

# Characterization of solitary wave dynamics in the nonlinear fractional Allen–Cahn model via two consistent analytical schemes

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## ABSTRACT

In this study, we aim to construct exact solitary-wave solutions of the time-fractional Allen–Cahn (AC) model using the conformable derivative framework. The AC model is effective for describing various physical phenomena, such as the mixing of two incompressible fluids, the motion of phase boundaries in crystalline solids, solid nucleation, vesicle membranes, and phase separation in iron alloys, among others. We initially identify the soliton solutions of the model under consideration using the generalized exponential rational function method and the improved modified simplest equation method. When applied with appropriate parameter values, the considered methods can generate trigonometric, hyperbolic, complex hyperbolic, rational, and mixed-type solutions via fractional wave transformation. The employed methods incorporate multiple parameters that can be varied to modify the properties of the resulting solutions. These results enhance our theoretical understanding of complex nonlinear systems and demonstrate the potential applications of advanced mathematical methods. Additionally, we analyze the influence of the fractional-order parameter on the obtained results using graphical visualization to reveal the underlying dynamics of the related phenomenon. We also perform a stability analysis of the model under consideration and critically assess the novelty of this study relative to the existing literature. The attained results demonstrate the effectiveness and robustness of the employed methods.

**Keywords:** Generalized exponential rational function method; Improved modified simplest equation method; Time-fractional Allen–Cahn model; Solitons



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## 1. Introduction

In current decades, nonlinear fractional partial differential equations (NLFDEs), which were primarily defined in 1695<sup>1</sup>, are regarded as efficient and elementary tools to describe the numerous properties of complex phenomena in our real life, from different fields such as quantum mechanics, kinematic model of neutron points, solid-state physics, oceanic spectacles, biochemistry, elasticity, nuclear physics, atomic physics, hydrology, optical physics, landscape evolution and many supplementary areas of engineering and physical sciences. Throughout this context, many scholars derived various effective nonlinear models. To realize the corporeal significance of these models, determining the exact solution or, in some cases, the approximate solution of these models plays a significant role. Researchers have developed various prominent definitions of NLFDEs<sup>2-4</sup> and effective methods such as: the Logarithmic transformation scheme<sup>5</sup>, the modified rational sinh-cosh function method<sup>6</sup>, the auxiliary equation scheme<sup>7-9</sup>, the  $\Phi^6$  model expansion approach<sup>10</sup>, the improved generalized Riccati equation mapping scheme<sup>11</sup>, the Hirota N-soliton conditions scheme<sup>12,13</sup>, the Shehu homotopy transform method<sup>14</sup>, the Unified Method<sup>15</sup>, the inverse scattering transformation method<sup>16</sup>, the generalized exponential rational function method<sup>17-19</sup>, the variation principle method<sup>20</sup>, the Hirota bilinear formation<sup>21</sup>, the generalized  $(G'/G)$ -expansion approach<sup>22,23</sup>, the extended tanh function technique<sup>24</sup>, the Binary darboux transformations scheme<sup>25</sup>, the Sardar-sub equation scheme<sup>26</sup>, the modified sub equation technique<sup>27</sup>, the  $(m + (G'/G))$ -expansion approach<sup>28</sup>, the  $(1/G')$ -expansion technique<sup>29</sup>, the  $\exp_a$  function scheme<sup>30</sup>, the extended tanh-coth method<sup>31</sup>, the modified auxiliary equation scheme<sup>32</sup>, the  $(G'/G, 1/G)$ -expansion approach<sup>33,34</sup>, the new extended direct algebraic method<sup>35,36</sup>, the improved modified simplest equation (IMSE) method<sup>37</sup> for examining solitary wave solutions of nonlinear models. Ghanbari *et al.*<sup>19</sup> proposed the generalized exponential rational function (GERF) method, and Al-Amin *et al.*<sup>37</sup> proposed the improved modified simplest equation (IMSE) method to explore traveling wave solutions of various NLFDEs. These methods have been successfully employed by many researchers to solve various nonlinear problems and derive solitary traveling wave solutions. This led us to use these methods to retrieve solitary-wave solutions of the time-fractional Allen–Cahn (AC) model. Phase-field models originating from phase transition theory have attracted considerable interest among many researchers. In general, there are four models: the Cahn–Hilliard model, the Penrose–Fife system, the AC model, and the Caginalp system.<sup>38</sup> Among them, the Cahn–Hilliard theory and the AC model are popular phase field models. The AC equation is a reaction–diffusion equation, originally introduced by Allen and Cahn<sup>39</sup>, to model phase separation and interfacial dynamics in multi-component alloys. Our considered AC model is also used to analyze various events related to the mixing of two incompressible fluids, the motion of anti-phase boundaries in crystalline

solids.<sup>39</sup> In this article, we examine the time-fractional AC model, which effectively describes the phase separation process in iron-based alloys<sup>40</sup>, in the form of Equation 1:

$$\frac{\partial^\gamma v(x,t)}{\partial t^\gamma} - \frac{\partial^2 v(x,t)}{\partial x^2} + v^3(x,t) - v(x,t) = 0 \quad (1)$$

where  $t > 0$ ,  $0 < \gamma \leq 1$ , and  $\gamma$  is a fractional order derivative.

The Adomian decomposition, B-spline, and modified Khater method<sup>40</sup>, the  $\exp_a$  function, hyperbolic function, and improved Bernoulli sub-equation function<sup>41</sup>, the double exp-function technique<sup>42</sup>, the first integral scheme<sup>43</sup>, the  $(G'/G)$ -expansion scheme<sup>44</sup>, the tanh function technique<sup>45</sup>, and the tanh-coth method<sup>46</sup>, are robust and leading approaches, employed to investigate the solitary wave solutions of the AC model. To the best of our knowledge, the influence of the fractional-order parameter on the solution behavior has not been thoroughly examined in the existing literature. Furthermore, a detailed stability analysis of the time-fractional AC model has yet to be fully established. To bridge this research gap, the present study analyzes the time-fractional AC model. A statistical assessment reveals that the model under consideration has not been studied using the GERF or IMSE methods in the existing literature.

The main findings of this study include the derivation of several novel solitary wave solutions for the considered model using the GERF and IMSE methods. We analyze the influence of the fractional-order parameter on the obtained results through graphical visualization to reveal the underlying dynamics of the phenomenon. Additionally, we perform a stability analysis of the model under consideration and critically examine the novelty of this study relative to the existing literature.

The rest of the article is organized as follows. Section 2 presents an outline of the conformable derivative. In Section 3, we briefly describe the GERF and IMSE methods. Section 4 is devoted to the construction of traveling wave solutions. In Section 5, the physical significance of the obtained solutions is discussed. Section 6 analyzes the stability of the model under consideration. In Section 7, we critically assess the novelty of this study relative to the existing literature. Finally, the article concludes with some concluding remarks.

## 2. Conformable derivative

The conformable derivative of order  $\mu > 0$  with respect to the variable  $t$ , is recognized by Khalil *et al.*<sup>2</sup>, is defined as follows:

Consider a function  $l: (0, \infty) \rightarrow \mathbb{R}$ , the conformable derivative of  $l$  is defined as.

$$\mathcal{K}_t^\mu l(t) = \lim_{\delta \rightarrow 0} \frac{l(t + \delta t^{1-\mu}) - l(t)}{\delta}$$

For all  $t > 0$ ,  $\mu \in (0, 1]$  The following theorems present the properties satisfied by the definition.

### 2.1. Theorem 1

Here,  $\mu \in (0,1]$  and  $l, M$  be  $\mu$ -differentiable at a point  $t$ . Then, the following properties hold:

- (i)  $\mathcal{K}_t^\mu (bl + aM) = b\mathcal{K}_t^\mu (l) + a\mathcal{K}_t^\mu (M)$
- (ii)  $\mathcal{K}_t^\mu (t^\tau) = \tau t^{\tau-\mu}, \forall \tau \in \mathbb{R}$
- (iii)  $\mathcal{K}_t^\mu (\theta) = 0, \forall \phi(t) = \theta$
- (iv)  $\mathcal{K}_t^\mu (Ml) = M\mathcal{K}_t^\mu (l) + l\mathcal{K}_t^\mu (M)$
- (v)  $\mathcal{K}_t^\mu \left( \frac{M}{l} \right) = \frac{l\mathcal{K}_t^\mu (M) - M\mathcal{K}_t^\mu (l)}{l^2}$
- (vi)  $\mathcal{K}_t^\mu M(t) = t^{1-\mu} \frac{dM}{dt}$

wherein  $t^{1-\mu}$  is the conformable function  $\forall b, a \in \mathbb{R}$ . The conformable derivative possesses several key properties, such as the chain rule, Taylor series expansion, and compatibility with the Laplace transform.

