

# New exact traveling wave solutions of the sharma–tasso–olver equation

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## ABSTRACT

This paper introduces a new analytical method for solving the Sharma–Tasso–Olver (STO) equation using the travelling wave technique coupled with the newly established homogeneous balance equation (HBM). The combination of both techniques makes it possible to systematically reduce the nonlinear STO equation to an exactly solvable form. In this paper, three separate cases resulting from the balance condition are examined carefully, and for each case, several sub-cases are investigated to cover various parameter arrangements and nonlinear properties. The resultant solutions are then generalized giving a complete family of traveling wave solutions that captures the intricate dynamics of the equation. Graphical representations are also presented to visualize the impact of different parameters on the solution behavior. The research not only generalizes the scope of classical traveling wave analysis but also illustrates the flexibility and capability of the HBM–Extended Riccati approach in solving sophisticated nonlinear differential equations.

**Keywords:** Sharma–Tasso–Olver equation; Travelling wave solutions; Homogeneous balance method; Extended Riccati equation; Nonlinear differential equations



## 1. Introduction

### 1.1. Classical nonlinear evolution equations

Numerous fields of applied mathematics, physics, and engineering employ nonlinear evolution equations (NLEEs). These equations represent complex dynamical processes in which nonlinear and dispersive effects combine to generate various wave structures, such as shock waves, solitons, rogue waves, breathers, and pattern formations.<sup>1–3</sup> Classical NLEEs such

as the Korteweg-de Vries (KdV), Boussinesq, Kadomtsev–Petviashvili (KP), Benjamin–Bona–Mahony (BBM), Benjamin–Bona–Mahony–Burgers (BBMB), and Sawada–Kotera (SK) equations play a significant role in modelling surface wave propagation, as they capture nonlinear wave dynamics across diverse physical settings. For example, although the KdV equation is commonly used to describe shallow-water wave behaviour, it becomes less accurate for waves with small wave numbers or long wavelengths.

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## 1.2. The role of higher-order KdV-type models and the STO equation and Literature

A particularly important higher-order extension of KdV-type models is the Sharma–Tasso–Olver (STO) equation, obtained by incorporating cubic nonlinear convection together with quadratic nonlinear dispersive terms. First introduced in the context of integrable nonlinear dispersive systems,<sup>4–6</sup> the STO equation is given by

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \quad (1)$$

where  $u = u(x, t)$  denotes the dependent field variable and  $\alpha$  is a nonzero parameter. The equation consists of a linear dispersive term  $\alpha u_{xxx}$  and two nonlinear contributions, namely  $\alpha(u^2)_{xx}$  and  $\alpha(u^3)_x$ , making it a rich model for studying higher-order nonlinear wave behaviour.

The study of exact solutions of NLEEs is of considerable importance, since the integrability of such equations is often closely linked to the existence of closed-form solutions. These exact solutions provide deeper insight into the physical processes being modelled, serve as benchmarks for testing numerical algorithms, and play a crucial role in stability analysis. Moreover, the presence of conservation laws for partial differential equations (PDEs) frequently indicates integrability and significantly influences analytical treatment.

Due to the inherent nonlinearity of NLEEs, researchers have developed a wide range of analytical techniques to obtain explicit travelling-wave solutions. Wang *et al.*<sup>7</sup> identified soliton fission and fusion phenomena using Hirota's direct method and Bäcklund transformations. Lian and Lou<sup>8</sup> applied repeated symmetry reductions to construct infinitely many symmetries and exact solutions. Johnpillai and Khalique<sup>9</sup> investigated the STO equation using Lie symmetry analysis. Chen<sup>10</sup> developed two fundamental Darboux transformations based on the Levi spectral problem, yielding multi-kink, soliton-fission, and soliton-fusion solutions.

