performed using the Qsys plug-in in the Quartus II software (Version 17.1, Altera, USA). The hardware implementation platform is shown in **Figure 15**.



**Figure 13.** Bifurcation diagrams of  $x_1$  versus the parameter  $\sigma$  showing a remerging Feigenbaum tree (bubbling). (A)  $\gamma = 1.1$ . (B)  $\gamma = 1.03$ . (C)  $\gamma = 1.0$ . (D)  $\gamma = 0.989$ .



**Figure 14.** Transient chaos behavior of **Equation 1**. (A) Phase portrait over  $t \in [0, 120]$ . (B) Phase portrait over  $t \in [120, 4,500]$ . (C) Time series of  $x_1$ . (D) Lyapunov exponents over  $t \in [0, 4,500]$ .

After completing the hardware configuration, the Nios II IDE is used for system programming in C. To balance discretization accuracy, resource consumption, and computational efficiency, the Euler method is adopted to discretize the differential equations of the chaotic system. The sampling interval is set to  $\Delta t = 0.001$ . The discretized form of the system is given by **Equation 16**:

$$\begin{cases} x_1(n+1) = x_2(n) \times \Delta t + x_1(n) \\ x_2(n+1) = \sigma x_3(n) \times \Delta t + x_2(n) \\ x_3(n+1) = (x_1(n) - \gamma x_2(n) - \mu x_3(n)) - \sinh(x_1(n)) |x_1(n)| \times \Delta t + x_3(n) \end{cases} \quad (16)$$

The chaotic data are converted using the DAC chip, which is mainly controlled by SOPC. Firstly, three arrays,  $x_1$ ,  $x_2$ , and  $x_3$ , are defined during implementation. Secondly, iterative computations are performed, updating the components of the three arrays based on **Equation 16**. Finally, the array elements are written to the DAC. After the digital-to-analog conversion, attractors can be obtained and observed by an oscilloscope. To allow comparison with the theoretical outcomes, the initial parameter values have been set to align with those discussed in Section 3. As the control parameter  $\sigma$  varies, the entire bifurcation sequence is shown experimentally in **Figure 16**, where the limit cycle undergoes multiple period-doubling bifurcations that lead to chaos. These observed bifurcation sequences align well with the theoretical analysis discussed in Section 3.1. Additionally, coexisting attractors have been observed in the experimental results.



**Figure 15.** The hardware implementation platform of the system.

**Figure 17** illustrates the simultaneous presence of four attractors, aligning with the theoretical analysis discussed in Section 3.2. This supports the practicality of developing the system outlined in this paper.

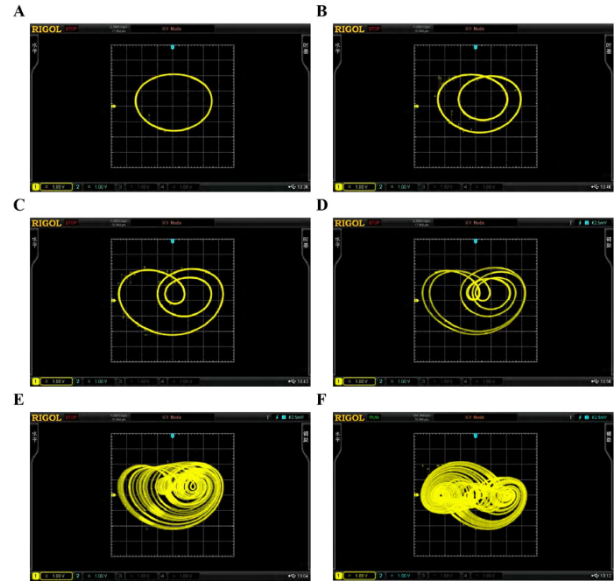
## 6. Application of the proposed system in image encryption

Chaos-based image encryption has gained considerable attention.<sup>36,37</sup> By exploiting the properties of chaotic sequences, it offers an effective approach for secure image encryption. In this section, an image encryption scheme based on **Equation 1** and an S-box is proposed, as illustrated in **Figure 18**.