### 2.2. Theorem 2

Let  $M = M(t)$  be a  $\mu$ -conformably differentiable function, and let  $l$  also be differentiable on the domain of  $M$ . Then, conformable differentiable function and also differentiable in the range of. Then  $\mathcal{K}_t^\mu (Mol)(t) = t^{1-\mu} l'(t) M'(l(t))$ .

### 3. Algorithm of the employed methods

Here, we briefly present the GERF and IMSE methods, which are competent and powerful, as outlined below.

#### 3.1. The generalized exponential rational function method

Ghanbari *et al.*<sup>19</sup> first introduced the GERF method. It is a newly proposed and effective approach for solving nonlinear partial differential equations. Its successful application in numerous studies encouraged us to adopt this method for addressing Model 1, which is examined in the present work.

Let a general NLFPE be in the following form:

$$\mathcal{F}(\psi, \mathcal{G}_t^\gamma \psi, \mathcal{G}_x^\alpha \psi, \mathcal{G}_y^\theta \psi, \mathcal{G}_t^{2\gamma} \psi, \mathcal{G}_x^{2\beta} \psi, \dots) = 0 \quad (2)$$

where  $\psi = \psi(t, x, y)$  is the wave function (depends on space and time).  $\mathcal{F}$  is a polynomial that contains  $\psi$  and some of its derivatives (first, second, etc.). The main stages for using this method are as follows:

- (i) Stage 1: Using the transformations as follows:

$$\psi(t, x, y) = \psi(\xi), \xi = m \frac{x^\alpha}{\alpha} + n \frac{y^\theta}{\theta} \pm \varepsilon \frac{t^\gamma}{\gamma} \quad (3)$$

where  $\varepsilon$  is the wave velocity. Equation 3 was used to convert Equation 2 into a nonlinear ordinary differential equation (NODE), as shown in Equation 4:

$$\mathcal{H}(\psi(\xi), \psi'(\xi), \psi''(\xi), \dots) = 0 \quad (4)$$

In Equation 4,  $\mathcal{H}$  is a polynomial of  $\psi, \psi', \psi'', \dots$ , wherein  $\psi'(\xi) = \frac{d\psi}{d\xi}$ .

- (ii) Stage 2: The **key assumption** of this technique is that the solution structure of Equation 4 is considered in the form of Equations 5 and 6:

$$\psi(\xi) = \mathcal{B}_0 + \sum_{i=1}^N \mathcal{B}_i f^i(\xi) + \sum_{i=1}^N \frac{\mathcal{A}_i}{f^i(\xi)} \quad (5)$$

where

$$f(\xi) = \frac{\sigma_1 e^{(\varrho_1 \xi)} + \sigma_2 e^{(\varrho_2 \xi)}}{\sigma_3 e^{(\varrho_3 \xi)} + \sigma_4 e^{(\varrho_4 \xi)}} \quad (6)$$

where  $\sigma_n, \varrho_n (1 \leq n \leq 4), \mathcal{B}_0, \mathcal{B}_i$ , and  $\mathcal{A}_i (1 \leq i \leq N)$  have to settle. The form of the solution Equation 5 always persuades Equation 5. By applying the homogeneous balance principle, the value of  $N$  can be determined.

- (iii) Stage 3: Substituting Equation 5 into Equation 4 with Equation 6, yields a polynomial equation  $S(P_1, P_2, P_3, P_4) = 0$  in terms of  $P_j = e^{(\varrho_j \xi)}$  for  $j = 1, \dots, 4$ . By equating all the coefficients of this polynomial, a system of nonlinear equations in terms of  $\sigma_n, \varrho_n$  and  $\mathcal{B}_0, \mathcal{B}_i, \mathcal{A}_i$  is generated.

- (iv) Stage 4: By solving the above system of equations using a computer program, the values of  $\sigma_n, \varrho_n, \mathcal{B}_0, \mathcal{B}_i$  and  $\mathcal{A}_i$  were obtained. Inserting these values in Equation 5 introduces the soliton solutions of NLEEs.

#### 3.2. The improved modified simplest equation method

The IMSE method<sup>37</sup> is a recently developed analytical technique. To assess the solution of Equation 2 using the IMSE technique, the subsequent stages were performed:

- (i) Stage 1: Let  $\mathcal{U}(x, y, t) = \mathcal{U}(\xi)$ ;

$$\xi = m \frac{x^\alpha}{\alpha} + n \frac{y^\theta}{\theta} \pm \varepsilon \frac{t^\gamma}{\gamma}, \quad (7)$$

where  $\varepsilon$  is the wave velocity.  $m$  and  $n$  are the real parameters, and  $0 < \gamma \leq 1$ .

- (ii) Stage 2: Equation 7 was used to convert Equation 2 into a nonlinear ordinary differential equation (NODE), as shown in Equation 8:

$$\mathcal{M}(\mathcal{U}, \mathcal{U}', \mathcal{U}'', \dots) = 0, \quad (8)$$

where the polynomial  $\mathcal{M}$  involved  $\mathcal{U}(\xi)$  and its derivatives.

- (iii) Stage 3: Equation 8 may be integrated term by term repeatedly, as required.

- (iv) Stage 4: According to the IMSE method, the analytical

solution of **Equation 8** can be structured as **Equation 9**:

$$\mathfrak{U}(\xi) = s_0 + \sum_{i=1}^N s_i \varphi^i(\xi), \quad (9)$$

where  $s_0$  and  $s_i$  are to be determined later, and  $s_N \neq 0$

Here,  $\varphi(\xi)$  assures the following auxiliary equation with the real parameter  $\eta$ .

$$\varphi'(\xi) = \varphi^2(\xi) + \eta \quad (10)$$

where  $\varphi'(\xi)$  means  $\frac{d\varphi}{d\xi}$ . The auxiliary equation, **Equation 10**, holds the following solutions for diverse cases of  $\eta$ :

a. Case 1: when  $\eta < 0$ , then we get:

$$\begin{aligned} \varphi(\xi) &= -\sqrt{-\eta} \tanh(\sqrt{-\eta} \xi) \\ \varphi(\xi) &= -\sqrt{-\eta} \coth(\sqrt{-\eta} \xi) \\ \varphi(\xi) &= \sqrt{-\eta} \left( -\tanh(2\sqrt{-\eta} \xi) \pm \operatorname{sech}(2\sqrt{-\eta} \xi) \right) \\ \varphi(\xi) &= \sqrt{-\eta} \left( -\coth(2\sqrt{-\eta} \xi) \pm \operatorname{csch}(2\sqrt{-\eta} \xi) \right) \\ \varphi(\xi) &= -\frac{\sqrt{-\eta}}{2} \left( \tanh\left(\frac{\sqrt{-\eta}}{2} \xi\right) + \coth\left(\frac{\sqrt{-\eta}}{2} \xi\right) \right) \\ \varphi(\xi) &= \frac{\sqrt{-\eta}}{\left( \tanh\left(\frac{\sqrt{-\eta}}{2} \xi\right) + \operatorname{csch}(\sqrt{-\eta} \xi) \right)} \\ \varphi(\xi) &= \frac{\sqrt{-\eta} \operatorname{sech}(\sqrt{-\eta} \xi) + \sqrt{3} \operatorname{csch}(\sqrt{-\eta} \xi)}{\operatorname{csch}(\sqrt{-\eta} \xi) + \sqrt{3} \eta \operatorname{sech}(\sqrt{-\eta} \xi)} \end{aligned}$$

b. Case 2: when  $\eta > 0$ , then we get:

$$\begin{aligned} \varphi(\xi) &= \sqrt{\eta} \tan(\sqrt{\eta} \xi) \\ \varphi(\xi) &= -\sqrt{\eta} \cot(\sqrt{\eta} \xi) \\ \varphi(\xi) &= \sqrt{\eta} \left( \tan(2\sqrt{\eta} \xi) \pm \sec(2\sqrt{\eta} \xi) \right) \\ \varphi(\xi) &= \sqrt{\eta} \left( -\cot(2\sqrt{\eta} \xi) \pm \csc(2\sqrt{\eta} \xi) \right) \\ \varphi(\xi) &= \frac{\sqrt{\eta}}{2} \left( \tan\left(\frac{\sqrt{\eta}}{2} \xi\right) - \cot\left(\frac{\sqrt{\eta}}{2} \xi\right) \right) \\ \varphi(\xi) &= \frac{-\sqrt{\eta}}{\left( \tan\left(\frac{\sqrt{\eta}}{2} \xi\right) - \csc(\sqrt{\eta} \xi) \right)} \\ \varphi(\xi) &= \frac{\sec(\sqrt{\eta} \xi) + \sqrt{3} \csc(\sqrt{\eta} \xi)}{\csc(\sqrt{\eta} \xi) - \sqrt{3} \sec(\sqrt{\eta} \xi)} \end{aligned}$$

c. Case 3: If  $\eta = 0$ , then:  $\varphi(\xi) = -\frac{1}{\xi}$

(v) Stage 5: The value of  $N$  in **Equation 9**, was set up using

the homogeneous balancing principle on **Equation 8**.