Subsequently, several analytical techniques were introduced to determine travelling-wave solutions of the STO equation, including the improved tanh method,<sup>11,12</sup> the Exp-function method,<sup>13</sup> the extended hyperbolic function method,<sup>14,15</sup> the auxiliary equation method,<sup>16</sup> the modified simple equation method (MSEM),<sup>17,18</sup> and the improved  $G'/G$ -expansion method.<sup>19</sup> These methods highlight the diversity of approaches developed for analysing nonlinear dispersive models such as STO.

## 1.3. Recent advances on STO (2020–2024)

Recent research has significantly expanded the analytical and numerical study of the Sharma–Tasso–Olver (STO) equation, contributing new perspectives to its solution structure and qualitative behaviour. Several works have focused on deriving exact travelling-wave solutions for generalized and variable-coefficient forms of the STO equation,<sup>20–22</sup> highlighting the adaptability of modern solution methodologies to parameter-dependent extensions of the model. Fractional and conformable versions of the STO equation have also been investigated, with exact solutions obtained through Kudryashov-type and related constructive techniques.<sup>23</sup> In addition to analytical results, recent studies have introduced advanced numerical and spectral schemes, including lattice-Boltzmann formulations and radial-basis-function pseudo-spectral approximations, to efficiently approximate

STO dynamics.<sup>24,25</sup> Furthermore, qualitative and geometric analyses of STO-type models, including STO–Burgers variants, have provided insight into soliton profiles, kink structures, and the associated phase-plane behaviour.<sup>26,27</sup> Collectively, these developments show that the STO equation remains a fertile model for both analytical investigation and numerical experimentation, motivating further refinement of solution techniques such as those explored in the present work.

## 1.4. Travelling-wave approaches and trial equation methodology

Most travelling-wave solution methods assume that the PDE admits a finite-series representation whose terms depend on a function satisfying a simpler auxiliary ODE (the *trial equation*, e.g., Riccati). The balancing principle is used to determine the series length by matching the highest-order linear and nonlinear derivatives of the PDE. Once the trial equation is fixed, the form of the solution follows directly.

A key result, expressed in Theorem 1 of,<sup>28</sup> is that the degree of the series must be consistent with the degree growth generated by the auxiliary Riccati-type equation. This explains why certain combinations of series expansions and trial equations succeed while others do not.

## 1.5. Motivation and contribution of the present study

In light of the above, this work introduces a travelling-wave-based analytical framework combined with a novel homogeneous balance method (HBM). The method aims to:

- (1) streamline and generalize the balancing process,
- (2) reduce inconsistent trial-equation choices,
- (3) derive rich classes of analytical solutions to the STO equation, complementing the recent advances in fractional, numerical, and generalized STO models.

In most existing literature on travelling-wave solutions of nonlinear partial differential equations (NLPDEs), it is typically assumed that the solution can be expressed as a finite series whose terms depend on a function that satisfies a simpler auxiliary differential equation—often referred to as a *trial equation* (e.g., the Riccati equation). The number of terms in this series is determined using the balancing principle, which equates the highest-order linear and nonlinear terms in the PDE. Once the trial equation is chosen, the entire structure of the travelling-wave solution is determined by that choice.

A key observation, formalised in Theorem 1 of,<sup>28</sup> is the essential consistency requirement between the degree of the assumed solution series and the degree growth induced by the Riccati equation. This matching condition significantly reduces unproductive trial choices and explains why certain combinations of series expansions and Riccati-type auxiliary equations yield successful and coherent solutions, while others do not.

## 2. Preliminaries

We begin by recalling several fundamental concepts that will be used in the construction of exact solutions for nonlinear partial differential equations (NLPDEs). Throughout this section, the independent variables are denoted by  $(x_1, x_2, \dots, x_n, t)$ , where  $t$  represents time.

