

Complex dynamics, electrocardiogram-like waveform generation, and hardware implementation of an extended three-dimensional Van der Pol system with non-smooth nonlinearities

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ABSTRACT

The Van der Pol (VdP) equation is a fundamental building block for modeling natural and engineering processes characterized by self-sustained oscillations. The present study introduces a novel three-dimensional autonomous chaotic system derived from a non-smooth extension of the classical VdP framework. The core innovation lies in the structural simplification of the nonlinear damping coefficient, replacing the standard quadratic term with a piecewise-linear absolute-value nonlinearity. This modification is coupled with the inclusion of a third state variable, modeled as a first-order dissipative system, which is fed back into the relaxation oscillator via a secondary absolute-value term. This topology significantly reduces computational overhead in hardware implementation while unlocking a higher-dimensional state-space. A comprehensive numerical analysis using bifurcation diagrams, Lyapunov exponent spectra, spectrograms, and Shannon entropy reveals complex dynamic behaviors, including chaos, antimonotonicity, and the coexistence of attractors. Furthermore, the proposed system can generate QRS-like waveforms that resemble human electrocardiogram patterns. The theoretical findings are validated through both an analog circuit implementation using off-the-shelf components and a digital realization on an STM32 microcontroller board using the Runge–Kutta numerical method. The agreement between numerical simulations, analog experiments, and digital implementation attests to the robustness of the proposed model across platforms for potential applications in biomedical signal emulation and chaos-based engineering.

Keywords: Modified Van der Pol; Chaotic dynamics; Antimonotonicity; Multistability; Electrocardiogram-like signal; Hardware implementation



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1. Introduction

Understanding complex rhythms in nature has long been a topic of interest in nonlinear science and engineering. In this context, mathematical models, particularly systems of ordinary differential equations (ODEs), have proven effective for capturing the dynamics of the underlying mechanisms governing natural processes.¹ Whether biologically inspired, phenomenological, or abstract, these models enable hypotheses to be developed and tested in scenarios that would be challenging or practically impossible through experimentation alone. A cornerstone of this scientific landscape is the Van der Pol (VdP) equation $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$, introduced in 1920 to model the oscillations in a vacuum tube circuit.² Featuring a nonlinear damping term that controls energy balance, the VdP equation has evolved far beyond electrical engineering to describe a universal class of systems exhibiting inherent periodicity. These range from mechanical vibrations and geological phenomena to the rhythmic behavior of biological processes.

Over the past few decades, the VdP equation has undergone extensive modifications to account for the complexities of real-world dynamical systems. Following the seminal work of VdP, which advanced our understanding of self-sustained oscillations, Cartwright and Littlewood investigated forced VdP oscillators and identified non-periodic solutions, paving the way for modern chaos theory.³ The FitzHugh–Nagumo model was derived from the Hodgkin–Huxley equations, by effectively using the VdP equation, through the addition of recovery variables to model the excitability and pulse transmission of nerve membranes.^{4–6} Shinriki et al.⁷ introduced the VdP–Duffing oscillator, a model accounting for oscillations with both nonlinear damping and frequency.⁷ The integration of a cubic stiffness term borrowed from the Duffing oscillator has enabled the exploration of the interaction between nonlinear damping and multi-well potentials, giving rise to more complex nonlinear behaviors, including chaos.^{8–10} Furthermore, the VdP equation has been extended to include additional dimensions, yielding higher-order autonomous versions with interesting features and dynamical properties.^{11–14} More recently, the integration of memristive elements into VdP oscillators and the exploration of collective dynamics in networks of locally coupled VdP systems have revealed a wide palette of behaviors, including multistability, hyperchaos, hidden attractors, chimera states, and synchronization.^{15–18} These advanced dynamics are increasingly recognized as essential features for modern chaos-based engineering applications and the accurate phenomenological modeling of complex biological processes, such as cardiac rhythms and neural activity.^{19–23}

Despite the maturity of the VdP framework, the continuous development of new models remains of significant interest. Such models enrich the repertoire of usable maps for chaos-based technological applications and offer fresh perspectives on system topology. The present study introduces a new mathematical model, an autonomous three-dimensional (3D), non-smooth extension of the VdP equation. The innovation of the proposed model lies in its structural simplification for hardware efficiency. The standard quadratic term in the VdP damping function is replaced by a non-smooth absolute value term, and a third state variable is coupled back into the core

oscillator through a similar absolute value function. The piecewise-linear terms add a slower timescale to the system and enable a richer interplay between state variables. This specific modification unlocks higher-order dynamical features while maintaining a lower computational profile than the forced VdP system or other existing autonomous versions. More specifically, the proposed 3D system features seven terms with a single multiplier and can generate a wide range of complex dynamics ranging from simple limit cycles to chaos and multistability. While the discovery of these complex dynamical regimes constitutes the primary focus, the present study also explores a compelling side application of the model: For certain parameter regimes, the system generates waveforms that bear a striking resemblance to the human electrocardiogram (ECG). This ability to emulate the QRS complex is inherited from the VdP dynamics and highlights the potential of the proposed system as a synthetic signal generator for data augmentation, hardware-in-the-loop testing, and other biomedical engineering applications. The contributions of the present work are as follows:

- (i) A 3D autonomous non-smooth chaotic system based on a modified VdP equation is introduced and described.
- (ii) Dynamical analysis: A comprehensive dynamic analysis of the proposed system is conducted using standard tools and methods to identify various regimes and nonlinear phenomena, including the emergence of antimonotonicity and multistability.
- (iii) Application to signal emulation: A qualitative comparison demonstrates the ability of the proposed system to generate ECG-like rhythms as a versatile byproduct of its chaotic dynamics.
- (iv) Hardware implementation: The practical feasibility is demonstrated through both an analog electronic circuit and an STM32 microcontroller implementation.

The paper is structured as follows: **Section 2** provides the mathematical description of the proposed system and stability analysis. **Section 3** explores the rich dynamics and route to chaos through numerical simulations. **Section 4** is dedicated to the emulation of ECG-like signals. **Section 5** presents the hardware design of the analog circuit and the microcontroller implementation, followed by a conclusion in **Section 6** that discusses potential applications in chaos-based engineering and data augmentation, and outlines future research directions.

1.1. Description of the proposed system

The VdP oscillator is described by the following nonlinear second-order ODE:²

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \quad (1)$$

which is commonly expressed in its equivalent first-order system form by considering $x_1 = x$ and $x_2 = dx/dt$ as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1 \end{cases} \quad (2)$$

The nonlinear coefficient $\mu(1 - x_1^2)$ is responsible for excitation and damping. For small magnitudes of x_1 , this term provides the energy input necessary to sustain oscillations; conversely, for large magnitudes, it dissipates energy, effectively driving the system toward a stable limit cycle. This mechanism defines the self-sustained oscillatory

regime characterizing VdP oscillators. The dynamics of the original VdP equation are fundamentally governed by the quadratic nonlinearity x_2^2 . In the present study, we propose substituting this term with the absolute-value nonlinearity $|x_1|$. This modification preserves the qualitative dynamics of the VdP equation while linearizing the damping coefficient piecewise, thereby reducing the computational complexity for hardware implementation. To expand the dynamical landscape beyond simple periodic oscillations, the system is extended to a 3D state-space by introducing a non-smooth forcing term $\delta|x_3|$ into the second equation. The third state variable, x_3 , is modeled as a dissipative first-order system driven by $-x_1$ with linear feedback. This configuration establishes a closed-loop chain of interactions: $x_1 \rightarrow x_3 \rightarrow x_2 \rightarrow x_1$. This 3D coupling introduces an additional layer of complexity, enabling a broader repertoire of dynamic behaviors. The resulting autonomous system is finally written as follows:

$$\begin{cases} \dot{x}_1 = \alpha x_2 \\ \dot{x}_2 = \mu(1 - |x_1|)x_2 - x_1 + \delta|x_3| \\ \dot{x}_3 = -x_1 - \eta x_3 \end{cases} \quad (3)$$

Equation 3, hereafter referred to as the modified VdP (mVdP) oscillator, is a seven-term autonomous 3D system with a single multiplier and two piecewise-linear nonlinearities. The system parameters are defined as follows:

- α : A scaling factor controlling the influence of x_2 on the velocity of x_1 .
- μ : The gain modulating the nonlinear damping, thereby controlling the energy balance within the core relaxation oscillator and determining the amplitude of the self-sustained oscillations as in the classical VdP equation.
- δ : A coupling coefficient that scales the magnitude of the energy injected into the core oscillator from the forcing term $|x_3|$.
- η : A damping coefficient providing local stability and energy dissipation of the auxiliary third state variable x_3 .

Effectively, as illustrated in **Figure 1**, the mVdP system represents a relaxation oscillator (VdP*) coupled to a first-order dissipative system (foDS), driven by $-x_1$, with a linearly adjustable feedback. The magnitude of the foDS output is supplied to the relaxation oscillator through an adjustable gain δ . The ratio (η/δ) regulates the balance between internal dissipation and external feedback.

Beyond the core structure of the VdP equation, the proposed architecture integrates non-smooth nonlinearities ($|x_1|$ and $|x_3|$) and extends the state-space to a 3D autonomous framework. These modifications suggest a greater potential for complex dynamics compared to the purely two-dimensional VdP system or previously reported smooth extensions. Furthermore, using absolute-value terms instead of polynomial products minimizes the computational and realization costs on both analog and digital platforms.

The dissipativity of a dynamical system characterizes the rate at which a volume of initial conditions expands or contracts in the state space. This is evaluated by computing the divergence of the vector field ∇V . For a system to be classified as dissipative, the computed value must be negative, implying that the phase-space volume asymptotically contracts toward a bounded region.

This constitutes a necessary condition for the existence of attracting sets. In the case of the mVdP system, defined in **Equation 3**, the divergence is computed as follows:

$$\nabla V = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = \mu(1 - |x_1|) - \eta \quad (4)$$

The dissipativity of the system is governed by the interplay between μ and η and the state variable x_1 , which continues to modulate the nonlinear damping term as in the original VdP equation. To satisfy the condition for a dissipative system ($\nabla V < 0$), the following relationship must hold:

$$|x_1| > 1 - \zeta \quad (5)$$

With $\zeta = \eta/\mu$ representing the damping ratio, it expresses the balance between the dissipation rate of the auxiliary first-order system and the intrinsic nonlinear damping of the relaxation oscillator VdP*. Thus, ζ acts as a threshold modulating both the global dissipativity and the resulting dynamics. When the magnitude of x_1 is sufficiently large to satisfy **Equation 5**, the system is dissipative, and the trajectories evolve toward a confined region of the state-space. In contrast, when the magnitude of x_1 is small such that the condition is violated, ∇V becomes positive, causing the trajectories to expand. This recurring interplay between expansion and contraction illustrates the oscillatory nature of the mVdP system and its ability to sustain self-oscillations. **Figure 2A** maps the boundaries between divergent and dissipative regions as a function of the damping ratio ζ and the magnitude of x_1 . In the divergent regions, the system's initial volume expands in the phase space. Conversely, in dissipative regions, volume contraction ensures the existence of confined and bounded solutions, providing the necessary condition for the formation of attractors. This dynamic equilibrium between expansion and contraction is a fundamental characteristic of systems exhibiting chaotic dynamics.

The global stability of the system exhibits a critical dependence on the ratio between parameters η and δ , which respectively modulate the coupling between the auxiliary foDS and its internal dissipation:

- When the condition $\delta > \eta$ holds, the energy injected into the system via the $|x_3|$ coupling exceeds the intrinsic dissipation rate. This imbalance results in a destabilizing feedback loop that can lead to sustained high-amplitude oscillations or a loss of amplitude regulation.
- In contrast, when $\eta > \delta$, dissipation dominates, and the dynamics converge asymptotically toward a bounded region of the state-space.

This transition is well illustrated by **Figure 2B**, which presents the average velocity magnitudes across the three dimensions, numerically computed for $\alpha = 1$ and $\mu = 5$. It is observed that for $\delta = 0$, the output of the first-order system is nullified, thereby recovering the classical relaxation regime of the original VdP equation.