(vi) Stage 6: Putting **Equation 9** and **Equation 10** into **Equation 8** gives a polynomial of  $\varphi^i(\xi)$ . A set of equations (algebraic) with constant parameters  $s_i$ ,  $m, n$  and  $\varepsilon$  will be originated by eliminating the powers  $\varphi^i(\xi)$ . By solving the equations, we obtain the relations among the unknown parameters. After obtaining the general solution of **Equation 10**, the relationships among  $s_i$  ( $i = 0, 1, 2, 3, \dots, N$ ),  $m$ ,  $n$ , and  $\varepsilon$  in **Equation 9** were determined, yielding additional classes of solutions for **Equation 8**.

#### 4. Analytical study of the nonlinear time-fractional Allen–Cahn model

In this segment, we investigate the time-fractional AC model to obtain a variety of novel, more general closed-form traveling wave solutions using the GERF and IMSE methods. Moreover, to further clarify the obtained results, graphical representations and physical interpretations are provided to illustrate the diverse characteristics of the resulting soliton solutions.

##### 4.1. Solution formulation based on the generalized exponential rational function method

The AC model is one of the most useful and renowned mathematical models. This was first proposed by Allen and Cahn in 1979 as the phenomenological clarification of anti-phase domain coarsening in such a binary alloy. The conformable time-fractional AC model<sup>39</sup> from **Equation 1** is:

$$D_t^\gamma v - v_{xx} + v^3 - v = 0,$$

where  $t > 0$ , and  $0 < \gamma \leq 1$ . Here,  $\gamma$  is the order of the fractional derivative.

Suppose the wave variable  $v(x, t) = v(\xi)$ , where  $\xi = \ell x + g \frac{t^\gamma}{\gamma}$ , then, **Equation 1** is converted to a NODE as:

$$g v' - \ell^2 v'' + v^3 - v = 0 \quad (11)$$

Applying the homogenous balancing principle on **Equation 11** gives us the balance number,  $N = 1$ . Thus, the solution structure of **Equation 11** is given by:

$$v(\xi) = a_0 + a_1 f(\xi) + \frac{b_1}{f(\xi)} \quad (12)$$

By following the required steps of the GERF method, the main results for the considered model are as follows:

(i) Group 1: Suppose  $\sigma = [-1, -1, 1, -1]$  and  $\varrho = [1, -1, 1, -1]$ , then:

$$f(\xi) = -\frac{\cosh(\xi)}{\sinh(\xi)} \quad (13)$$

Segment 1.1: For  $g = \frac{3}{8}$ ,  $\ell = \pm \frac{1}{4\sqrt{2}}a_0 = \frac{1}{2}$ ,  $a_1 = -\frac{1}{4}$ ,  $b_1 = -\frac{1}{4}$ , the solution becomes:

$$v(x, t) = \frac{1}{2} + \frac{1}{4} \left( \frac{\coth\left(\pm \frac{x}{4\sqrt{2}} + \frac{3t^\gamma}{8\gamma}\right) + \tanh\left(\pm \frac{x}{4\sqrt{2}} + \frac{3t^\gamma}{8\gamma}\right)}{2} \right) \quad (14)$$

(ii) Group 2: Suppose  $\sigma = [1, -3, -1, 1]$ , and  $\varrho = [1, -1, 1, -1]$  then:

$$f(\xi) = \frac{\cosh(\xi) - 2\sinh(\xi)}{\sinh(\xi)} \quad (15)$$

Segment 2.1: When  $\ell = \frac{1}{2\sqrt{2}}$ ,  $g = \frac{-3}{4}$ ,  $a_0 = \frac{3}{2}$ ,  $a_1 = 0$ ,  $b_1 = \frac{3}{2}$ , the solution takes the form:

$$v(x, t) = \frac{3}{2} + \left( \frac{3\sinh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right)}{2\left(\cosh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right) - 2\sinh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right)\right)} \right) \quad (16)$$

(iii) Group 3: Suppose  $\sigma = [-2, 0, -1, 1]$ , and  $\varrho = [1, -1, 1, -1]$  then:

$$f(\xi) = \frac{\cosh(\xi) + \sinh(\xi)}{\sinh(\xi)} \quad (17)$$

Segment 3.1: When  $\ell = \frac{1}{2\sqrt{2}}$ ,  $g = \frac{-3}{4}$ ,  $a_0 = 1$ ,  $a_1 = -\frac{1}{2}$ ,  $b_1 = 0$ , the solution becomes:

$$v(x, t) = 1 - \left( \frac{\cosh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right) + \sinh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right)}{2\sinh\left(\frac{x}{2\sqrt{2}} - \frac{3t^\gamma}{4\gamma}\right)} \right) \quad (18)$$

(iv) Group 4: Suppose  $\sigma = [2 - i, -2 - i, -1, 1]$  and  $\varrho = [i, -i, i, -i]$  then:

$$f(\xi) = \frac{-2\sin(\xi) + \cos(\xi)}{\sin(\xi)} \quad (19)$$

Segment 4.1: For  $g = \pm \frac{3i}{4}$ ,  $\ell = \pm \frac{i}{2\sqrt{2}}$ ,  $a_0 = \frac{1}{2} \pm i$ ,  $a_1 = \pm \frac{i}{2}$ ,  $b_1 = 0$ , the solution can be expressed as:

$$v(x, t) = \frac{1}{2} \pm i \left( 1 + \frac{1}{2} \left( \frac{-2\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right) + \cos\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)}{\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)} \right) \right) \quad (20)$$

(v) Group 5: Suppose  $\sigma = [-1 - i, 1 - i, -1, 1]$  and  $\varrho = [i, -i, i, -i]$  then:

$$f(\xi) = \frac{\sin(\xi) + \cos(\xi)}{\sin(\xi)} \quad (21)$$

Segment 5.1: When

$g = \pm \frac{3i}{4}$ ,  $\ell = \pm \frac{i}{2\sqrt{2}}$ ,  $a_0 = \frac{1}{2} \pm i$ ,  $a_1 = 0$ ,  $b_1 = \pm i$ , the solution takes the form:

$$v(x, t) = \frac{1}{2} \pm i \left( \frac{1}{2} + \frac{\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)}{\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right) + \cos\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)} \right) \quad (22)$$

(vi) Group 6: Suppose  $\sigma = [-2 - i, 2 - i, -1, 1]$  and  $\varrho = [1, -1, 1, -1]$ , then:

$$f(\xi) = \frac{\cos(\xi) + 2\sin(\xi)}{\sin(\xi)} \quad (23)$$

Segment 6.1: When  $g = \pm \frac{3i}{4}$ ,  $\ell = \pm \frac{i}{2\sqrt{2}}$ ,  $a_0 = \frac{1}{2} \pm i$ ,  $a_1 = 0$ ,  $b_1 = \pm 5i$ , the solution becomes:

$$v(x, t) = \frac{1}{2} \pm i \left( 1 + \frac{i5\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)}{\cos\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right) + 2\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)} \right) \quad (24)$$