**Definition 1** (Nonlinear Partial Differential Equation (NLPDE)). A nonlinear partial differential equation is an equation involving an unknown function

$$u = u(x_1, \dots, x_n, t)$$

and its partial derivatives, where the dependence on  $u$  or its derivatives is nonlinear. A general representation is

$$\mathcal{F}(x_1, \dots, x_n, t, u, \partial u, \partial^2 u, \dots) = 0, \quad (2)$$

where  $\mathcal{F}$  denotes a nonlinear operator acting on  $u$  and its derivatives.

**Definition 2** (Travelling Wave Transformation). Let  $u(x_1, \dots, x_n, t)$  satisfy a nonlinear PDE. A travelling wave transformation introduces a new variable

$$\xi = k_1 x_1 + k_2 x_2 + \dots + k_n x_n - \omega t,$$

where  $k_i$  ( $i = 1, \dots, n$ ) are wave numbers and  $\omega$  is the frequency. Under this transformation, the solution is assumed to be of the form

$$u(x_1, \dots, x_n, t) = U(\xi),$$

which converts the original PDE into an ordinary differential equation (ODE) in the single variable  $\xi$ .

**Definition 3** (Solution Ansatz). A travelling-wave solution is approximated by a finite series expansion,

$$U(\xi) = \sum_{i=0}^N q_i \phi^i(\xi), \quad (3)$$

where  $\{q_i\}_{i=0}^N$  are constants to be determined and  $\phi(\xi)$  is an auxiliary function governed by a trial equation.

**Definition 4** (Trial Equation). The auxiliary function  $\phi(\xi)$  is assumed to satisfy a first-order ODE of the general form

$$\phi'(\xi) = a \phi^m(\xi) + b \phi^n(\xi) + c, \quad (4)$$

where  $a, b, c$  are constants and  $m, n$  are integers selected in accordance with the homogeneous balancing condition.

**Definition 5** (Degree of  $\phi(\xi)$ ). Let  $\phi(\xi)$  be a function satisfying a trial (auxiliary) equation. The degree of a term involving  $\phi$ , denoted by  $\mathcal{D}(\phi)$ , is defined as the highest power of  $\phi(\xi)$  appearing in that term. For example,

$$\mathcal{D}(\phi^k(\xi)) = k.$$

## 2.1. Theorem 1: Balancing equation<sup>28</sup>

General nonlinear. Consider a nonlinear partial differential equation (NLPDE) of the form

$$\mathcal{F}(\mathbf{x}, t, \mathbf{u}, D^\alpha \mathbf{u}) = 0,$$

where  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{u} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^p$  is the unknown function,  $D^\alpha \mathbf{u}$  denotes partial derivatives of  $\mathbf{u}$  with respect to  $\mathbf{x}$  and  $t$  up to a certain order, and  $\mathcal{F}$  may be nonlinear in all of its arguments.

Let  $\mathbf{u}^{(r)}(\xi)$  denote the highest-order linear derivative term in the reduced NLPDE, and let

$$(\mathbf{u}^{(p)}(\xi))^s \mathbf{u}(\xi)^q$$

represent the highest-order nonlinear term. Then, the degree of the trial function  $\phi(\xi)$  in the solution ansatz is determined by the balancing relation

$$\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{E}) \left( \frac{s+q-1}{r-sp} \right) + 1,$$

where

- $\mathcal{D}(\mathcal{R})$  is the degree of  $\phi(\xi)$  appearing in the reducing (balance) equation,
- $\mathcal{D}(\mathcal{E})$  is the degree of  $\phi(\xi)$  in the expansion function.

General form of the reducing equation. In the homogeneous balance method, we assume a trial solution of the form

$$u(\xi) = \sum_{k=0}^N a_k \phi(\xi)^k, \quad \phi'(\xi) = P(\phi(\xi)),$$

where  $N = \mathcal{D}(\mathcal{R})$  and  $P(\phi)$  is a polynomial in  $\phi$ . The reducing equation is therefore written as

$$P(\phi) = \alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2 + \dots + \alpha_m \phi^m,$$

where the coefficients  $\alpha_i$  are obtained by substituting the ansatz into the given NLPDE and matching coefficients of equal powers of  $\phi$ .