2. Equilibrium and stability

The dynamic behavior of the proposed mVdP system is characterized by determining its equilibrium points and evaluating their stability. This analysis is fundamental to understanding how the system behaves when perturbed around these steady-state conditions. The equilibrium

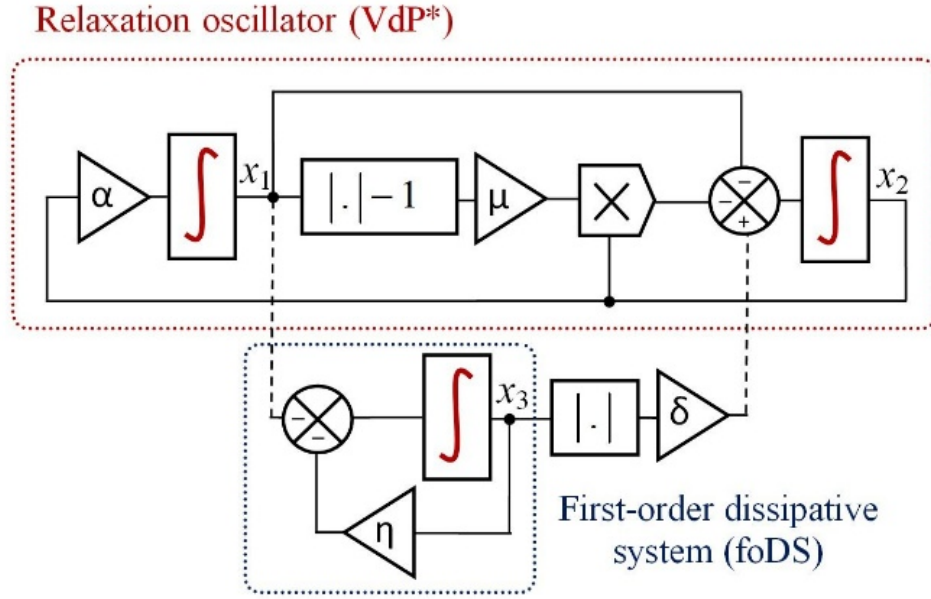


Figure 1. Block diagram of the mVdP system. The VdP* block represents the VdP equation with an absolute value nonlinearity. Abbreviations: mVdP: Modified Van der Pol; VdP*: Relaxation oscillator.

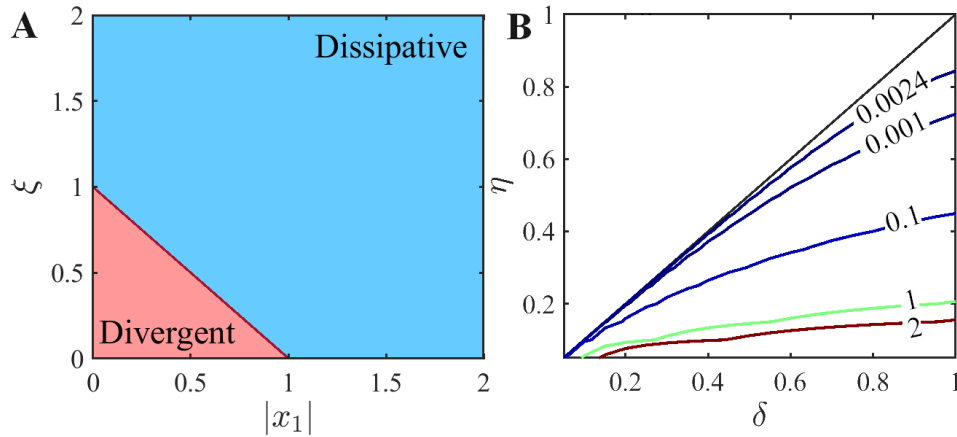


Figure 2. Stability boundaries of the mVdP system: (A) Mapping of divergent/dissipative regions in the $(\xi, |x_1|)$ plane; (B) Mapping the average velocity magnitudes across the state-space in the (δ, η) plane, for $\alpha = 1$ and $\mu = 5$. Abbreviation: mVdP: modified Van der Pol.

points of the system are computed by assuming zero velocity. This is equivalent to setting the right-hand side of **Equation 3** to zero, which simplifies the problem to solving the following transcendental equation:

$$\eta x_3^* + \delta |x_3^*| = 0 \quad (6)$$

where $x_1^* = -\eta x_3^*$ and $x_2^* = 0$. The existence and nature of the solutions depend on the relationship between δ and η , which complements the analysis of the previous section.

- When $\eta \neq \delta$, the system possesses a unique equilibrium point $S_0(0, 0, 0)$
- When $\eta = \delta$, the system exhibits a continuum of equilibrium along the ray $(-\eta x_3^*, 0, x_3^*)$ for $x_3^* \leq 0$.

In the following, we will focus exclusively on the situation where $\eta > \delta$, ensuring S_0 is the sole fixed point and allowing for an asymptotic convergence of trajectories. In the limit where $\eta \gg \delta$, the system enters a strong dissipation regime dominated by the third dimension. The

local stability of S_0 , is analyzed by linearizing the system via the Jacobian matrix J . The general Jacobian is computed as:

$$J = \begin{bmatrix} 0 & \alpha & 0 \\ -\mu x_2^* \text{sign}(x_1^*) - 1 & \mu(1 - |x_1^*|) & \delta \text{sign}(x_3^*) \\ -1 & 0 & -\eta \end{bmatrix} \quad (7)$$

At the origin S_0 , the Jacobian simplifies to:

$$J_{S_0} = \begin{bmatrix} 0 & \alpha & 0 \\ -1 & \mu & 0 \\ -1 & 0 & -\eta \end{bmatrix} \quad (8)$$

The stability of this unique equilibrium is determined by the characteristic equation $\det(J - \lambda I) = 0$, with I being the 3×3 identity matrix, which yields:

$$(\lambda + \eta)(\lambda^2 - \mu\lambda + \alpha) = 0 \quad (9)$$

The eigenvalues of the system are:

$$\lambda_1 = -\eta \text{ and } \lambda_{2,3} = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - \alpha}$$

Since all parameters are positive, the system displays a stable manifold along one axis, as suggested by the presence of a negative real eigenvalue λ_1 , while $\lambda_{1,2}$ could be real and equal ($\mu^2 = 4\alpha$), distinct real ($\mu^2 > 4\alpha$), or complex conjugate ($\mu^2 < 4\alpha$). However, in all cases, the real parts of $\lambda_{1,2}$ are positive, implying unstable trajectories. The origin is therefore an unstable saddle-focus around which the system oscillates.

3. Numerical analyses

This section aims to investigate the dynamics of the proposed system through extensive numerical simulations. More specifically, the parameter space is scrutinized using a suite of standard tools from nonlinear science, including bifurcation diagrams, Lyapunov exponent (LE) spectra, spectrograms, Shannon entropy (H), and Poincaré maps. Due to the non-smooth piecewise nonlinearities and the stiff nature of the mVdP model, the system was integrated using a modified Rosenbrock formula (MATLAB ode23s solver). This second-order stiff solver ensures sufficient numerical stability and prevents the accumulation of integration artifacts over a long time span, as required for spectral analysis.²⁴ Simulations are conducted over a total time interval of 10^5 , with a time step of 0.001. For all cases, the transient regime is discarded to ensure that the obtained data represent the asymptotic behavior of the system.

3.1. Methods

3.1.1. Local maxima and spectral analysis

The frequency spectrum and the distribution of time series local maxima are important tools for constructing a preliminary image of a dynamical system in both time and frequency domains. Spectral analysis reveals the dominant oscillatory modes by deconstructing a time series into its individual frequency components, while the extraction of local maxima provides a time-domain depiction of the attractor's folding mechanism. Periodic and quasi-periodic motions are characterized by distinct peak frequencies and a finite, repeating set of local maxima, whereas chaotic dynamics correspond to a broadband, continuous spectrum and a non-repeating sequence of local maxima spread over a continuous range.

3.1.2. Bifurcation diagram and Lyapunov spectrum

The bifurcation diagram provides a comprehensive visualization of the qualitative changes in the exhibited dynamics as a specific control parameter is varied. It represents the local maxima of the time series for increasing or decreasing values of the investigated parameter.²⁵ The LEs provide a quantitative measure of the system's sensitivity to initial conditions. LEs characterize the average rate of divergence or convergence of trajectories that are initially nearby. By linearizing the system around its equilibrium and tracking the evolution of small perturbations, the Wolf algorithm quantifies the LEs in all directions, constituting the Lyapunov spectrum from which the dynamics are completely characterized.²⁶ The sign of the largest LE (LLE) provides fundamental information on the qualitative behavior of the system. An LLE > 0 indicates chaotic behavior, where infinitesimal perturbations grow exponentially over time, rendering long-term predictions impossible. LLE $= 0$ corresponds to marginal stability, typical of a limit cycle, and LLE < 0 suggests asymptotic

stability, where trajectories converge toward a fixed point.

3.1.3. Shannon entropy

The H quantifies the information-theoretic complexity of the generated time series. This metric is a robust, commonly used proxy for evaluating system unpredictability. It consists of quantifying the uncertainty in the phase portrait, thereby revealing the disorder in the evolution of the state variables. This approach has been successfully used to characterize the chaotic behavior of nonlinear systems and circuits.^{27,28} Generally, large values of entropy correspond to a high level of disorder and chaotic dynamics, in which trajectories seem to wander erratically through phase space, while low values of entropy are associated with regular and periodic dynamics. The evaluation of H is based on a two-dimensional projection of the phase space partitioned into an $N \times N$ grid, yielding N^2 disjoint cells $(X_{i=1\dots N}, Y_{j=1\dots N})$, corresponding to the "possible states". The normalized H is then computed as follows:

$$H = -\frac{1}{\log(N^2)} \sum_{i=1}^N \sum_{j=1}^N p(X_i, Y_j) \log(p(X_i, Y_j)) \quad (10)$$

where p is the probability of the trajectory visiting a specific cell. Values of $H \approx 1$ indicate a near-uniform distribution, corresponding to a high level of disorder, while low values indicate the predominance of one specific pattern, associated with regular dynamics. In the following, we will consider $N = 30$ to ensure a sufficiently fine resolution and computational efficiency.

3.1.4. Spectrograms

While a static power spectrum identifies frequency components, a spectrogram shows how the main frequency components shift, emerge, or disappear under varying control parameters. This captures bifurcation points and phase transitions between the various qualitative dynamic behaviors, presenting the discrete spectral lines for periodic regimes and broadband spectral windows for chaotic regimes.

3.1.5. Poincaré section

The Poincaré map is obtained by intersecting the system's trajectory in the state-space with a lower-dimensional transversal hyperplane known as the Poincaré section. The points at which the trajectory intersects the Poincaré section in a specified direction are utilized to construct a discrete map that preserves essential topological properties of the original continuous-time system. On the Poincaré section, chaotic dynamics appear as a multitude of points whose arrangement reflects the attractor's underlying structure, whereas periodic motion is recorded as a finite number of points.

3.2. Dynamics and complexity

Figure 3 illustrates the transitions within the proposed system as the control parameter α increases, with the other parameters arbitrarily fixed at $\mu = 3$, $\delta = 2.5$, and $\eta = 2.8$. While the bifurcation diagram (**Figure 3A**) and the LE spectrum (**Figure 3B**) track potential shifts in qualitative behavior, the H (**Figure 3C**) and the parameter-dependent spectrogram (**Figure 3D**) offer additional detail regarding signal complexity and frequency response.