(vii) Group 7: Suppose  $\sigma = [1 - i, -1 - i, -1, 1]$  and  $\varrho = [i, -i, i, -i]$  then:

$$f(\xi) = \frac{\cos(\xi) - \sin(\xi)}{\sin(\xi)} \quad (25)$$

Segment 7.1: For  $g = \pm \frac{3i}{4}$ ,  $\ell = \pm \frac{i}{2\sqrt{2}}$ ,  $a_0 = -\frac{1}{2} \pm \frac{i}{2}$ ,  $a_1 = \pm \frac{i}{2}$ ,  $b_1 = 0$ , the solution takes the form:

$$v(x, t) = -\frac{1}{2} \pm \frac{i}{2} \left( 1 + \frac{\cos\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right) - \sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)}{\sin\left(\pm \frac{ix}{2\sqrt{2}} \pm \frac{3it^\gamma}{4\gamma}\right)} \right) \quad (26)$$

(viii) Group 8: Suppose  $\sigma = [-3, -2, 1, 1]$  and  $\varrho = [0, 1, 0, 1]$ , then:

$$f(\xi) = \frac{-3 - 2e^\xi}{1 + e^\xi} \quad (27)$$

Segment 8.1: When  $g = -\frac{3}{2}$ ,  $\ell = \pm \frac{1}{\sqrt{2}}$ ,  $a_0 = -3$ ,  $a_1 = 0$ ,  $b_1 = -6$ , we attained the solution:

$$v(x, t) = -3 \left( 1 + \frac{2 \left( 1 + e^{\left( \pm \frac{x}{\sqrt{2}} - \frac{3t^\gamma}{2\gamma} \right)} \right)}{-3 - 2e^{\left( \pm \frac{x}{\sqrt{2}} - \frac{3t^\gamma}{2\gamma} \right)}} \right) \quad (28)$$

(ix) Group 9: Suppose  $\sigma = [-1, -2, 1, 1]$  and  $\varrho = [1, 0, 1, 0]$ , then:

$$f(\xi) = \frac{-e^\xi - 2}{e^\xi + 1} \quad (29)$$

Segment 9.1: For  $g = \frac{3}{2}, \ell = \pm \frac{1}{\sqrt{2}}, a_0 = -2, a_1 = -1, b_1 = 0$ , the solution can be expressed as:

$$v(x, t) = -2 + \left( \frac{e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 2}{e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 1} \right) \quad (30)$$

(x) Group 10: Suppose  $\sigma = [2, 1, 1, 1]$  and  $\varrho = [1, 0, 1, 0]$ , then:

$$f(\xi) = \frac{2e^\xi + 1}{e^\xi + 1} \quad (31)$$

Segment 10.1: When  $g = -\frac{3}{2}, \ell = \pm \frac{1}{\sqrt{2}}, a_0 = -2, a_1 = 1, b_1 = 0$ , we achieved the solution:

$$v(x, t) = -2 + \left( \frac{2e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 1}{e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 1} \right) \quad (32)$$

(xi) Group 11: Suppose  $\sigma = [3, 2, 1, 1]$  and  $\varrho = [1, 0, 1, 0]$ , then:

$$f(\xi) = \frac{3e^\xi + 2}{e^\xi + 1} \quad (33)$$

Segment 11.1: For  $g = -\frac{3}{2}, \ell = \pm \frac{1}{\sqrt{2}}, a_0 = 2, a_1 = 0, b_1 = -6$ , the solution can be expressed as:

$$v(x, t) = 2 \left( 1 - \frac{3 \left( e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 1 \right)}{3e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 2} \right) \quad (34)$$

(xii) Group 12: Suppose  $\sigma = [-1, 0, 1, 1]$  and  $\varrho = [1, 0, 1, 0]$ , then:

$$f(\xi) = -\frac{1}{e^\xi + 1} \quad (35)$$

Segment 12.1: When  $g = \frac{3}{2}, \ell = \pm \frac{1}{\sqrt{2}}, a_0 = -1, a_1 = -1, b_1 = 0$ , the solution takes the form:

$$v(x, t) = -1 + \left( \frac{1}{e^{\left(\pm \frac{x}{\sqrt{2}} + \frac{3t'}{2\gamma}\right)} + 1} \right) \quad (36)$$

Remarkably, the soliton solutions obtained for the considered model are more general, novel, and practically significant, and are absent in previous literature.

## 4.2. Solution formulation based on the improved modified simplest equation method

The solution of Equation 1 using the IMSE method is structured as:

$$w = s_0 + s_1 \varphi(\xi). \quad (37)$$

By substituting the results of Equations 37 and 10 into Equation 11, and equating the coefficients of like powers of  $\varphi'(\xi)$  to zero, a system of algebraic equations for  $s_0, s_1, l$ , and  $g$  was obtained. Solving this system using Maple 13 yielded the values of  $s_0, s_1, l$ , and  $g$ , which are presented below:

$$g = \pm \frac{3}{4\sqrt{-\eta}}, l = \pm \frac{1}{2\sqrt{-2\eta}}, s_0 = \frac{1}{2}, s_1 = \pm \frac{1}{2\sqrt{-\eta}}$$

(i) Case 1: When  $\eta < 0$ , then we get:

$$k_1(x, t) = \frac{1}{2} \left( 1 \mp \tanh(\sqrt{-\eta}\xi) \right). \quad (38)$$

$$k_2(x, t) = \frac{1}{2} \left( 1 \mp \tanh(\sqrt{-\eta}\xi) \right). \quad (39)$$

$$k_3(x, t) = \frac{1}{2} \left( 1 \pm \left( -\tanh(2\sqrt{-\eta}\xi) \pm \operatorname{sech}(2\sqrt{-\eta}\xi) \right) \right). \quad (40)$$

$$k_4(x, t) = \frac{1}{2} \left( 1 \pm \left( -\coth(2\sqrt{-\eta}\xi) \pm \operatorname{csch}(2\sqrt{-\eta}\xi) \right) \right). \quad (41)$$

$$k_5(x, t) = \frac{1}{2} \left( 1 \mp \frac{1}{2} \left( \tanh\left(\frac{\sqrt{-\eta}}{2}\xi\right) + \coth\left(\frac{\sqrt{-\eta}}{2}\xi\right) \right) \right). \quad (42)$$

$$k_6(x, t) = \frac{1}{2} \left( 1 \pm \frac{1}{\left( \tanh\left(\frac{\sqrt{-\eta}}{2}\xi\right) + \operatorname{csch}(\sqrt{-\eta}\xi) \right)} \right). \quad (43)$$

$$k_7(x, t) = \frac{1}{2} \left( 1 \pm \frac{\sqrt{-\eta} \operatorname{sech}(\sqrt{-\eta}\xi) + \sqrt{3} \operatorname{csch}(\sqrt{-\eta}\xi)}{\sqrt{-\eta} (\operatorname{csch}(\sqrt{-\eta}\xi) + \sqrt{-3\eta} \operatorname{sech}(\sqrt{-\eta}\xi))} \right) \quad (44)$$

(ii) Case 2: When  $\eta > 0$ , then we get:

$$k_8(x, t) = \frac{1}{2} \left( 1 \mp i \left( \tan(\sqrt{\eta}\xi) \right) \right) \quad (45)$$

$$k_9(x, t) = \frac{1}{2} \left( 1 \pm i \left( \cot(\sqrt{\eta}\xi) \right) \right) \quad (46)$$

$$k_{10}(x, t) = \frac{1}{2} \left( 1 \mp i \left( \tan(2\sqrt{\eta}\xi) \pm \sec(2\sqrt{\eta}\xi) \right) \right) \quad (47)$$

$$k_{11}(x, t) = \frac{1}{2} \left( 1 \mp i \left( -\cot(2\sqrt{\eta}\xi) \pm \csc(2\sqrt{\eta}\xi) \right) \right) \quad (48)$$

$$k_{12}(x, t) = \frac{1}{2} \left( 1 \mp \frac{i}{2} \left( \tan \left( \frac{\sqrt{\eta}}{2} \xi \right) - \cot \left( \frac{\sqrt{\eta}}{2} \xi \right) \right) \right) \quad (49)$$

$$k_{13}(x, t) = \frac{1}{2} \left( 1 \pm \frac{i}{\left( \tan \left( \frac{\sqrt{\eta}}{2} \xi \right) - \csc(\sqrt{\eta} \xi) \right)} \right) \quad (50)$$

$$k_{14}(x, t) = \frac{1}{2} \left( 1 \pm \frac{\sec(\sqrt{\eta} \xi) + \sqrt{3} \csc(\sqrt{\eta} \xi)}{\sqrt{-\eta} (\csc(\sqrt{\eta} \xi) - \sqrt{3} \sec(\sqrt{\eta} \xi))} \right) \quad (51)$$