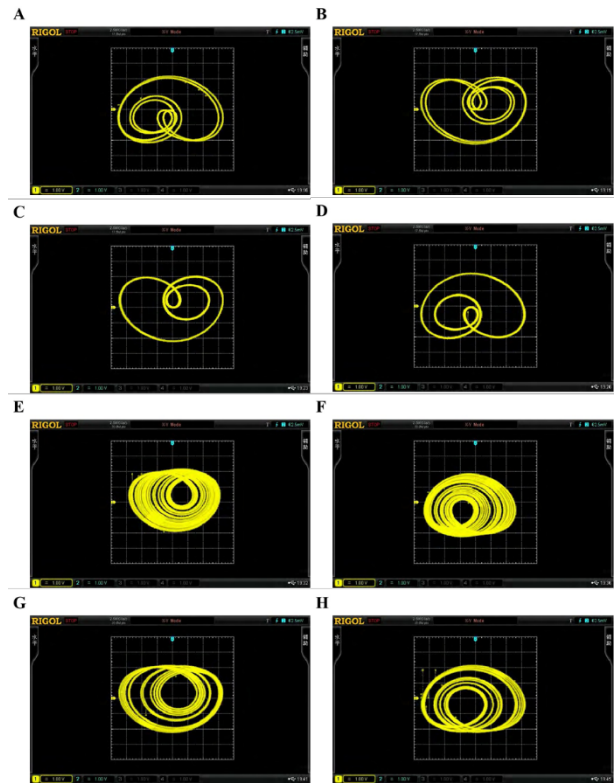
### 6.1. Experimental results

If the initial values change, the system can exhibit different coexisting attractors, each with distinct encryption effects. To compare the performance of different attractors, two keys are used to encrypt and decrypt the image in this section. The initial condition plays a crucial role. The key 1 is  $(0, 0.5, 0)$ , corresponding to the periodic state; and the key 2 is  $(0.8, 0, 0)$ , corresponding to the chaotic state. The system settings for the experiment are shown in **Table 2**,

which is the same as **Figure 6**. The test results are shown in **Figure 19**. The encrypted image effectively conceals the plaintext image's information, appearing random and unrecognizable. The decrypted image closely matches the original plaintext image, demonstrating the effectiveness of the proposed encryption and decryption method.

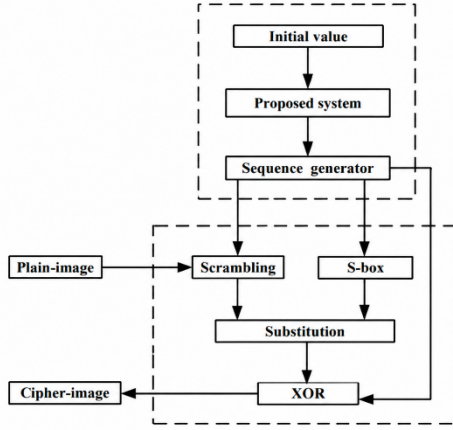


**Figure 16.** Field-programmable gate array implementation results showing the phase-space trajectories leading to chaos. (A)  $\sigma = 34$ . (B)  $\sigma = 30$ . (C)  $\sigma = 25$ . (D)  $\sigma = 21$ . (E)  $\sigma = 18$ . (F)  $\sigma = 12$ .



**Figure 17.** Field-programmable gate array implementation results showing the phase-space trajectories leading to chaos. (A)  $\sigma = 21.9$ . (B)  $\sigma = 21.9$ . (C)  $\sigma = 21.9$ . (D)  $\sigma = 21.9$ . (E)  $\sigma = 22.5$ . (F)  $\sigma = 22.5$ . (G)  $\sigma = 24$ . (H)  $\sigma = 24$ .





**Figure 18.** Block diagram of the image encryption algorithm. Abbreviation: XOR: Exclusive OR.

**Table 2.** The system settings for the experiment

| Scheme   | Parameters                                | Initial value      | Behavior |
|----------|-------------------------------------------|--------------------|----------|
| Scheme 1 | $\sigma = 22.5, \gamma = 0.55, \mu = 0.7$ | Key 1: (0, 0.5, 0) | Periodic |
| Scheme 2 | $\sigma = 22.5, \gamma = 0.55, \mu = 0.7$ | Key 2: (0.8, 0, 0) | Chaotic  |