As α is varied from 7 to 15, the system navigates a diverse landscape of chaotic and periodic regimes. For

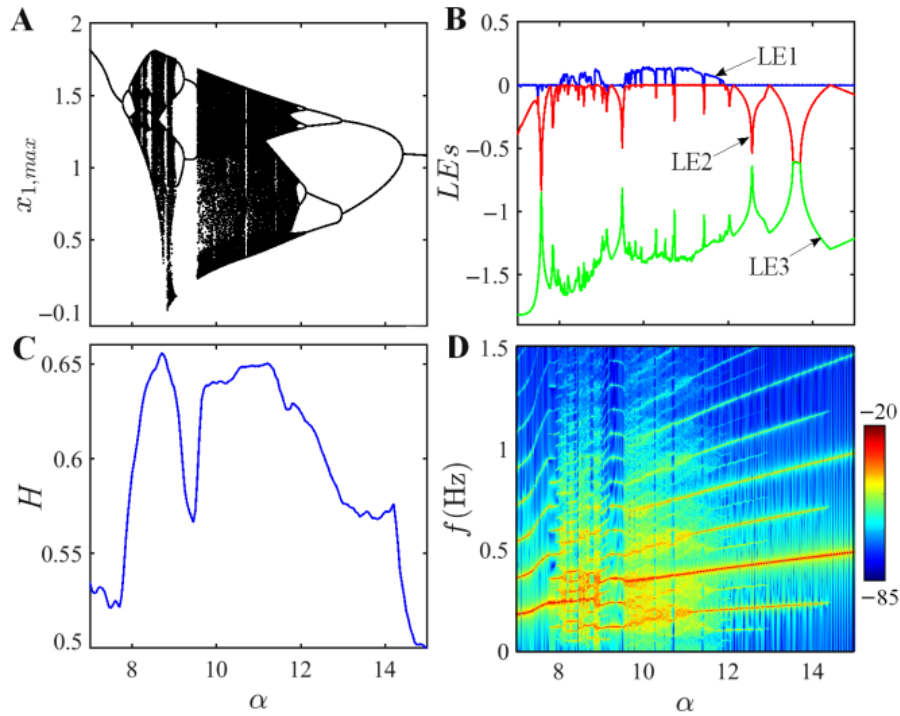


Figure 3. Characterization of dynamics and complexity of the mVdP system for increasing values of α , with $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$, and initial condition $(0.1, 0, 0.1)$. (A) Bifurcation diagram. (B) Lyapunov exponent (LE) spectrum. (C) Normalized Shannon entropy (H). (D) Parameter-dependent spectrogram.

Abbreviation: mVdP: modified Van der Pol.

$7 < \alpha < 7.74$, the dynamics are periodic, characterized by a relatively low entropy ($H \approx 0.53$) and a discrete frequency spectrum. From $\alpha \approx 7.74$, the system undergoes a period-doubling cascade, progressively leading to a chaotic regime at $\alpha \approx 8$, which persists until $\alpha \approx 9$ ($H \approx 0.65$). Within this domain, the system alternates between periodic and chaotic windows, which are well resolved in the bifurcation diagram and spectrogram.

From $\alpha \approx 9$, the system undergoes a cascade of reverse period-doubling bifurcations, leading to a notable loss of complexity in the generated time series, reflected in the progressive drop in normalized entropy. The oscillatory modes constituting the broadband spectrum are progressively destroyed, and the spectrogram gradually converges toward well-defined, discrete, and identifiable frequency components. Within this window, the dynamics return to regularity, as evidenced by the reappearance of discrete spectral peaks and a notable reduction in entropy ($H \approx 0.56$).

However, beyond $\alpha \approx 9.55$, the system abruptly transitions back to a chaotic state, which extends to approximately $\alpha = 12$. This phase is characterized by a sudden increase in entropy ($H \approx 0.635$) and the re-emergence of a broadband spectrum for $\alpha > 9.55$. This resurgence of chaos and the spontaneous expansion of the attractor size suggest an internal crisis, underscoring the intricacy of the interactions taking place within the system.

Finally, another cascade of reverse period-doubling bifurcations is observed for α exceeding 12, gradually leading to the collapse of the chaotic attractor. This is indicated by the progressive decrease in entropy and the persistence of a discrete frequency spectrum corresponding to regular dynamics. The system settles into periodic

oscillations, and the entropy returns to lower values ($H \approx 0.5$). This sequence of transitions from period-doubling to chaos, through periodic windows, crises, and an ultimate return to periodicity, underscores the rich and complex dynamical landscape of the mVdP system.

Figure 4 illustrates a palette of different dynamic behaviors exhibited by the system for arbitrarily selected discrete values of α , with $\mu = 3$, $\delta = 2.5$, and $\eta = 2.8$. To ensure consistent comparison, the initial conditions are fixed as $(0.1, 0, 0.1)$.

Figure 4 A presents the chaotic attractor obtained for $\alpha = 8.5$, quantitatively confirmed by a positive LLE ≈ 0.08 . The corresponding time series (**Figure 4B**) shows a dense, erratic distribution of local maxima, a signature of chaos. As α is increased to 9.5, the chaotic dynamic is suppressed through a sequence of reverse period-doubling bifurcations, leading to a stable period-2 limit cycle (**Figure 4C**) with LLE = 0, and a time series showing a rhythmic and predictable arrangement of local maxima (**Figure 4D**). Following this stable window, the system undergoes an abrupt transition back to chaos, giving rise to a one-scroll chaotic attractor shown in **Figure 4E** (plotted for $\alpha = 10$, LLE ≈ 0.13), with the corresponding time series (**Figure 4F**). Beyond this point, a subsequent cascade of reverse-period doubling bifurcations progressively shifts the system towards regular dynamics. This transition is clearly captured by the period-4 limit cycle and the corresponding time series in **Figure 4G** and **Figure 4H** (plotted for $\alpha = 12.5$, LLE = 0), respectively.

Figure 5 provides a multi-domain diagnostic of the system behavior for $\alpha = 8.5$, $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$, and initial conditions $(0.1, 0, 0.1)$. For this specific parameter setting, the mVdP system is chaotic, as illustrated by the frequency spectrum of x_1 (**Figure 5A**), which shows a continuous, broadband profile, and the local maxima

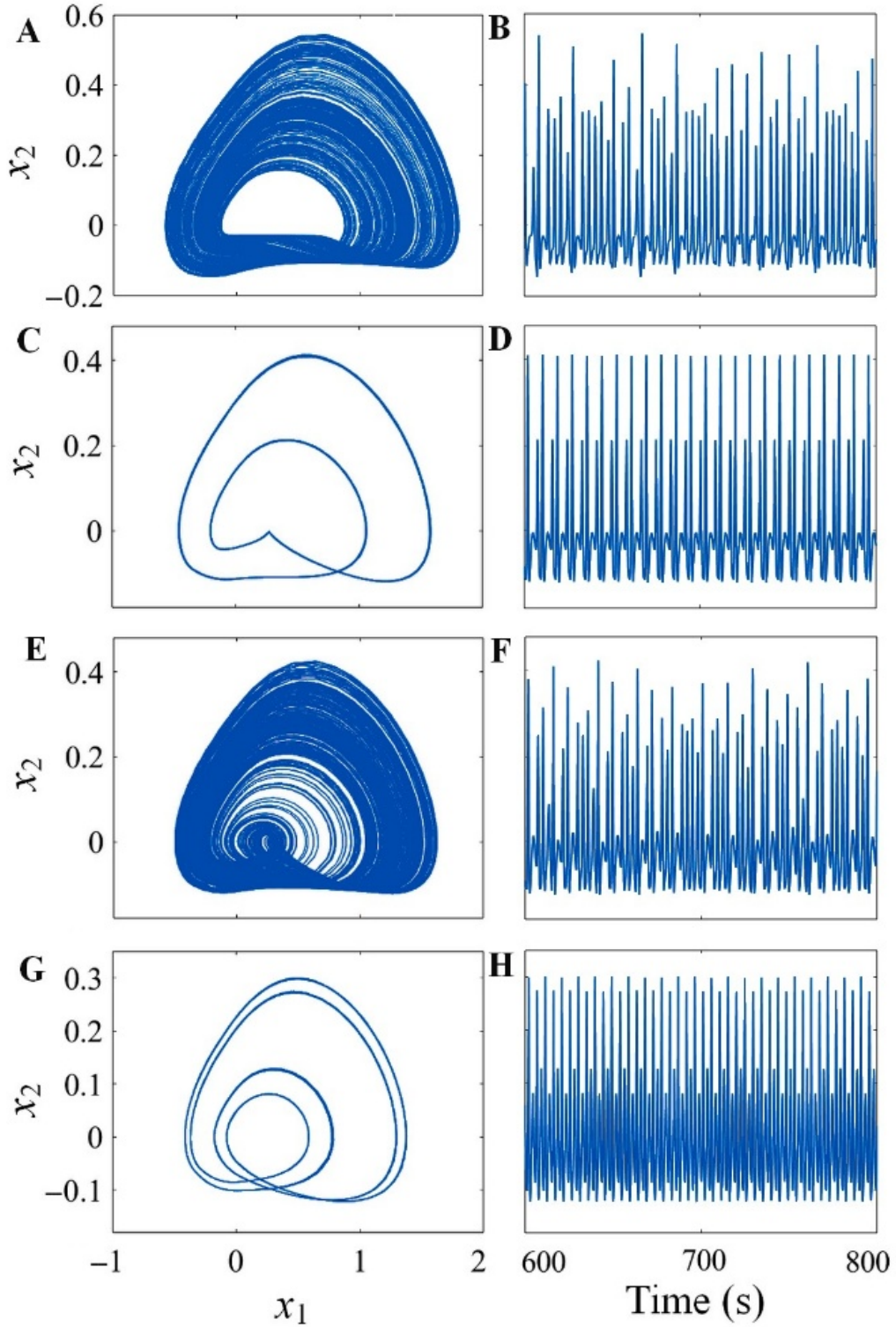


Figure 4. Palette of dynamical behavior exhibited by the mVdP system for discrete values of α , with $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$ and initial condition $(0.1, 0, 0.1)$: Phase portraits in the x_1 - x_2 plane and x_2 time series for $\alpha = 8.5$ (A, B), $\alpha = 9.5$ (C, D), $\alpha = 10$ (E, F), and $\alpha = 12.5$ (G, H). Abbreviation: mVdP: modified Van der Pol.

(Figure 5B) exhibit an irregular, unpredictable sequence. To provide a geometric appreciation, the Poincaré section is captured at the transverse plane $x_3 = 0$ (Figure 5C).

The resulting map depicts a structured cloud of points that highlights the flow's stretching and folding mechanism. Together, these three perspectives provide a comprehensive

view of the chaotic attractor for the specified parameter set.