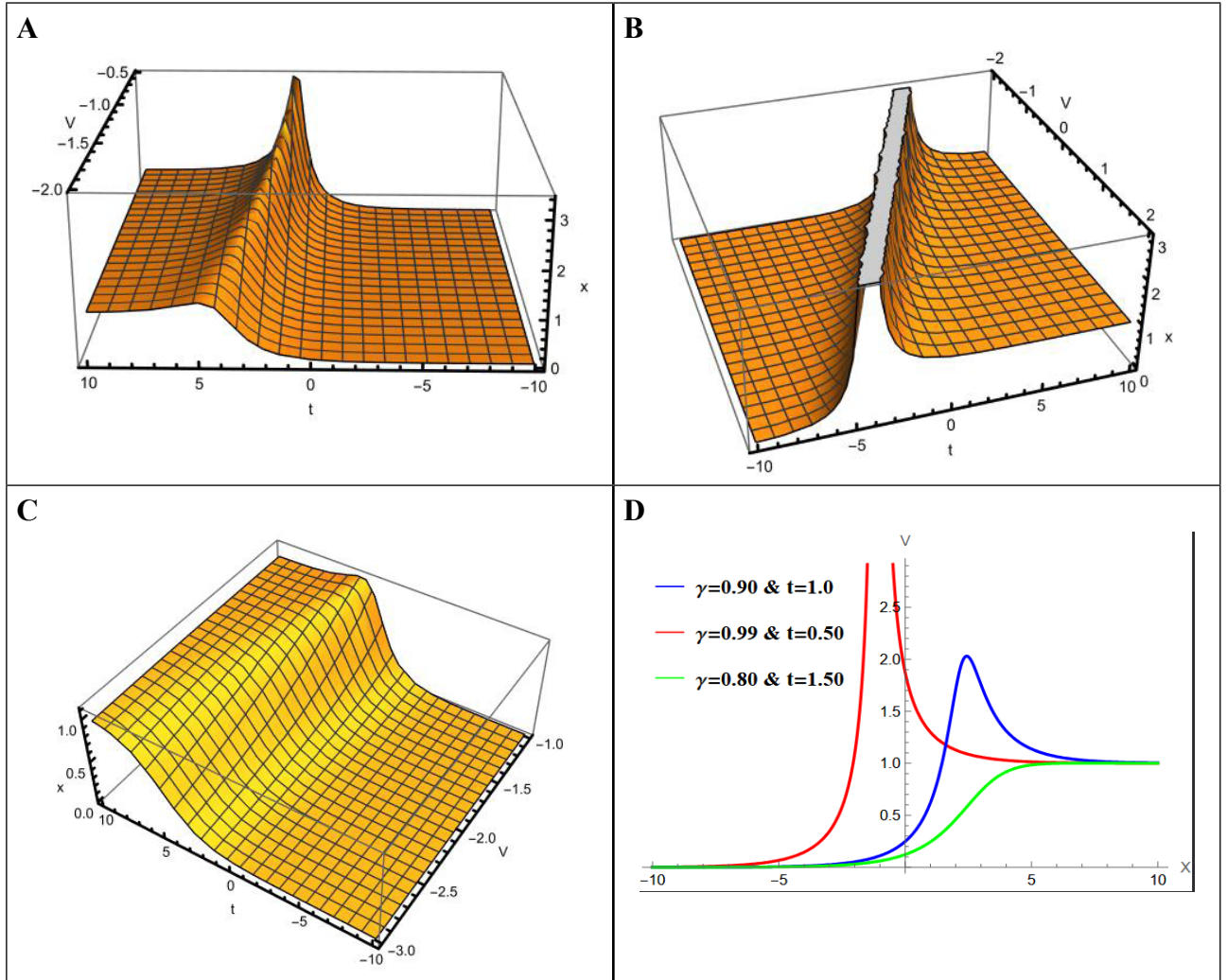
(iii) Case 3: When  $\eta = 0$ , then we attain a trivial solution which is physically meaningless.

### 5. Graphical illustration and physical implications of the results

In this section, we discuss the physical significance of the obtained solutions for the proposed model. Furthermore, various plots of the closed-form solitary wave solutions are

presented to illustrate their physical behavior. The three-dimensional and associated two-dimensional plots of the results obtained are presented to illustrate the effect of dynamical changes in the parameters within an appropriate interval. The shapes of the obtained solutions include bell-shaped, singular bell-shaped, multiple singular, multiple periodic, kink-type, and other types of solitons. For minimalism, we graphically present only a selection of the obtained soliton solutions of the considered model, namely **Equations 14, 24, 36, 41, 43, and 51**, while the plots of the remaining solutions are omitted. The practical applicability of the traveling wave solutions of the considered model is examined here using Mathematica programs, as presented below:

**Figure 1** symbolizes the traveling wave shape of **Equation 14** for  $\gamma = 0.99$ ,  $\gamma = 0.90$ , and  $\gamma = 0.80$  respectively. Here, the fractional parameter was selected arbitrarily to simulate the dynamical behavior of the soliton solution, which shows that the soliton shape transitions from a semi-kink to a bright-kink soliton as the fractional parameter decreases, indicating that the soliton stabilizes. These shapes symbolize the influence of different fractional



**Figure 1.** Solitonic construction of **Equation 14**. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 90$ . (C) The three-dimensional shape for  $\gamma = 80$ . (D) The two-dimensional united plot.



orders on the soliton profiles, illustrating how variations in  $\gamma$  affect their behavior.

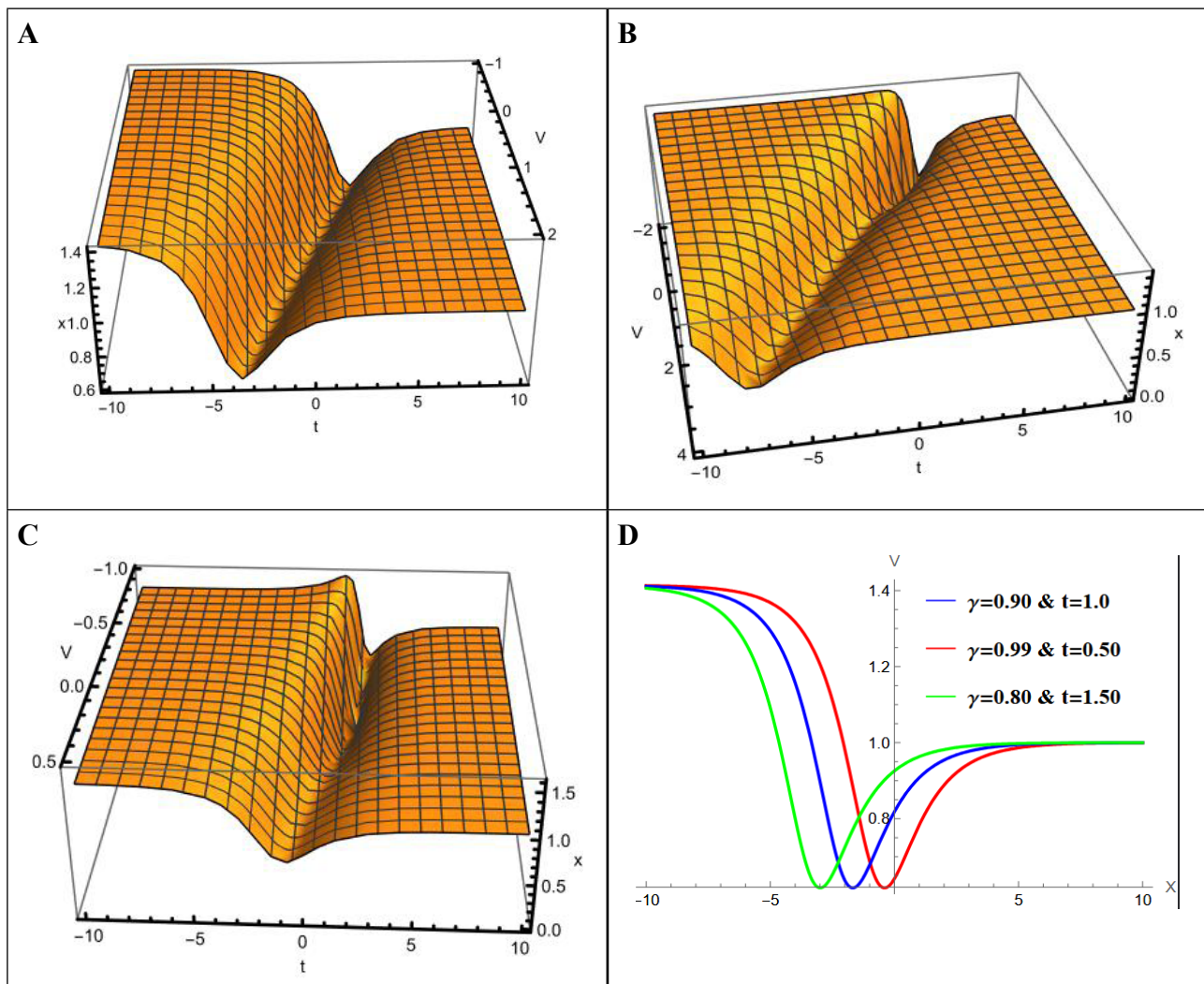
**Figure 2** illustrates the traveling wave profiles of Equation 24 for diverse values of the fractional parameter, namely  $\gamma = 0.99$ ,  $\gamma = 0.90$ , and  $\gamma = 0.80$ , correspondingly and the fractional parameter values are chosen arbitrarily to examine the dynamical behavior of the soliton solution. It is seen that the soliton structure gradually transforms from an anti-bell shape to a semi-bell-shaped compact soliton as the value of  $\gamma$  decreases. This transition indicates that the soliton's stability diminishes as the fractional parameter decreases. Moreover, these variations clearly demonstrate the significant influence of the fractional order on the soliton profiles, highlighting how changes in  $\gamma$  modify both the shape and the dynamical characteristics of the solutions.