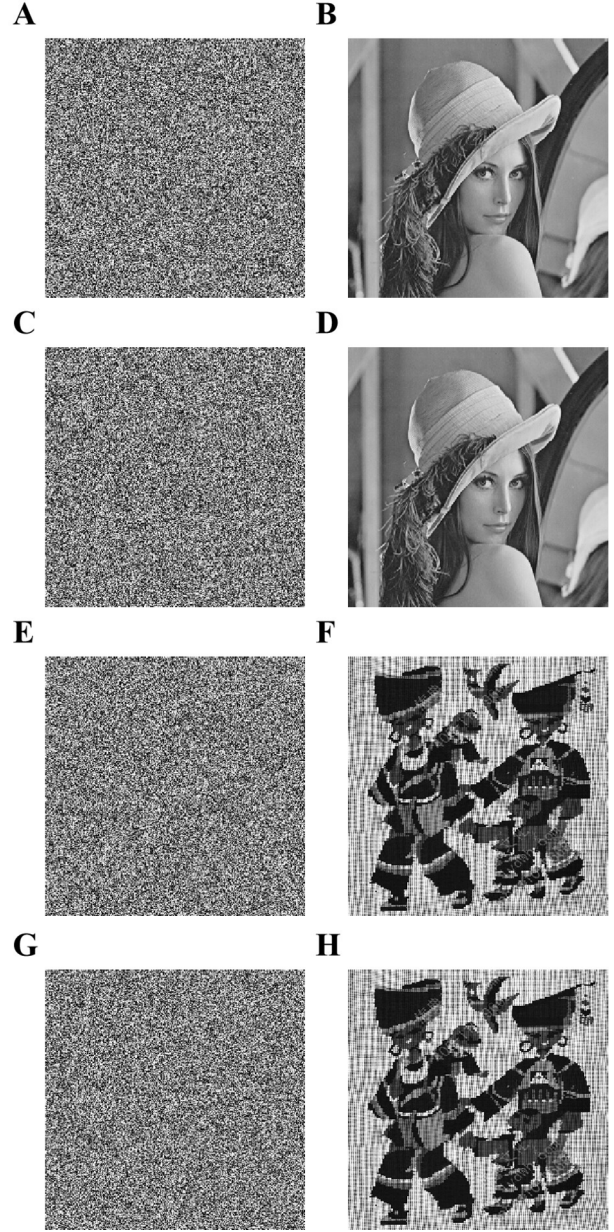
## 6.2. Key sensitivity analysis

Key sensitivity reflects the response of the encryption algorithm to variations in the secret key. A robust scheme should exhibit high sensitivity, such that even slight changes in the key produce significant differences in the ciphertext, while incorrect keys fail to recover the original image. The number of pixel change rate (NPCR) and the unified average changing intensity (UACI) are commonly used to quantify this property, and are defined as Equation 17:

$$\begin{cases} NPCP = \frac{1}{M \times N} \sum_{i,j}^{M,N} D(i,j) \times 100\% \\ D(i,j) = \begin{cases} 1, & C_1(i,j) \neq C_2(i,j) \\ 0, & C_1(i,j) = C_2(i,j) \end{cases} \\ UACI = \frac{1}{M \times N} \left[ \sum_{i,j}^{M,N} \frac{|C_1(i,j) - C_2(i,j)|}{255} \right] \times 100\% \end{cases} \quad (17)$$

In this equation,  $M$  and  $N$  denote the image's width and height, respectively, while  $C_1$  and  $C_2$  signify the ciphertext images prior to and following key variation. To assess key sensitivity in the encryption process, the initial value  $x_1(0)$  is altered by  $1.0 \times 10^{-15}$ , while the other key parameters remain constant. The NPCR and UACI values for the resultant ciphertext images of Lena and Brocade are computed. The findings in Table 3 indicate that the suggested algorithm is highly sensitive to variations in the key, with Key 1 providing greater security than Key 2. In the decryption phase, even a minor alteration in the key ( $1.0 \times 10^{-15}$ ) hinders accurate image recovery, highlighting the decryption

process's strong sensitivity to key modifications.



**Figure 19.** Encryption and decryption results for Lena and Brocade images using Key 1 (periodic state) and Key 2 (chaotic state). Lena: (A) Ciphertext with Key 1; (B) Decrypted with Key 1; (C) Ciphertext with Key 2; (D) Decrypted with Key 2. Brocade: (E) Ciphertext with Key 1; (F) Decrypted with Key 1; (G) Ciphertext with Key 2; (H) Decrypted with Key 2.

**Table 3.** NPCR and UACI values between two images encrypted before and after the key  $x_0$  changed

| Images             | Lena   |        | Brocade |        |
|--------------------|--------|--------|---------|--------|
|                    | NPCR   | UACI   | NPCR    | UACI   |
| Scheme 1           | 99.65% | 33.48% | 99.59%  | 33.57% |
| Scheme 2           | 99.58% | 33.48% | 99.61%  | 33.40% |
| Ref. <sup>38</sup> | 99.60% | 33.35% | —       | —      |

Abbreviations: NPCR: Number of pixel change rate;  
UACI: Unified average changing intensity.

### 6.3. Correlation analysis

Natural images often exhibit strong dependencies among neighboring pixels, leading to significant redundancy. Such statistical characteristics may be exploited by attackers. To enhance security, it is necessary to weaken these correlations. For quantitative evaluation, 1,000 pairs of neighboring pixels are randomly sampled from both plaintext and ciphertext images along horizontal, vertical, and diagonal directions, and the corresponding correlation coefficients are computed as **Equation 18**:

$$\begin{cases} E(x) = \frac{1}{K} \sum_{i=1}^K x_i \\ D(x) = \frac{1}{K} \sum_{i=1}^K (x_i - E(x))^2 \\ \text{Cov}(x, y) = \frac{1}{K} \sum_{i=1}^K (x_i - E(x))(y_i - E(y)) \\ r_{xy} = \frac{\text{Cov}(x, y)}{\sqrt{D(x)} \cdot \sqrt{D(y)}} \end{cases} \quad (18)$$