As previously highlighted, the mVdP model is equivalent to a VdP-like relaxation oscillator (VdP*) coupled to a foDS, with x_1 as input, and providing x_3 as output, whose magnitude directly influences the evolution of the state variable x_2 . To obtain a comprehensive view of the

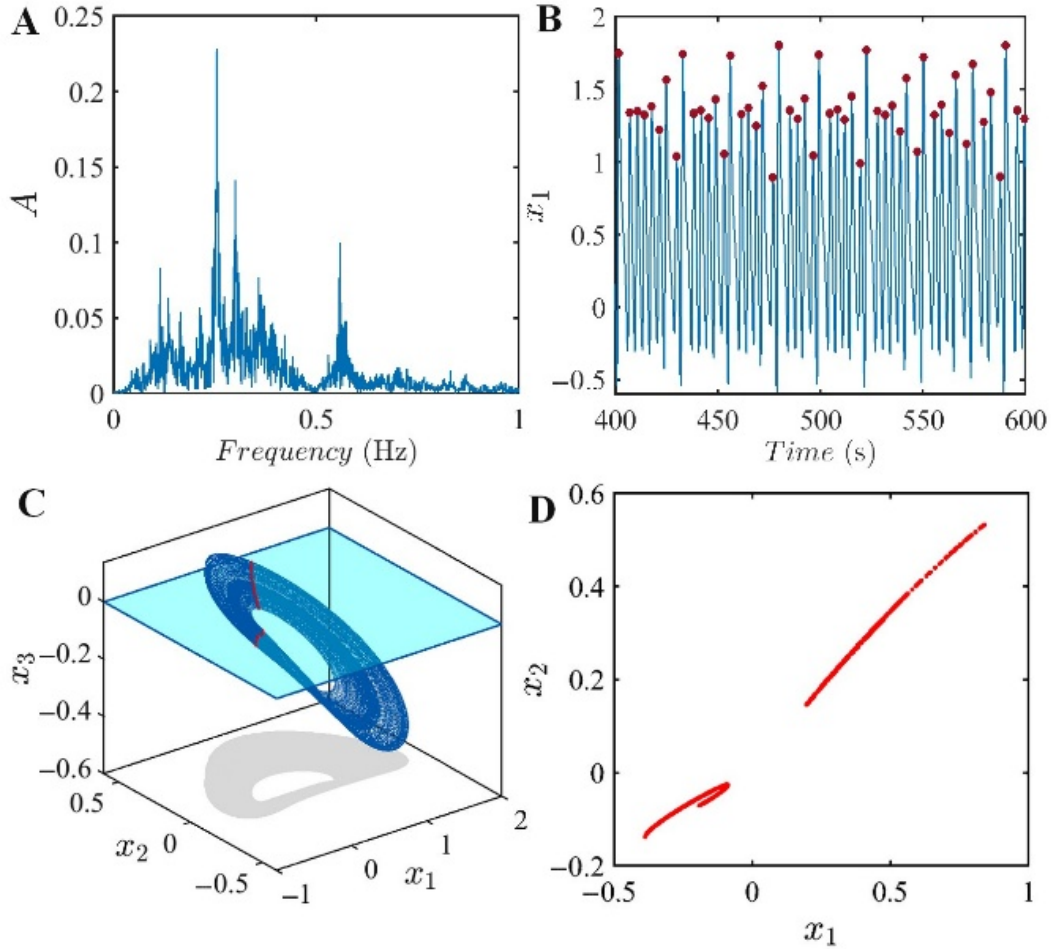


Figure 5. Cross-domain validation of chaotic dynamics for $\alpha = 8.5$, $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$, and initial condition (0.1, 0, 0.1). (A) Broadband frequency spectrum. (B) Non-repeating distribution of local maxima. (C) Poincaré section at $x_3 = 0$. (D) Poincaré map revealing the underlying structure of the chaotic attractor.

dynamics, it is essential to explore the effects of additional parameters such as η , which controls dissipation in the auxiliary foDS, and to understand how this parameter alters the global dynamics under different values of α . Furthermore, the critical dependence of the system on the balance between δ and η is investigated to understand how the interplay between the intrinsic linear feedback of the foDS (η) and the foDS-VdP* coupling (δ) shapes the dynamic behavior of the mVdP system. To explore these dependencies, η versus α and η versus δ 2D bifurcation, as well as LLE maps, are plotted (Figure 6). Figure 6A and 6B shows the number of unique local maxima, whereas Figure 6C and 6D shows the LLE for each pair of control parameters.

As η is varied from 2.5 to 3, the bifurcation sequence of the mVdP system for increasing values of α exhibits various shifts in the transition points. However, for each value of η , the system exhibits qualitatively similar bifurcation sequences: a preliminary zone of rapid period-doubling leading to chaos, followed by a reverse scenario, an eventual abrupt appearance of chaotic dynamics, and a subsequent slower reverse period-doubling cascade that converges into a period-1 limit cycle. The system occasionally exhibits enhanced complexity for lower values of α ($7 < \alpha < 9$), where periodic and chaotic zones alternate rapidly.

The (δ, η) maps are marked by an extensive unbounded region where the system diverges. This corresponds to cases where $\delta > \eta$, triggering a destabilizing feedback loop in which the perturbation introduced by x_3 overcomes the system's dissipative capacity, leading to a loss of amplitude regulation. Conversely, for $\delta < \eta$, the dynamics remain bounded, allowing the system to exhibit either periodic or chaotic behavior, depending on the other parameters and initial conditions.

3.3. Antimonotonicity

A particularly compelling feature of the mVdP system is the emergence of antimonotonicity. As defined by Dawson et al.,²⁹ antimonotonicity is a nonlinear phenomenon characterized by the recurrent creation and annihilation of periodic orbits through forward and reverse period-doubling bifurcations. Unlike monotonic transitions, the system's response to a gradual change in a control parameter does not follow a consistent, predictable trend. The system then exhibits unexpected reversals or counterintuitive outcomes. Typically, this phenomenon manifests as bubble-like structures in the bifurcation diagram. As a control parameter is varied, a stable periodic orbit may destabilize through a forward period-doubling cascade into chaos, only to subsequently undergo a reverse period-doubling cascade that annihilates the complex

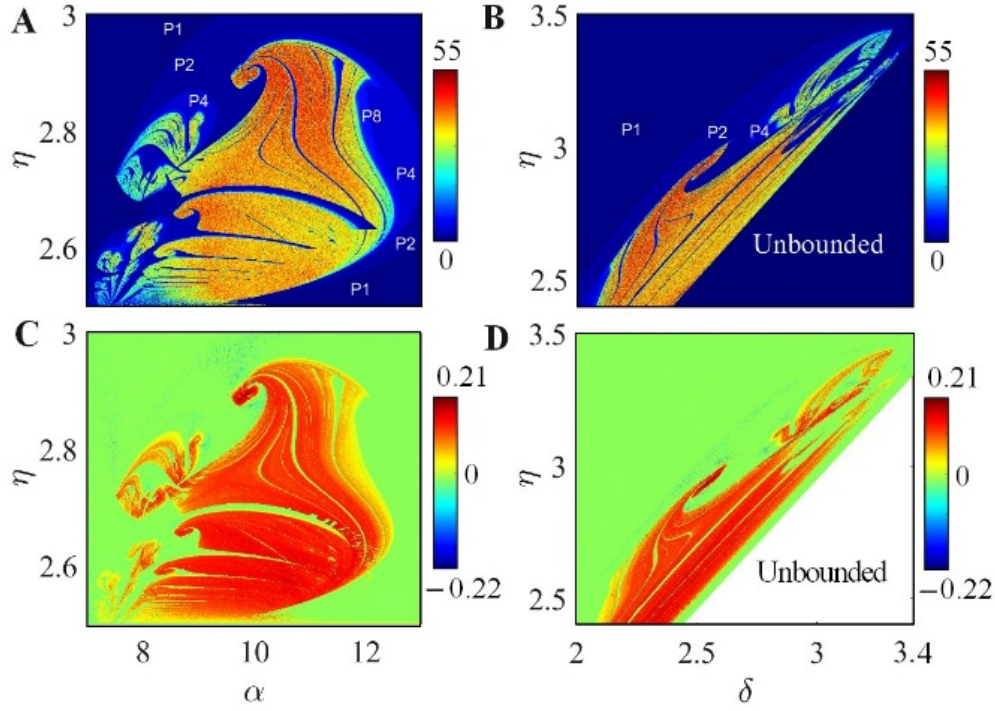


Figure 6. Two-dimensional parametric analysis of the mVdP system dynamics: (A) and (C) represent the number of unique local maxima of x_1 and LLE maps in the (α, η) plane, respectively; (B) and (D) represent the number of unique local maxima of x_1 and LLE maps in the (δ, η) plane, respectively. When not investigated, other parameters and initial conditions are fixed at $\alpha = 8.5$, $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$, and initial condition $(0.1, 0, 0.1)$.

Abbreviations: LLE: Largest Lyapunov exponent; mVdP: modified Van der Pol.

dynamics and returns the system to its initial periodic state. As these bubbles organize into fractal-like structures, the Feigenbaum tree reemerges, reflecting the universal scaling laws governing the transition between order and complexity.³⁰ Each of these bubbles corresponds to a stable periodic orbit, and their arrangement reveals the underlying structure of the chaotic attractor. The presence of multiple critical points in the first-return map of the system's local maxima serves as the fundamental topological folding mechanism responsible for these unpredictable reversals.³¹

Antimonotonicity was observed in the mVdP system for $2.9 \leq \eta \leq 3$, as α was varied while $\mu = 3$ and $\delta = 2.5$ were held. **Figure 7** illustrates this behavior, for arbitrarily selected values of η within the specified range, showing bubbles that transit from fundamental periodicity (**Figure 7C** and **7D**) to increasingly complex dynamic regimes (**Figure 7A** and **7B**).

To formally characterize the observed antimonotonicity, the first return map of the time series local maxima ($x_{i,max}$) is analyzed. Plotting the first return maps of $x_{1,max}$ (**Figure 8A**) and $x_{3,max}$ (**Figure 8B**), for $\alpha = 11.85$, $\mu = 3$, $\delta = 2.5$ and $\eta = 2.9$, allows us to identify two distinct critical points C1 and C2, which confirms that the mVdP system possesses the necessary folding mechanisms for the occurrence of antimonotonicity.

This further underscores the rich dynamics of the proposed model and its high sensitivity to parameter settings.

3.4. Occurrence of coexisting attractors

Multistability is a fundamental phenomenon in nonlinear systems wherein coexisting attractors emerge for a fixed set of parameters under different initial conditions. This

behavior further highlights the effect of the system's initial state on its long-term evolution, often leading to entirely distinct dynamical regimes. Bistability is the elementary case in which two attractors coexist simultaneously.⁷ This was identified in the mVdP system by plotting the superimposed bifurcation diagrams (**Figure 9A**) and LLE (**Figure 9B**) as functions of α , for two different initial conditions: IC1 $(0.4, 0.4, 0.8)$ in blue and IC2 $(0.2, 0.2, 0.15)$ in magenta, with remaining parameters held at $\mu = 3$, $\delta = 2.4$ and $\eta = 2.5$.

This analysis reveals how the system can settle into qualitatively different states, with representative instances of coexisting attractors observed for $\alpha = 9.4$ (chaos–periodicity in **Figure 10A** and **10B**) and $\alpha = 9.24$ (chaos–chaos in **Figure 10C** and **10D**). The topological asymmetry of these attractors further demonstrates the richness of the system's dynamics; notably, their distinct geometries are not the result of inherent symmetries in the governing equations but rather emerge from the flow.

To delineate the regions of state-space leading to each dynamical regime, the basins of attraction were computed for $x_1(0)$ and $x_2(0)$ in the range $[-1, 1]$, with $x_3(0) = 0$. These maps illustrate how one attractor may colonize the phase space at the expense of another as parameter settings are varied.

The basins of attraction in **Figure 10B** and **10C** are populated with a seemingly random dissemination of attractors in the $x_1(0)$ – $x_2(0)$ plane. The absence of continuous domains for each attractor is a critical finding, highlighting a high sensitivity to initial conditions, such that a small perturbation in the initial state can trigger a drastic shift in the qualitative behavior of the system. This complex basin

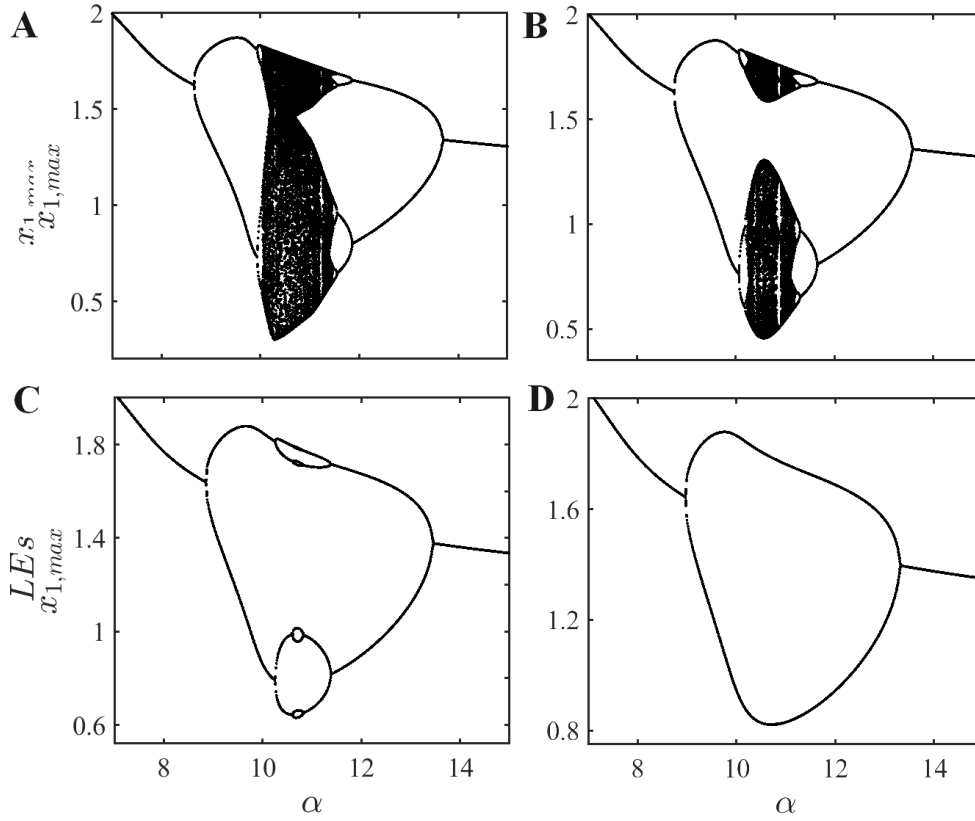


Figure 7. Reemerging Feigenbaum tree (bubbling) for $\mu = 3$, $\delta = 2.5$ and (A) $\eta = 2.94$ (full Feigenbaum reemerging tree), (B) $\eta = 2.95$ (full Feigenbaum reemerging tree), (C) $\eta = 2.96$ (period-8 bubble), and (D) $\eta = 2.97$ (period-8 bubble)
Abbreviation: LE: Lyapunov exponent