**Figure 3** exhibits the traveling wave shape of Equation 36 for  $\gamma = 0.99$ ,  $\gamma = 0.80$ , and  $\gamma = 0.70$ , respectively, and the fractional parameter was chosen arbitrarily to simulate its influence on the dynamical behavior of the soliton solution. All the depicted three-dimensional figures are varied with respect to the value of the fractional parameter  $\gamma$  varies.

The soliton shapes turn from a bright kink shape to a semi-kink shape soliton. Thus, the soliton's stability decreases as the fractional parameter decreases. These shapes signify the influence of different fractional orders on the soliton profiles, explaining how changes in  $\gamma$  alter shapes and characteristics.

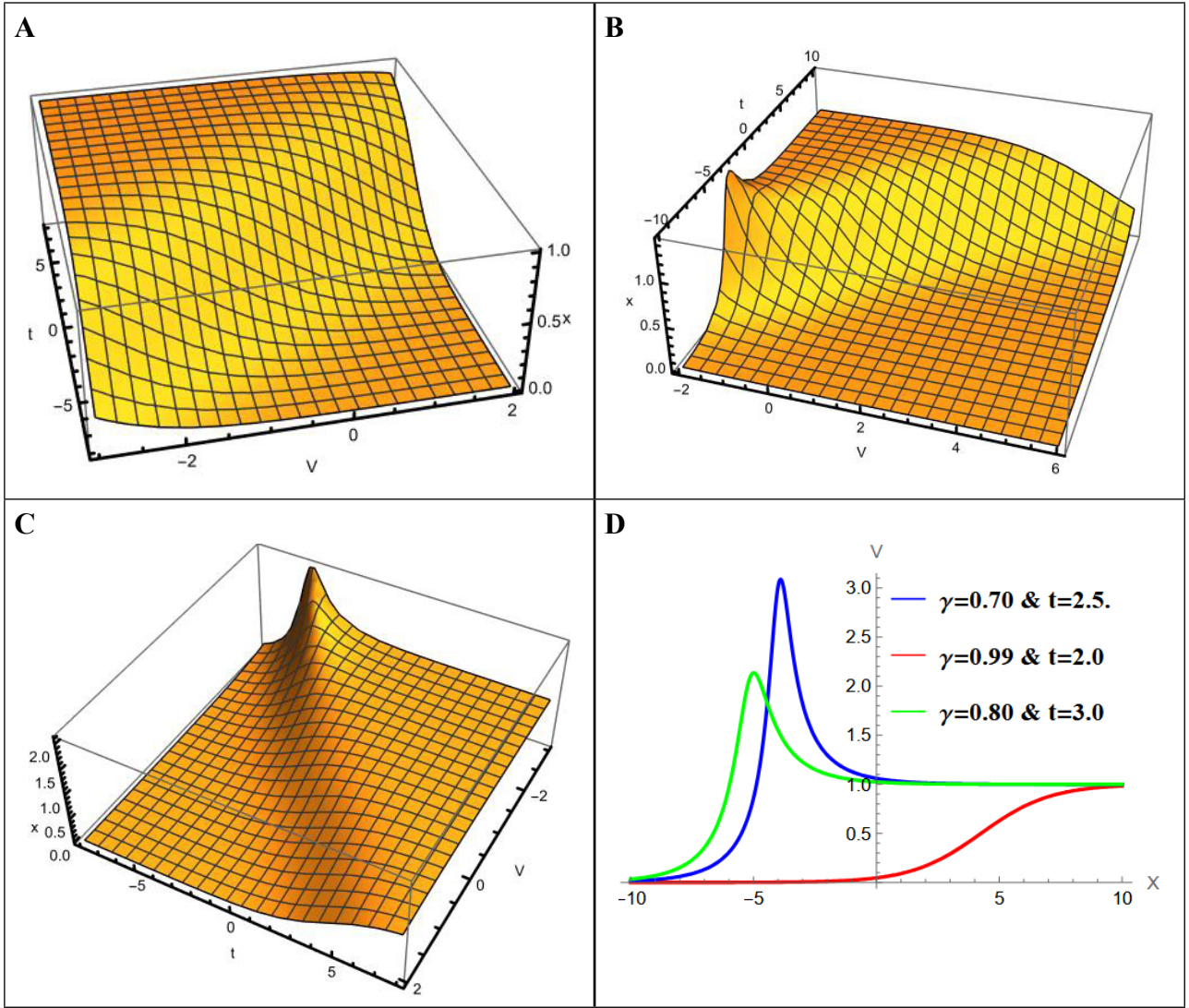
**Figure 4** displays the soliton shape of Equation 41 for  $\gamma = 0.99$ ,  $\gamma = 0.70$ , and  $\gamma = 0.50$ , respectively, and an arbitrary value of the fractional parameter was considered to simulate the dynamical behavior of the soliton solution. The soliton shapes turned from a periodic soliton shape to a singular periodic soliton shape, indicating that the stability of the soliton decreases as the fractional parameter value decreases. These shapes signify the effect of different fractional orders on the soliton profiles, demonstrating how variations in  $\gamma$  alter shapes and characteristics.

**Figure 5** displays the soliton shape of Equation 43 for  $\gamma = 0.99$ ,  $\gamma = 0.90$ , and  $\gamma = 0.50$ , respectively, and an arbitrary value of the fractional parameter was taken to replicate the dynamical behavior of the soliton solution. The form of the soliton changed from multiple periodic to a semi-kink



**Figure 2.** Solitonic construction of Equation 24. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 90$ . (C) The three-dimensional shape for  $\gamma = 80$ . (D) The two-dimensional united plot.





**Figure 3.** Solitonic construction of Equation 36. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 80$ . (C) The three-dimensional shape for  $\gamma = 70$ . (D) The two-dimensional united plot.

soliton shape with the change of the fractional parameter  $\gamma$ .

**Figure 6** exhibits the shape of Equation 51 for  $\gamma = 0.99$ ,  $\gamma = 0.80$ , and  $\gamma = 0.50$ , correspondingly, and the fractional parameter values were chosen arbitrarily to scrutinize the dynamical activities of the soliton solution. The soliton shapes turned from a bell shape to a semi-bell shape compaction soliton, indicating that the soliton stability decreases as the fractional parameter value decreases. These shapes signify the influence of diverse fractional order on the soliton profiles, explaining how variations in  $\gamma$  alter shapes and characteristics.