The findings are presented in **Table 4**. The correlation coefficients for all plaintext images are above 0.53, indicating a strong relationship among neighboring pixels. In contrast, the correlation coefficients for ciphertext images in all directions are nearly 0, indicating that pixel correlations have been effectively disrupted. Compared to previous studies, the correlation coefficients of the ciphertext images generated by the proposed scheme are closer to 0.<sup>39-41</sup> These findings support the conclusion that the proposed encryption method substantially reduces or eliminates the inherent correlations in the original images. Additionally, the correlations achieved with the Key 1 approach are closer to 0 than those obtained with Key 2.

### 6.4. Information entropy analysis

Information entropy quantifies the randomness in the distribution of gray levels. A larger entropy value corresponds to a more uniform intensity distribution. It can be expressed as **Equation 19**:

$$H(S) = -\sum_{i=1}^L p(x_i) \log_2 p(x_i) \quad (19)$$

where  $H(S)$  represents the entropy of image  $S$ , and  $p(x_i)$  denotes the probability of the pixel value  $x_i$ . For an 8-bit grayscale image, the theoretical entropy is 8, with higher values indicating greater randomness. As shown in **Table 5**, the ciphertext entropy exceeds 7.997, with a deviation below 0.003 from the theoretical maximum, outperforming previous results.<sup>39-42</sup> This demonstrates that the encrypted images exhibit near-random characteristics. These results confirm that the proposed method achieves a highly uniform gray-level distribution. Moreover, Key 1 provides better performance than Key 2.

In addition, the encryption efficiency of our algorithm was assessed on a  $256 \times 256$  grayscale Lena image using MATLAB. The average encryption time over 50 independent tests was 45.50 ms, confirming its good computational performance. The algorithm also works for images of any size without modification, as chaotic sequences are generated based on the total pixel count  $M \times N$ .

## 7. Conclusion

This paper proposes a simple autonomous jerk system with nonlinearities composed of hyperbolic sines and absolute values. Its dynamics are analyzed through standard nonlinear tools, revealing inverse period-doubling, crises, antimonotonicity, periodic windows, and multistability with coexisting attractors. FPGA implementation confirms these behaviors experimentally. A chaos-based image

**Table 4.** Correlation coefficients between the original images and their encrypted images

| Images | Plaintext | Ciphertext          |                    |                    |                    |
|--------|-----------|---------------------|--------------------|--------------------|--------------------|
|        |           | The proposed scheme | Ref. <sup>39</sup> | Ref. <sup>40</sup> | Ref. <sup>41</sup> |
| Lena   | 7.4832    | 7.9971 (key 1)      | 7.9965             | 7.9959             | 7.9951             |
|        |           | 7.9970 (key 2)      |                    |                    |                    |

**Table 5.** Comparison of image information entropy

| Images  | Plaintext          | Horizontal direction |           | Vertical direction |           | Diagonal direction |           |
|---------|--------------------|----------------------|-----------|--------------------|-----------|--------------------|-----------|
|         |                    | Ciphertext           | Plaintext | Ciphertext         | Plaintext | Ciphertext         | Plaintext |
| Lena    | Scheme 1           | 0.9491               | 0.0061    | 0.9073             | 0.0065    | 0.8577             | -0.0549   |
|         | Scheme 2           | 0.9491               | 0.0277    | 0.9073             | 0.0082    | 0.8577             | 0.0542    |
| Brocade | Scheme 1           | 0.8561               | -0.0055   | 0.6336             | -0.0038   | 0.5374             | 0.0248    |
|         | Scheme 2           | 0.8561               | -0.0448   | 0.6336             | -0.0149   | 0.5374             | -0.0423   |
| Lena    | Ref. <sup>41</sup> | -                    | -0.0066   | -                  | -0.0089   | -                  | 0.0424    |
| Lena    | Ref. <sup>42</sup> | -                    | -0.0086   | -                  | -0.1020   | -                  | 0.0125    |

encryption scheme is further developed, and security analyses demonstrate its effectiveness. The results indicate that chaotic states provide superior encryption performance compared to periodic states. The proposed system offers both theoretical insights and practical potential for applications of chaos.

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## Conflict of interest

The authors declare no conflicts of interest.

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*Investigation:* All authors

*Methodology:* Song Liu, Zihan Li, Da Qiu, Min Luo

*Writing—original draft:* All authors

*Writing—review & editing:* Song Liu, Zihan Li, Guoping Zhang

## Availability of data

Data used in this work is available from the corresponding author upon reasonable request.

## AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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