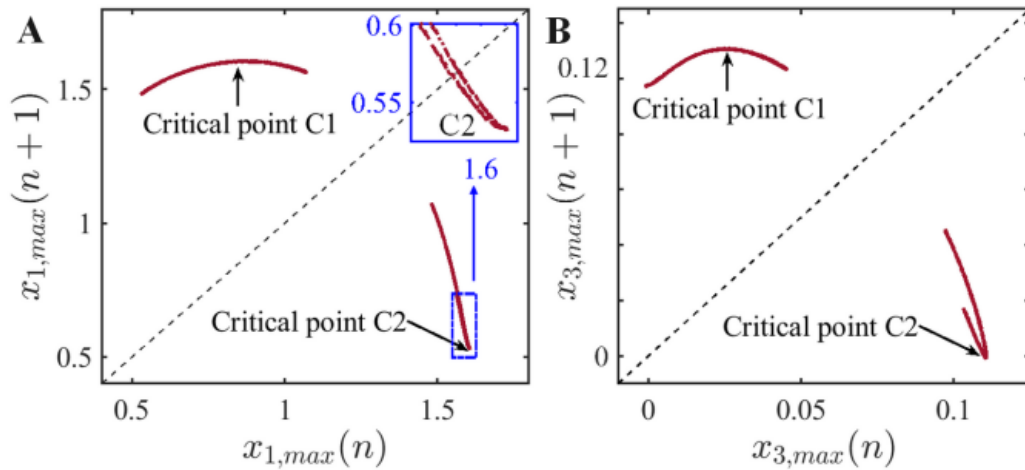


Figure 8. First-return maps of local maxima of (A) x_1 and (B) x_3 for $\alpha = 11.85$, $\mu = 3$, $\delta = 2.5$, $\eta = 2.9$, and initial conditions $(0.1, 0, 0.1)$.

structure serves as further evidence of the sophisticated nonlinear interactions within the mVdP system.^{33,34}

4. Potential application: Emulation of cardiac-like rhythms

The VdP equation has been extensively used to replicate natural rhythms, most notably cardiac electrical activity. As the mVdP system retains the fundamental mechanisms of the VdP equation, this section explores

its capacity to produce phenomenological ECG-like waveforms.

4.1. The cardiac rhythm

The heart is a myogenic muscle, controlled by coordinated electrical impulses that facilitate blood circulation.³⁵ The cardiac system is often assimilated to a network of self-excited oscillators operating in coordination to maintain a stable, homeostatic rhythm. The sinoatrial node initiates electrical impulses that propagate through

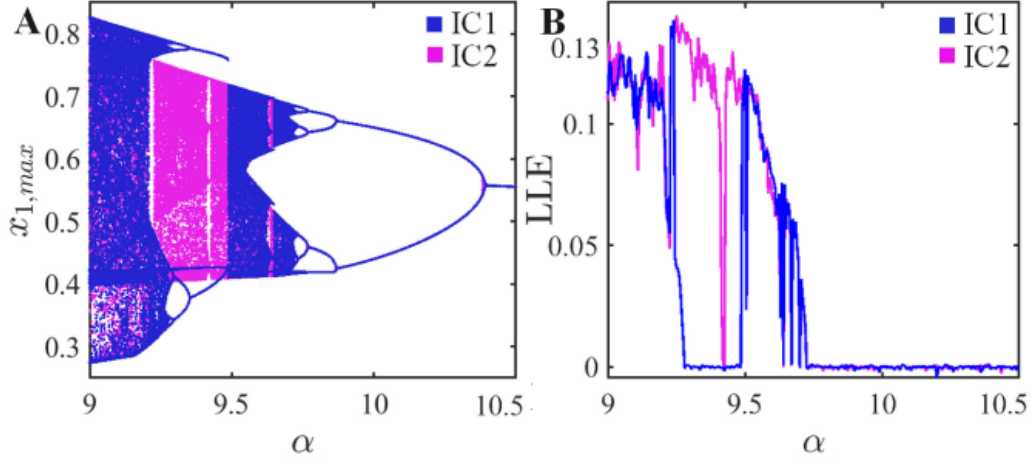


Figure 9. Bistability in the mVdP system: (A) Bifurcation diagram and (B) LLE for increasing values of α , with $\mu = 3$, $\delta = 2.4$, $\eta = 2.5$, and two distinct initial conditions: IC1 (0.4, 0.4, 0.8) in blue and IC2 (0.2, 0.2, 0.15) in magenta. Abbreviation: mVdP: modified Van der Pol.

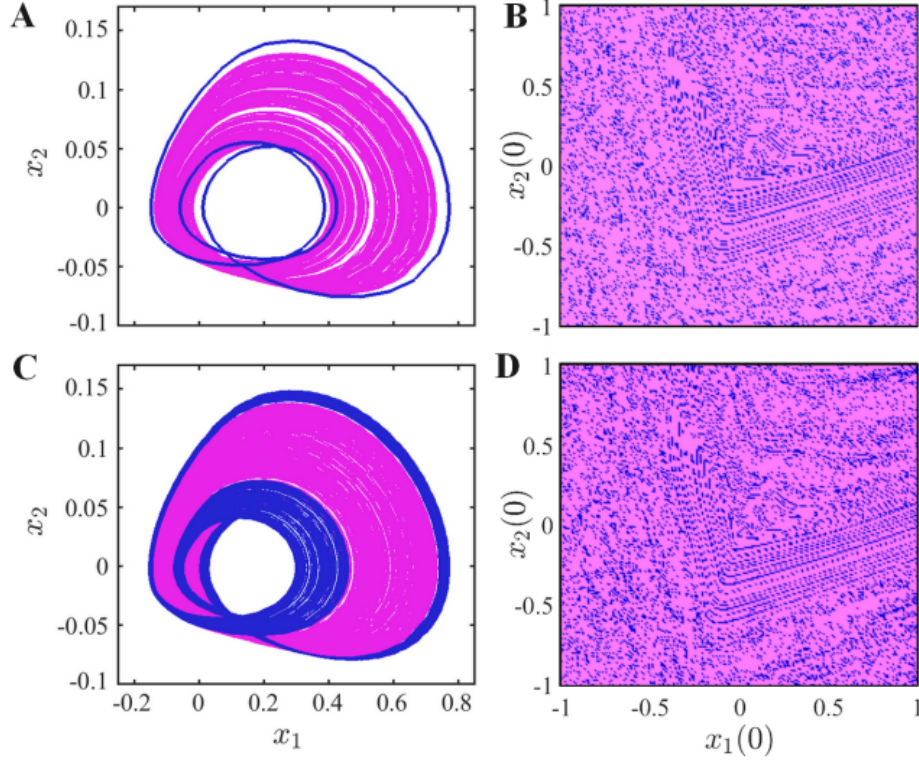


Figure 10. Coexisting attractors and corresponding basin of attraction in the mVdP system, for $\mu = 3$, $\delta = 2.4$, $\eta = 2.5$, and two distinct initial conditions: IC1 (0.4, 0.4, 0.8) in blue and IC2 (0.2, 0.2, 0.15) in magenta. (A) and (B) $\alpha = 9.4$; (C) and (D) $\alpha = 9.24$. Abbreviation: mVdP: modified Van der Pol.

the atria before reaching the atrioventricular node. The signal is then channeled through the His–Purkinje system, which consists of conducting fibers that trigger the simultaneous contraction of the ventricles. **Figure 11** illustrates this process and sketches the profile of the resulting electrical signal, referred to as an ECG, which comprises a series of waves (P, QRS, and T) indicative of specific electrophysiological phases.^{36,37}

The P wave corresponds to atrial depolarization, the QRS complex represents ventricular depolarization, with the R-peak typically dominating in amplitude, and the T wave reflects ventricular repolarization.

4.2. Generation and comparison of ECG-like waveforms

As a common feature of relaxation oscillators, the mVdP system can generate spikes that closely resemble ECG waveforms under specific parameter tuning. To maintain focus on this potential application, the mVdP system is solved with $\alpha = 6.3$, $\mu = 2.6$, $\delta = 2.378$, $\eta = 2.39$, and the numerical solution, specifically the state variable x_2 (renamed as s_1), is compared to clinical data (s_2) from the Massachusetts Institute of Technology–Beth Israel Hospital (MIT–BIH) Arrhythmia Database, freely accessible from the PhysioNet service.^{38,39} Following temporal rescaling and unit-range normalization, the numerical solution of

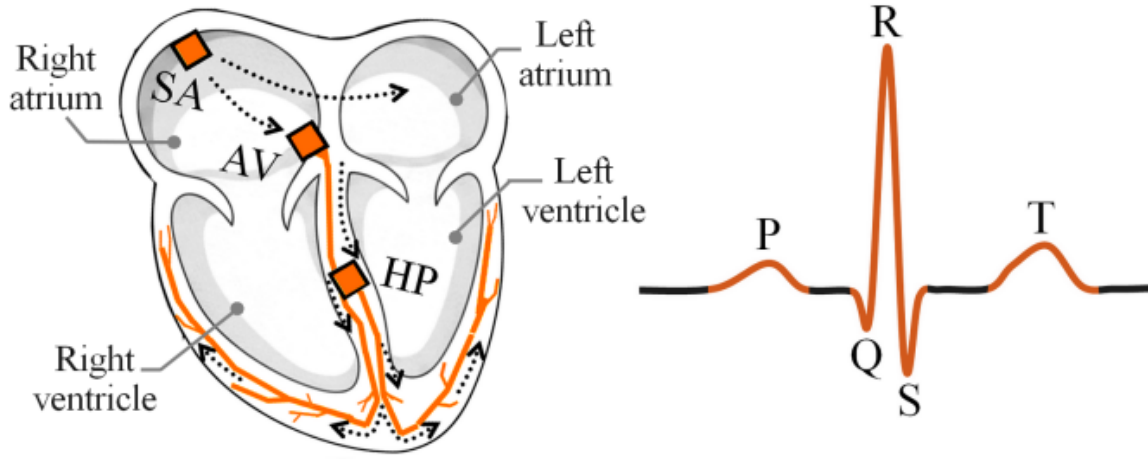


Figure 11. Functional diagram of the cardiac conduction system and its characteristic P–QRS–T waveform
Abbreviations: AV: Atrioventricular; HP: His–Purkinje; SA: Sinoatrial

the mVdP system s_1 is benchmarked against real ECG fragments associated with normal sinus rhythm.