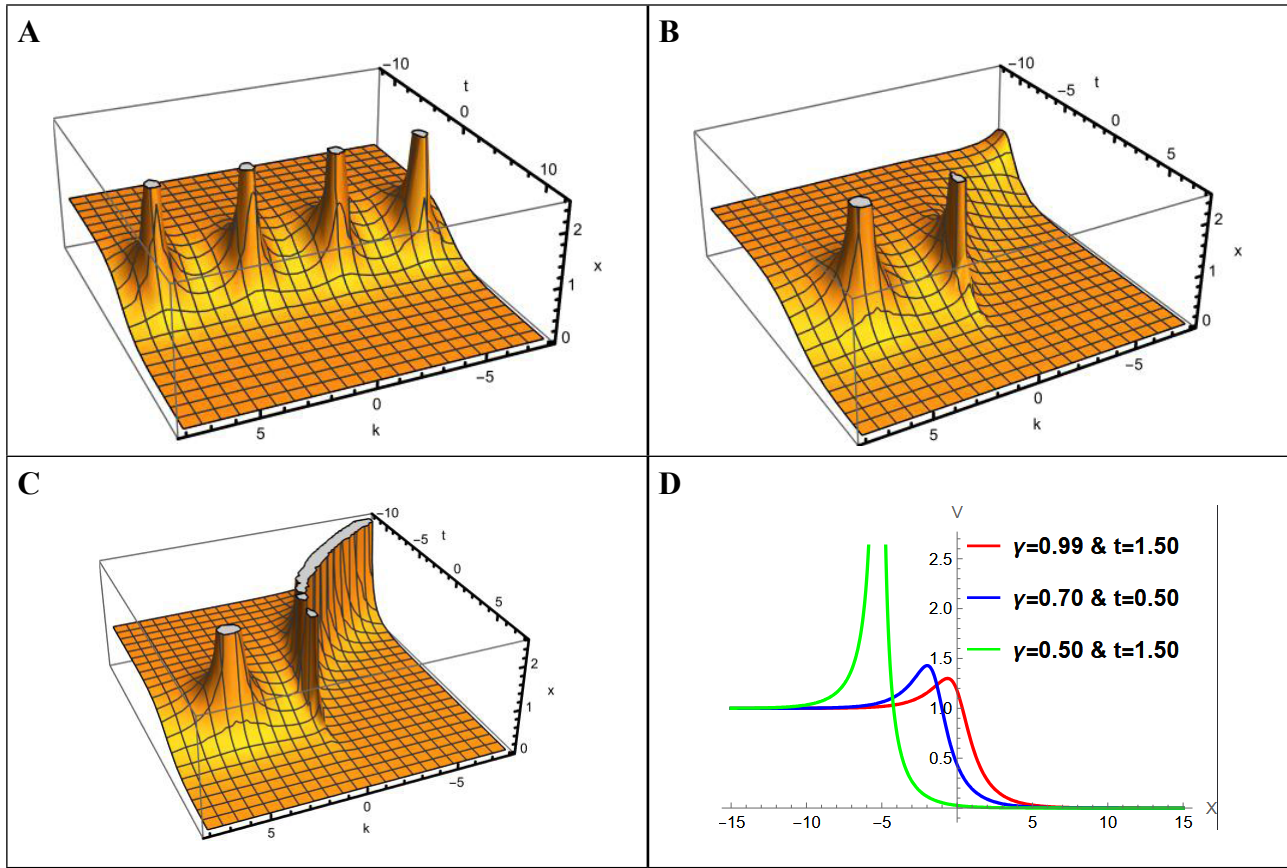
Our obtained soliton solutions may effectively describe the underlying mechanisms of the physical phenomena associated with the considered model and reveal several novel features of complex processes across various fields, including nuclear physics, optical physics, plasma physics, atomic physics, solid-state physics, the mixing of two incompressible fluids, the motion of phase boundaries

in crystalline solids, nucleation processes, and vesicle membranes. From the above graphical representations, it is evident that the behavior of wave propagation in the model is significantly influenced by the fractional parameter  $\gamma$ .

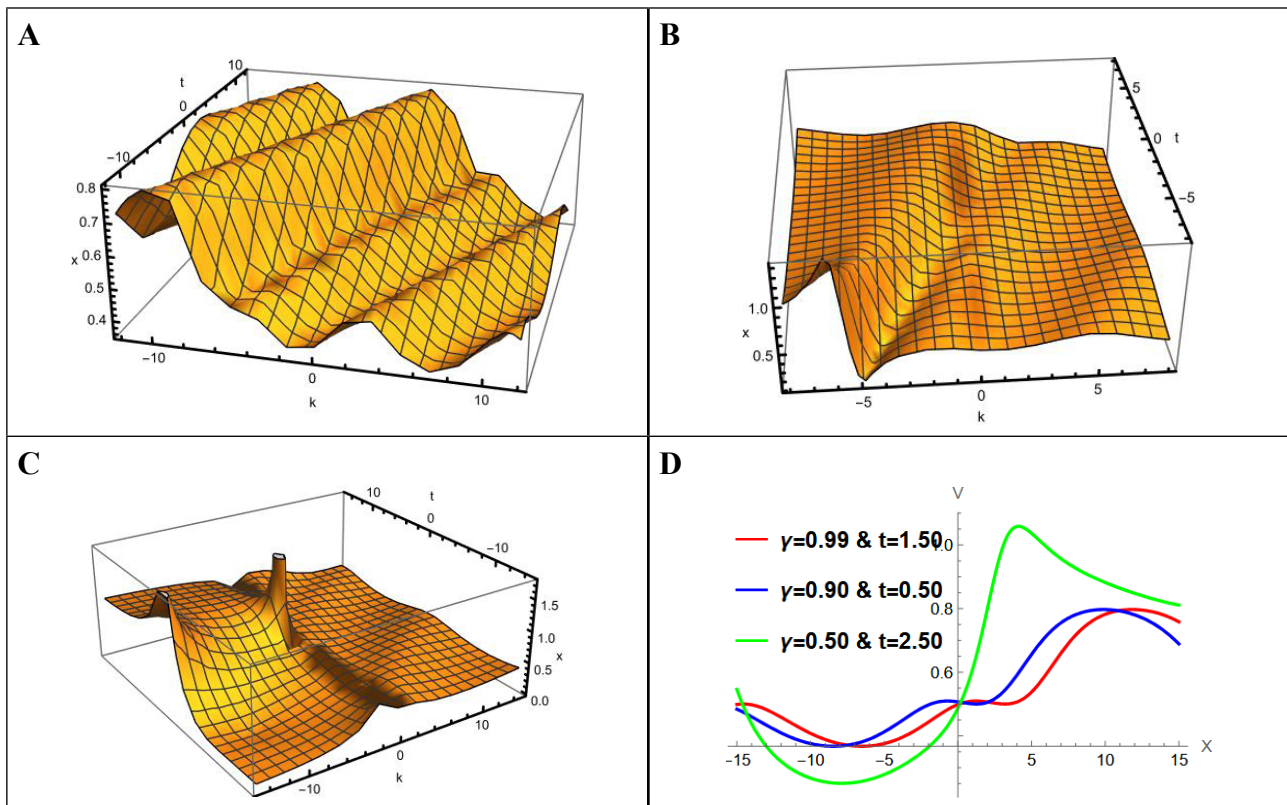
## 6. Stability analysis of the considered model

A system is considered stable if small perturbations or adjustments do not substantially disrupt its overall functionality or performance. Stability analysis poses several significant challenges for researchers in control, signal processing, and systems.

The primary objective of stability analysis for a delayed system is to determine the regions in the delay parameter space where the system remains stable. Although stability analysis is widely applied in disciplines such as mathematics and ecology to investigate complex systems, it is often overlooked in pharmacometrics, where computational approaches are more commonly used to study system behavior. For this purpose, the momentum and Hamiltonian of the time-fractional AC model