As illustrated in **Figure 12A**, s_1 captures the macrostructure of the cardiac rhythm, including discernible P wave, PR segment, and QRS complex. However, a notable discrepancy arises due to the anticipated T-wave and the absence of the ST segment—a physiological marker of the ventricular depolarization–repolarization interval. Despite this structural simplification, both signals share comparable structure and spectral signature, as evidenced by their first-return maps (**Figure 12B**) and spectrograms (**Figure 12C** and **12D**). To further evaluate the mVdP system as a phenomenological emulator, a comparative analysis was conducted between its numerical solution (ODE) and five normal sinus rhythm fragments from the MIT–BIH Arrhythmia Database (**Figure 13**).

For the selected parameter settings, the RR interval (0.749 ms) and heart rate (80.15 bpm) fall within the range of clinical observations. The skewness (??) and spectral entropy (??) are consistent with the general complexity and asymmetry of the clinical data. However, the morphological differences include the P-peak (??), which is roughly twice the clinical average (~ 0.086), suggesting an overestimation of the atrial depolarization amplitude. Furthermore, although the kurtosis (??) and PR interval (0.135 ms) comply with the broad variance of the considered ECG fragments, the system fails to capture the precise ST-segment morphology. Specifically, the current mathematical model lacks the high-dimensional degrees of freedom necessary to fully replicate the stochastic heart rate variability observed in clinical data. These results show that while the mVdP system can serve as a functional template for generating ECG-like pacing, it remains a simple model that clearly requires additional topological complexity and parameter tuning to achieve better morphological accuracy.

5. Analog circuit design and microcontroller implementation

To bridge the gap between theoretical modeling and practical implementation, this section presents both analog and microcontroller-based realizations of the proposed system.

5.1. Analog circuit design and simulation program with integrated circuit emphasis simulation

The electronic circuit corresponding to the mVdP system, illustrated in **Figure 14**, is designed using off-the-shelf components, including resistors, capacitors, 1N4148 diodes, TL082 operational amplifiers, and a single AD633 analog multiplier. The circuit is powered by $\pm 15V$ symmetric supply rails and functions as an analog computer that physically implements the state equations in **Equation 3**.

The cross-product term is realized via the AD633 multiplier, while the absolute-value terms are implemented using diode-based configurations (D1 and D2). Basic arithmetic operations, such as addition, subtraction, and integration, are performed by the TL082 operational amplifiers. The state variables x_1 , x_2 , and x_3 are represented by the voltages across capacitors C_1 , C_2 , and C_3 , respectively. By applying Kirchhoff's laws to this circuit, where $R_2 = R_3 = R_4 = R_5 = R_6 = R_7 = R_8 = R_{10} = R_{12} = R_{14} = R_{15} = R$, $C_1 = C_2 = C_3 = C$ and $V_p = 1V$, the following circuit equations are obtained:

$$\begin{cases} \dot{x}_1 = \frac{1}{RC} \int \left(\frac{R}{R_1} x_2 \right) dt \\ \dot{x}_2 = \frac{1}{RC} \int \left(-\frac{R}{R_9} (|x_1| - 1) x_2 - x_1 + \frac{R}{R_{11}} |x_3| \right) dt \\ \dot{x}_3 = \frac{1}{RC} \int \left(-x_1 - \frac{R}{R_{13}} x_3 \right) dt \end{cases} \quad (11)$$

Differentiating these expressions yields a direct mapping to the proposed mathematical model. The control parameters are defined as $\alpha = R/R_1$, $\mu = R/R_9$, $\delta = R/R_{11}$, and $\eta = R/R_{13}$, where the product RC determines the time constant of the integrator stages. The designed circuit was simulated using a high-fidelity simulation program with an integrated circuit emphasis environment using realistic component models, providing a reliable bridge between numerical simulation and physical implementation. Simulations were conducted with R and C fixed at 100 k Ω and 10 nF, respectively.

The phase portraits and time series in **Figure 15** were obtained by varying R_1 , while setting $R_9 = 33.33$ k Ω , $R_{11} = 40$ k Ω , and $R_{13} = 35.71$ k Ω . Initial capacitor voltages were set to: $V_{C1}(0) = 0.1$ V, $V_{C2}(0) = 0$, and $V_{C3}(0) = 0.1$ V.

Multistability was verified in the designed analog circuit by setting $R_9 = 38.46$ k Ω , $R_{11} = 40$ k Ω , and $R_{13} = 41.66$ k Ω and by considering two distinct sets of initial conditions. For $R_1 = 10.63$ k Ω , the circuit exhibits coexisting attractors;

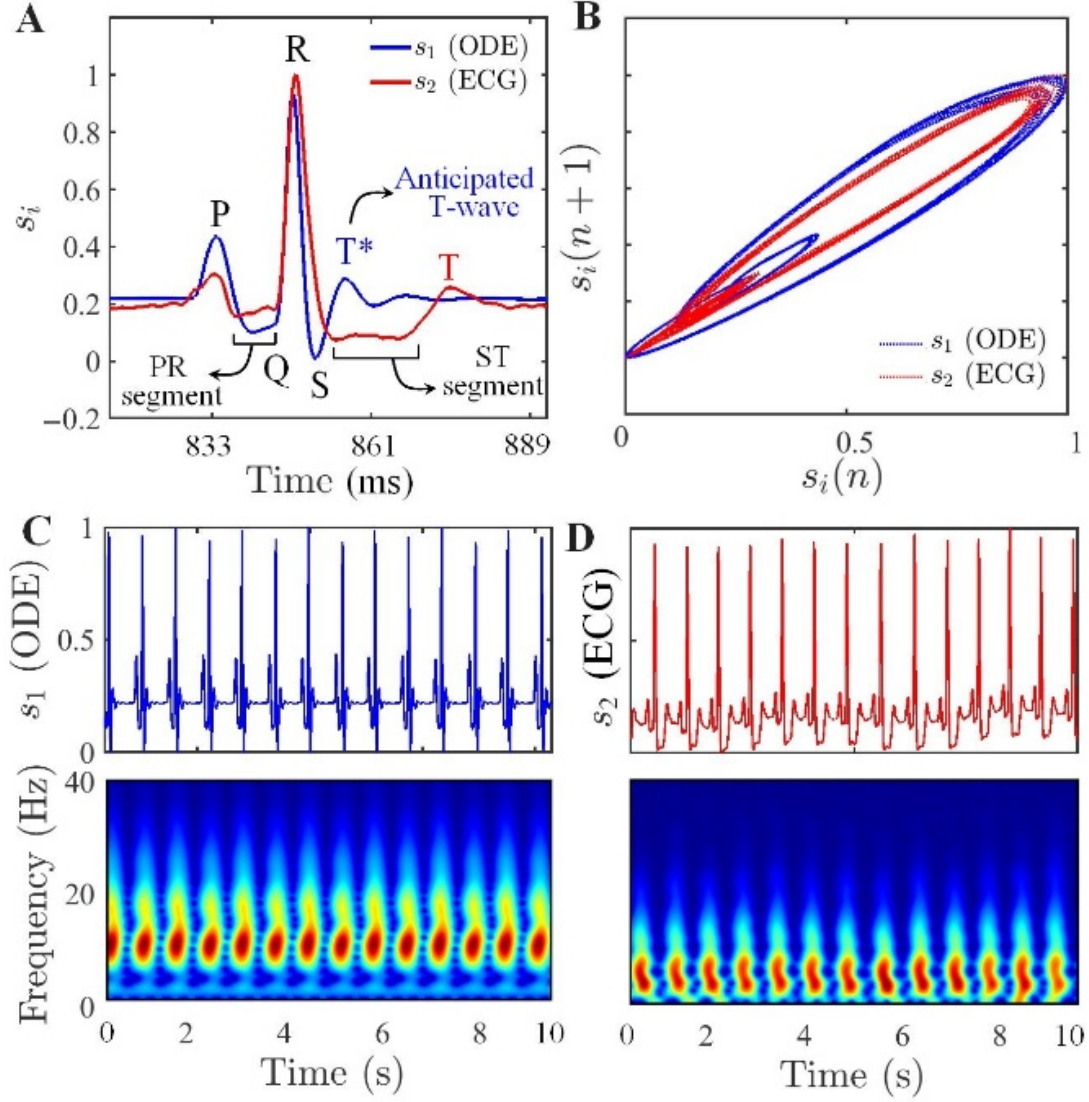


Figure 12. Qualitative comparison between mVdP solution computed for $\alpha = 6.3$, $\mu = 2.6$, $\delta = 2.378$, $\eta = 2.39$, and clinical data from Massachusetts Institute of Technology–Beth Israel Hospital Arrhythmia database (105 m fragment). (A) Time-domain morphological comparison, (B) first-return maps, (C) mVdP system time series (x_2) and spectrogram, (D) ECG fragment and corresponding spectrogram. Abbreviations: ECG: Electrocardiogram; mVdP: modified Van der Pol; ODE: Ordinary differential equations

specifically, a chaotic attractor is obtained for $V_{C1}(0) = 0.2V$, $V_{C2}(0) = 0.2$, and $V_{C3}(0) = 0.4V$ (**Figure 16A and 16B**), and a periodic regime for $V_{C1}(0) = 0.4V$, $V_{C2}(0) = 0.4V$, and $V_{C3}(0) = 1V$ (**Figure 16C and 16D**).

Furthermore, setting the circuit parameters to $R_1 = 15.8\text{ k}\Omega$, $R_9 = 33.33\text{ k}\Omega$, $R_{11} = 40\text{ k}\Omega$, and $R_{13} = 35.71\text{ k}\Omega$ allows for the successful replication of the ECG-like signals depicted in **Figure 16E**.

Beyond merely validating the proposed mathematical model, this analog circuit provides a versatile platform for real-time exploration of complex dynamic behaviors in the mVdP system and the development of chaos-based engineering applications.

5.2. Microcontroller-based implementation

In contrast to analog emulators, which are prone to performance degradation due to aging and environmental factors, digital implementations offer robustness, portability, and reprogrammability. This digital framework enables greater accuracy and flexibility in defining control

parameters and executing advanced mathematical functions. Such capabilities streamline the design process and enhance the feasibility of implementing topologically complex chaotic maps.

In this study, the mVdP system is implemented on an STM32F407VG microcontroller unit (MCU) featuring dual 12-bit digital-to-analog converters (DACs) and running at a clock frequency of 180 MHz. **Figure 17** illustrates the practical setup used for programming and visualization. This setup integrates a personal computer, running the STM32CubeIDE environment for programming, and provides a stable power supply to the STM32 board via USB. Although the non-smooth nature of the mVdP system introduces numerical stiffness, a fixed-step fourth-order Runge–Kutta scheme was preferred to ensure predictable execution timing and minimize computational overhead on the MCU ($dt = 0.005$).

The computed digital values of the state variables are converted into analog signals via the on-chip DACs and output to two distinct channels (CH1 and CH2) of a digitizing oscilloscope through pins PA4 and PA5.

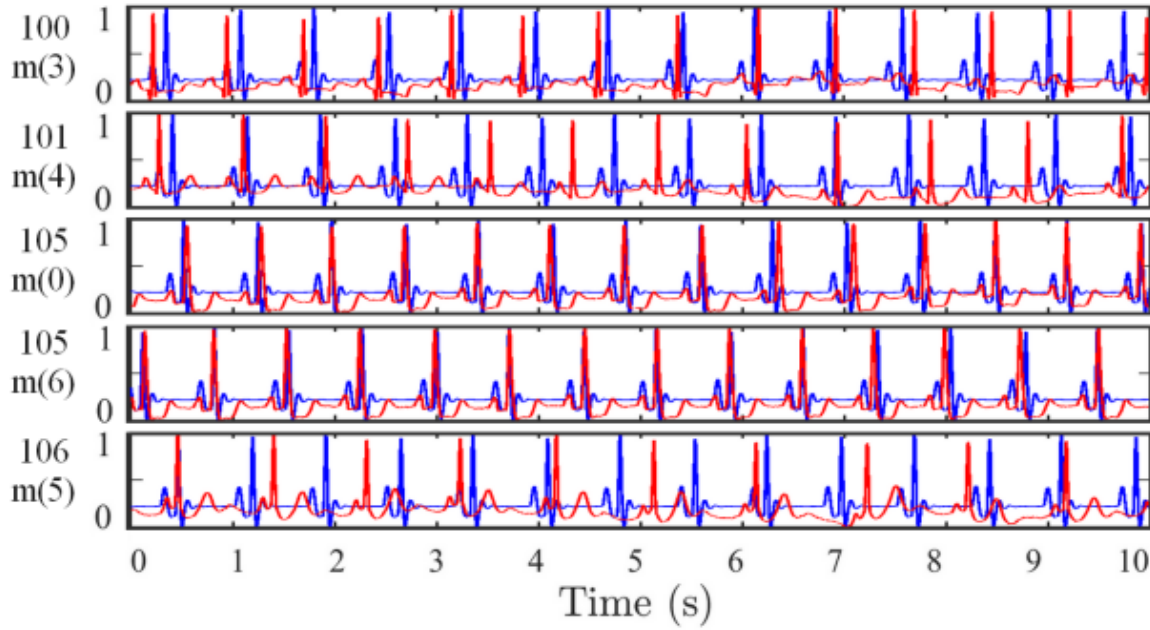


Figure 13. Comparative gallery of the mVdP system output (in blue) and five NSR signals from the Massachusetts Institute of Technology–Beth Israel Hospital Arrhythmia database (red).
Abbreviations: mVdP: modified Van der Pol; NSR: Normal sinus rhythm

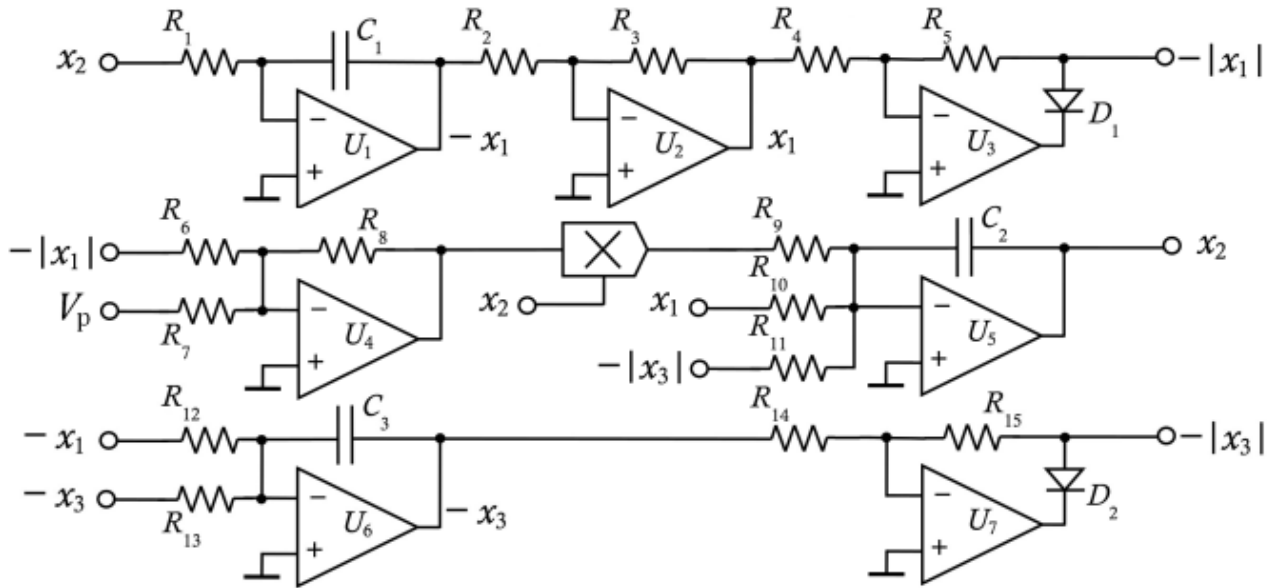


Figure 14. Schematic diagram of the analog circuit designed to emulate the mVdP system Abbreviations: mVdP: modified Van der Pol

The amplitudes of the generated signals are dynamically adjustable over 0–3.3 V to comply with the STM32 I/O voltage requirements.

As shown in **Figure 18**, the phase portraits and time series from the microcontroller-based implementation were obtained by assigning state variables x_1 and x_2 to oscilloscope channels CH1 and CH2, respectively. Consistent with the analog implementation, both the coexistence of attractors and the generation of ECG-like waveforms were verified on the digital platform, with results illustrated in **Figure 19**.

The phase portraits and time series obtained from the microcontroller-based implementation, the analog

circuit, and the numerical simulations demonstrate excellent agreement. This underscores the potential of the proposed model for both theoretical frameworks and practical deployment in engineering and biomedical applications.

6. Conclusion

In this study, we introduced a novel non-smooth 3D autonomous chaotic system derived from the classical VdP equation. The dynamics of the proposed system were explored through numerical analyses, revealing a diverse and rich landscape of nonlinear phenomena, including

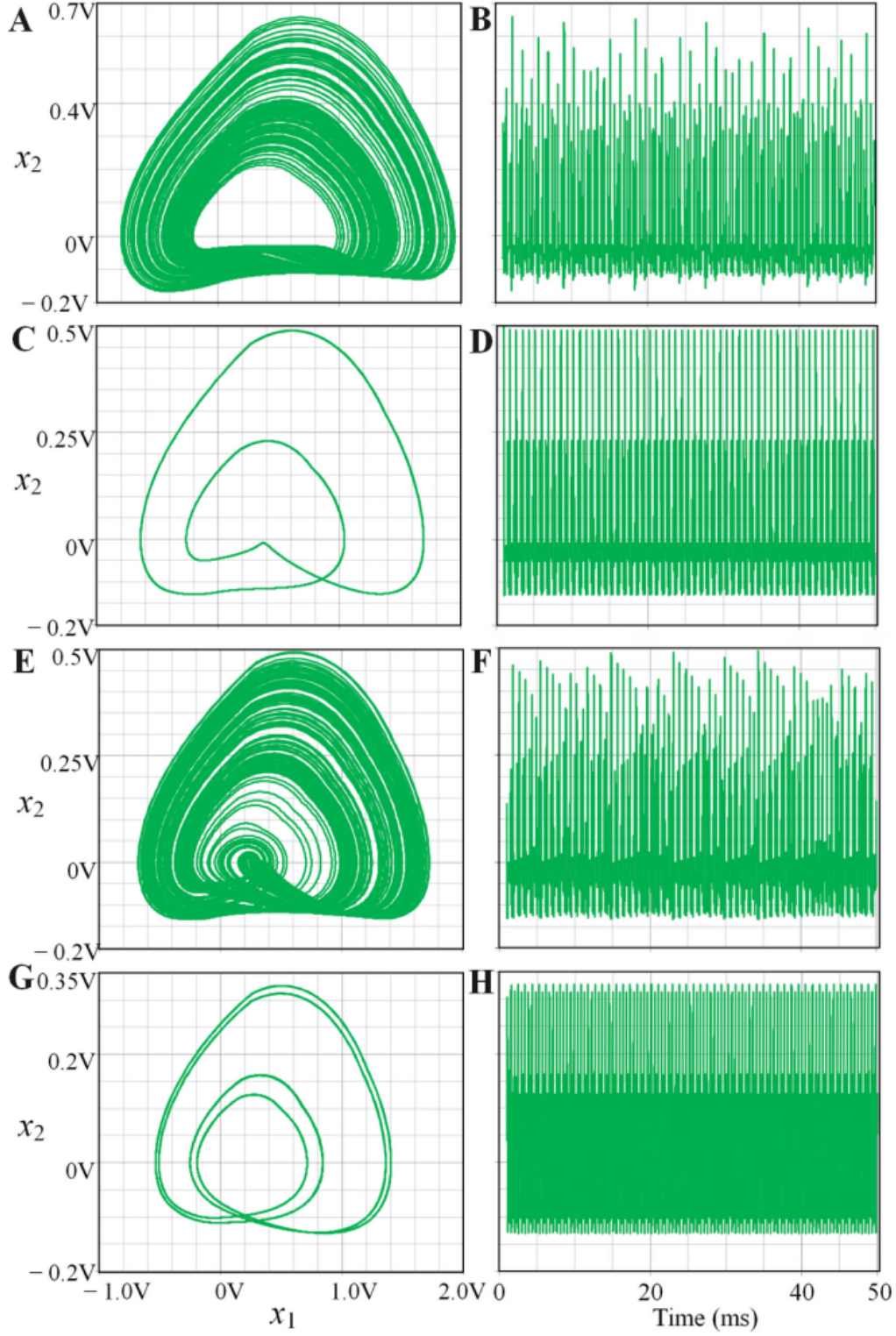


Figure 15. Palette of dynamical behaviors of the mVdP system replicated through SPICE simulations for discrete values of resistor R_1 , with $R_9 = 33.33 \text{ k}\Omega$, $R_{11} = 40 \text{ k}\Omega$ and $R_{13} = 35.71 \text{ k}\Omega$. Phase portraits in the x_1 - x_2 plane and x_2 time series for: $R_1 = 11.76 \text{ k}\Omega$ (A, B), $R_1 = 10.52 \text{ k}\Omega$ (C, D), $R_1 = 10 \text{ k}\Omega$ (E, F), and $R_1 = 8 \text{ k}\Omega$ (G, H)

Abbreviations: mVdP: modified Van der Pol; SPICE: Simulation program with integrated circuit emphasis

antimonotonicity and multistability. To bridge the gap between theoretical modeling and practical feasibility, analog circuit and microcontroller-based implementations were presented, both of which accurately replicate the system dynamics. The high degree of congruence

between numerical simulations, the analog circuit, and the STM32-based digital implementation confirms the robustness of the proposed model across different platforms. The model's versatility enables technological applications such as chaos-based secure communication, chaos-driven

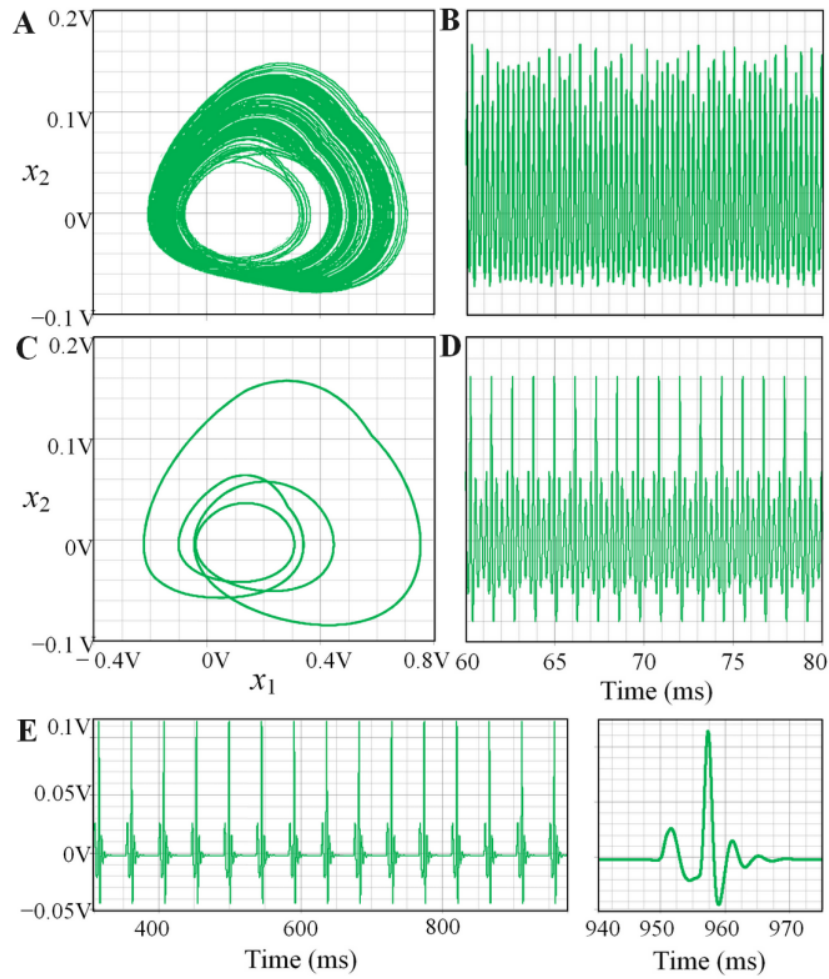


Figure 16. Coexisting attractors and corresponding time series obtained for $R_1 = 10.63 \text{ k}\Omega$, $R_9 = 38.46 \text{ k}\Omega$, $R_{11} = 40 \text{ k}\Omega$, $R_{13} = 41.66 \text{ k}\Omega$ with (A, B) $(V_{C1}(0), V_{C2}(0), V_{C3}(0)) = (0.2 \text{ V}, 0.2 \text{ V}, 0.4 \text{ V})$ and (C, D) $(V_{C1}(0), V_{C2}(0), V_{C3}(0)) = (0.4 \text{ V}, 0.4 \text{ V}, 1 \text{ V})$. (E, F) ECG-like waveform generation replicated through SPICE simulations for $R_1 = 15.8 \text{ k}\Omega$, $R_9 = 33.33 \text{ k}\Omega$, $R_{11} = 40 \text{ k}\Omega$, and $R_{13} = 35.71 \text{ k}\Omega$. Abbreviations: ECG: Electrocardiogram; SPICE: Simulation program with integrated circuit emphasis