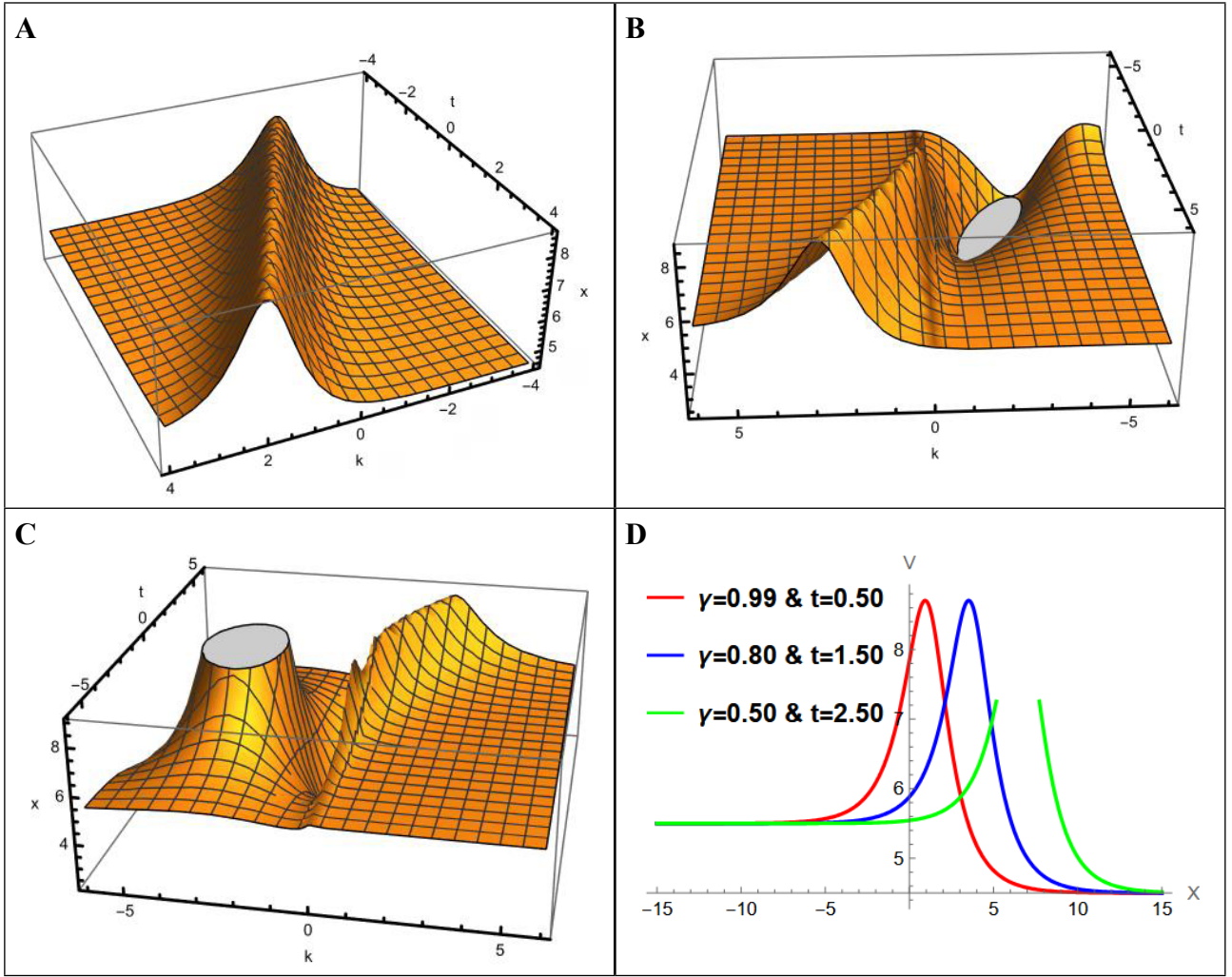


**Figure 4.** Solitonic construction of Equation 41. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 70$ . (C) The three-dimensional shape for  $\gamma = 50$ . (D) The two-dimensional united plot.



**Figure 5.** Solitonic construction of Equation 43. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 90$ . (C) The three-dimensional shape for  $\gamma = 50$ . (D) The two-dimensional united plot.





**Figure 6.** Solitonic construction of Equation 51. (A) The three-dimensional shape for  $\gamma = 99$ . (B) The three-dimensional shape for  $\gamma = 80$ . (C) The three-dimensional shape for  $\gamma = 50$ . (D) The two-dimensional united plot.

(Equation 1) are presented below:

$$E = \frac{1}{2} \int_{p_1}^{p_2} v^2(\psi) d\psi \quad (52)$$

where  $v(\psi)$  is the solution function of the model under consideration.  $p_1$  and  $p_2$  are arbitrary constants such that they must satisfy  $p_1 < p_2$  must satisfy. Thus, the necessary condition for the stability of this solution is derived as follows:

$$\frac{\partial E}{\partial g} > 0 \quad (53)$$

where  $g$  is the velocity of the traveling wave, the stability of the AC model takes the subsequent steps. Substituting Equation 38 into Equation 52, we obtain:

$$E = -\frac{\tanh(2\sqrt{-\eta})}{8\sqrt{-\eta}} - \frac{\ln(\tanh(2\sqrt{-\eta})-1)}{4\sqrt{-\eta}} + \frac{\tanh(\sqrt{-\eta})}{8\sqrt{-\eta}} + \frac{\ln(\tanh(\sqrt{-\eta})-1)}{4\sqrt{-\eta}} \quad (54)$$

and thus:

$$\frac{\partial E}{\partial g} \Big|_{g=0.5} = 0.3550844262 > 0$$

Consequently, this solution is stable. The parameter  $g$  is was chosen arbitrarily, and any positive value of  $g$  can be used to examine the system's stability. Applying the same procedures to the other obtained solutions also satisfies the stability properties of the considered model. Therefore, the model considered is stable.

## 7. Novelty and significance of the established results

Zafar *et al.*<sup>41</sup> investigated the time-fractional AC model using the  $exp_a$ -function method. However, their approach differs from that of the present study in methodology, objectives, results, and overall contributions. The following comparison is therefore conducted to critically examine these differences and to highlight the novelty and improved effectiveness of this study.

- (i) Utilized method: This study adopted the GERF technique along with the IMSE method, both of which

provided a broader framework for constructing exact solutions. On the other hand, Zafar *et al.*<sup>41</sup> employed  $exp_a$ -function method, which is comparatively more restrictive in exploring a wide variety of solution structures.

(ii) Soliton formation: Our investigation yielded a broader variety of well-known soliton structures, including singular periodic waves, regular periodic waves, bell-shaped solitons, kink-type waves, and singular kink solitons. This represents a more diverse set of coherent structures compared to the study by Zafar *et al.*<sup>41</sup> In contrast, their work reported only a limited range of soliton forms, primarily kink and singular kink solitons, indicating a comparatively narrower spectrum of wave structures.

(iii) Corporeal insights: This study provides more comprehensive insights into solitary wave dynamics through the use of two recently developed analytical methods. In contrast, Zafar *et al.*<sup>41</sup> present a relatively limited analysis of the results and their physical interpretations.

(iv) Contributions: The present study offers a substantial contribution to the understanding of nonlinear wave evolution. In contrast, the work of Zafar *et al.*<sup>41</sup> demonstrates relatively limited novelty in this regard.

In summary, this study offers a comprehensive and rigorous investigation of the AC model through the use of a broader set of advanced analytical techniques, providing additional insights into nonlinear wave phenomena beyond those reported by Zafar *et al.*<sup>41</sup>

## 8. Conclusion

In this study, the GERF and IMSE methods were successfully applied to explore novel closed-form traveling-wave solutions of the nonlinear time-fractional AC model that have not been reported in prior literature. By applying these methods, we constructed a wide range of advanced, unique, and generalized solitary traveling wave solutions, including trigonometric, hyperbolic, complex hyperbolic, mixed-type, and rational functions. The solutions exhibited various forms, including multiple singular periodic waves, multiple periodic waves, bell-shaped solitons, singular bell-shaped solitons, kink waves, and more. The graphical illustration of the obtained solutions explains the different dynamical manners of the model, and it is clear that the behavior of wave propagation related to the model is significantly influenced by the fractional parameter  $\gamma$ . It is further apparent that the model under consideration is mathematically stable. This study is significant for phase separation and interfacial dynamics in multi-component systems, capturing the evolution of interfaces and phase segregation (e.g., in alloys). These phenomena, relevant to grain growth, crystal growth, and image edge detection, are driven by the minimization of the system's total free energy.

Finally, it can be concluded that the GERF and IMSE

methods are effective, straightforward, and powerful mathematical tools that offer several precise solutions to nonlinear models arising in mathematical physics, engineering, shallow-water wave mechanics, solid-state physics, nonlinear science, and other related areas.

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## Conflict of interest

There is no conflict of interest.

## Author contributions

*Conceptualization:* Md. Mizanur Rahman

*Formal analysis:* Md. Al-Amin

*Investigation:* Md. Sajib Ali

*Methodology:* Md. Mizanur Rahman

*Writing—original draft:* Md. Mizanur Rahman

*Writing—review & editing:* Md. Nurul Islam, Md. Ali Akbar

## Availability of data

Data will be made available upon request to the corresponding author.

## AI tools statement

We confirm that no AI tools were utilized in the preparation of this document.

## Further disclosure

None.

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