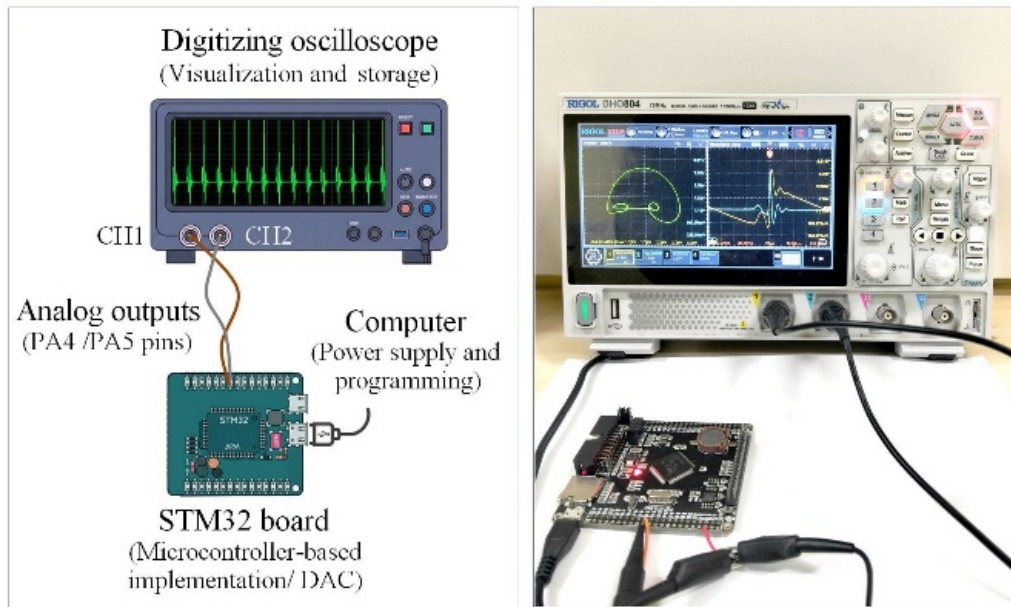


Figure 17. Minimal setup for the microcontroller-based implementation of the mVdP system. Abbreviation: mVdP: modified Van der Pol

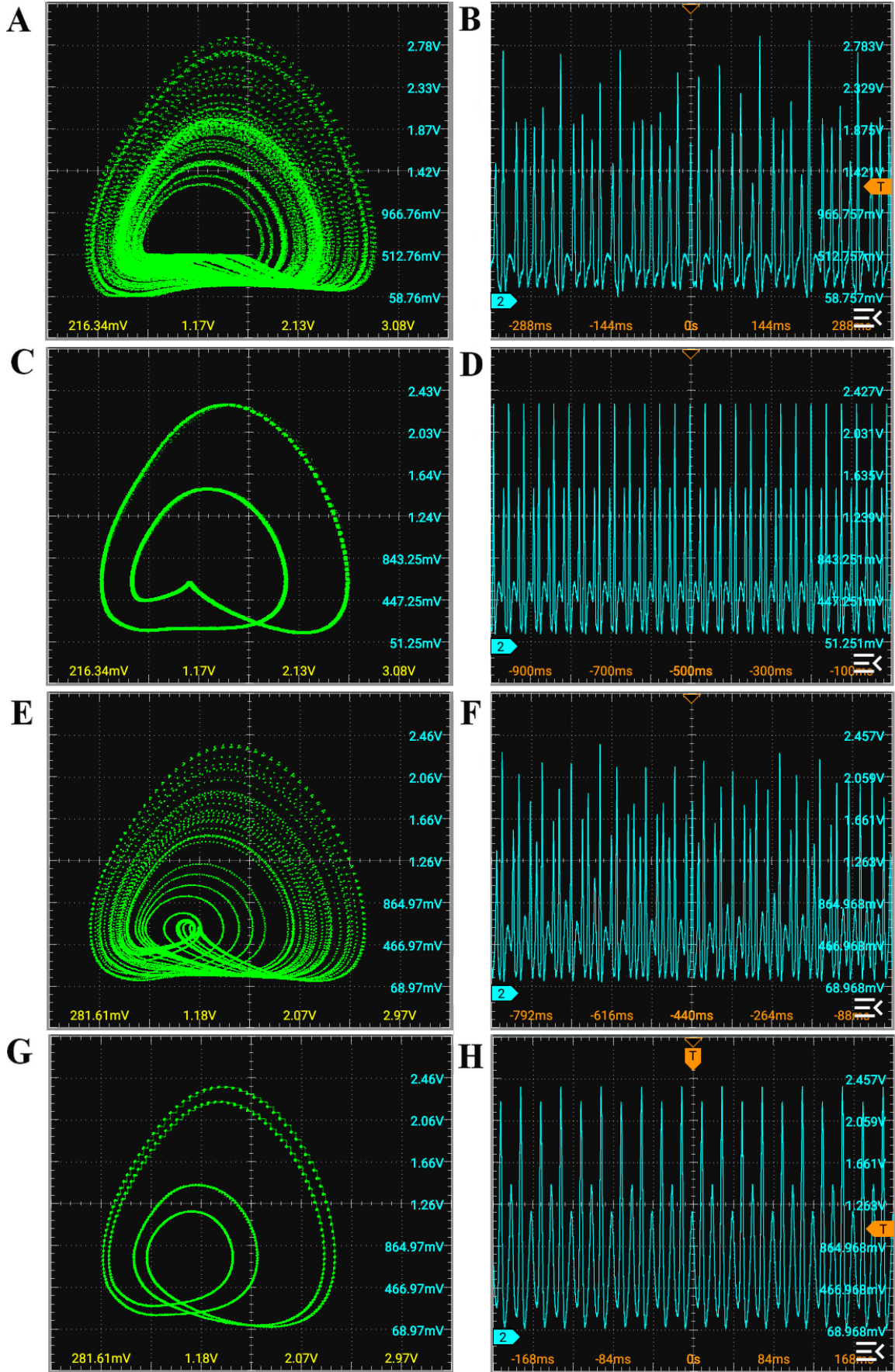


Figure 18. Palette of dynamical behaviors replicated with microcontroller-based implementation of the mVdP system for discrete values of α , with $\mu = 3$, $\delta = 2.5$, $\eta = 2.8$ and initial condition $(0.1, 0, 0.1)$: Phase portraits in the $x_1 - x_2$ plane and x_2 time series for $\alpha = 8.5$ (A, B), $\alpha = 9$ (C, D), $\alpha = 10$ (E, F), and $\alpha = 12.5$ (G, H)

Abbreviation: mVdP: modified Van der Pol

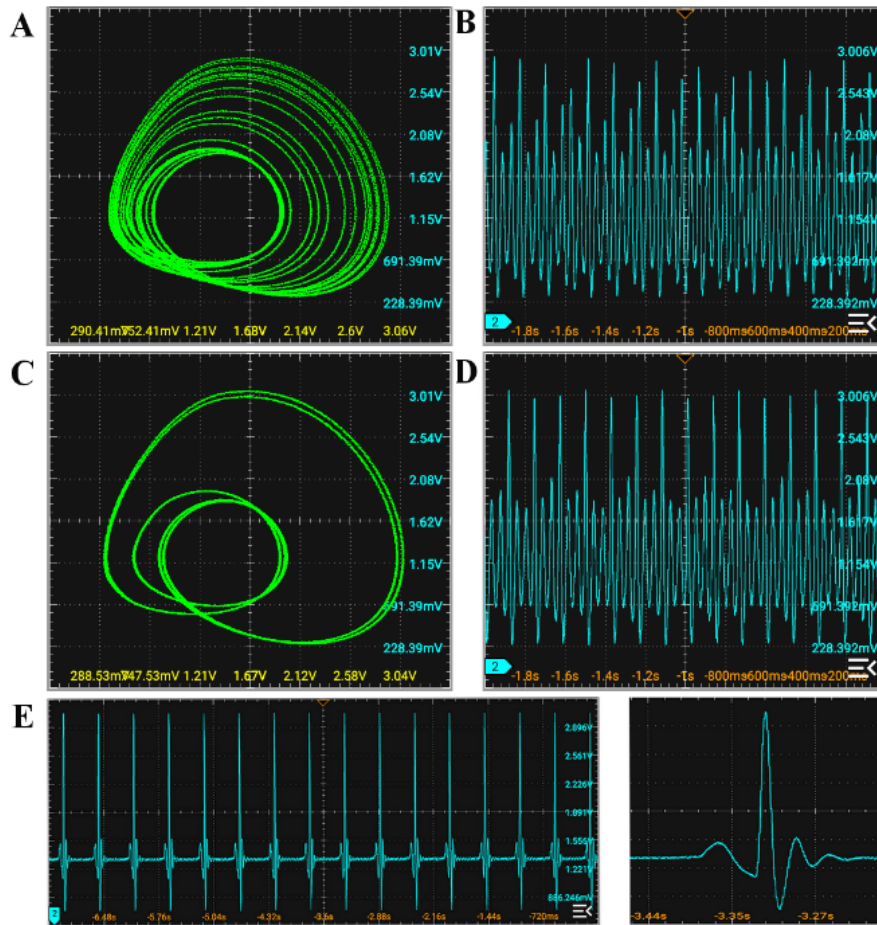


Figure 19. Results from microcontroller-based implementation: Coexisting attractors and corresponding time series for $\alpha = 9.4$, $\mu = 3$, $\delta = 2.4$, $\eta = 2.5$ and initial conditions IC1 (0.4, 0.4, 0.8) (A, B) and IC2 (0.2, 0.2, 0.15) (C, D); (E) ECG-like signal generated for $\alpha = 6.3$, $\mu = 2.6$, $\delta = 2.378$ and $\eta = 2.39$

Abbreviation: ECG: Electrocardiogram

cryptography, ECG-like waveform emulation for medical instrumentation testing, and synthetic data generation to augment medical datasets. However, it is important to acknowledge that, in its current form, the model does not yet capture the full physiological complexity of ECG signals. Nevertheless, its alignment across mathematical theory, analog circuitry, and digital hardware underscores its potential for synthetic signal generation, with future works targeting higher dimensionality and topological complexity to model complete cardiac cycles and support hardware-in-the-loop testing for pacemaker design.

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Conflict of interest

Paul Didier Kamdem Kuate is an Editorial Board Member of this journal, but was not in any way involved in the editorial and peer-review process conducted for this paper, directly or indirectly. Separately, other authors declared that they have no known competing financial interests or personal

relationships that could have influenced the work reported in this paper.

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Availability of data

The data used in this study were retrieved from the Massachusetts Institute of Technology–Beth Israel Hospital Arrhythmia database accessible from the PhysioNet service (<https://archive.physionet.org/physiobank/database/mitdb/>).

AI Tools Statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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