**Remark.** The balancing equation ensures compatibility between the highest-order linear derivative term and the dominant nonlinear term of the NLPDE. This determines the maximal admissible degree  $N$  of the polynomial expansion in  $\phi(\xi)$ , thereby reducing the original nonlinear PDE to an ordinary differential equation in  $\phi(\xi)$ . The method prevents either the linear part or the nonlinear part from dominating, and guarantees that the expansion is closed under substitution.

## 2.2. Theorem 2: Polynomial structure of solutions<sup>28</sup>

**Theorem 2.** Assume that the solution of a nonlinear partial differential equation is expanded up to the degree  $\mathcal{D}(\mathcal{R})$  obtained from the balancing equation. Then, the admissible solutions correspond to the  $(\mathcal{D}(\mathcal{R}) - 1)^{\text{th}}$  roots of an associated algebraic Riccati-type equation of the form

$$\phi'(\xi) = \alpha_0 + \alpha_1 \phi(\xi) + \alpha_2 \phi(\xi)^2,$$

subject to the following constraints:

- The constant term satisfies  $\alpha_0 = 0$  (ensuring homogeneity of the reduced equation),
- The remaining coefficients  $\alpha_1, \alpha_2, \dots$  are selected to maintain consistency with the original NLPDE after substitution.

In particular, when  $\mathcal{D}(\mathcal{R}) = 2$ , the solution reduces to the roots of the standard Riccati equation

$$\phi'(\xi) = \alpha_1 \phi(\xi) + \alpha_2 \phi(\xi)^2,$$

which governs the reduced form of the solution expansion.

**Remark.** This theorem reveals that the structure of admissible travelling-wave solutions is governed entirely by the degree prescribed by the balance equation. Therefore, once  $\mathcal{D}(\mathcal{R})$  is known, the underlying Riccati equation—and hence the solution class—is uniquely determined.

## 3. Methodology: Homogeneous balance method

Our approach is based on the well-established Homogeneous Balance Method (HBM), first introduced in the early 1980s for constructing exact solutions of nonlinear partial differential equations (PDEs), especially those arising in mathematical physics. When combined with a trial (auxiliary) equation technique, HBM becomes a highly effective framework for obtaining closed-form solutions of nonlinear PDEs.

The core idea of HBM is the *balancing principle* in which the highest-order linear derivative term in a PDE is balanced with the highest-order nonlinear term. This balance determines the degree and structure of the solution ansatz, which is usually expressed as a

polynomial or rational expansion in terms of auxiliary functions (e.g., tanh, exponential, or elliptic functions). Over the years, HBM has proven to be one of the most reliable and straightforward tools for generating soliton, periodic, and rational solutions. Its flexibility has also led to the development of hybrid approaches such as **HBM combined with auxiliary or trial equation methods**, thereby improving both accuracy and generality in solving complex nonlinear systems.

Since our methodology is based directly on this balancing mechanism, we begin with a concise overview of the key steps of the Homogeneous Balance Method.

- (1) **Travelling Wave Transformation:** Reduce the given nonlinear PDE to an ordinary differential equation (ODE) by introducing the transformation

$$u(x, t) = U(\xi), \quad \xi = x - \omega t,$$

where  $\omega$  is the wave velocity to be determined.

- (2) **Homogeneous Balance:** Establish the balance between the highest-order derivative term and the highest-order nonlinear term in the transformed ODE. This yields the appropriate degree  $N$  of the solution expansion (see Theorem 1). The trial solution is assumed in the form

$$U(\xi) = \sum_{i=0}^N q_i \phi^i(\xi),$$

where  $\phi(\xi)$  is an auxiliary function, typically satisfying a Riccati-type equation, and  $N = \mathcal{D}(\mathcal{R})$  arises from the balancing relation.

- (3) **Substitution into the ODE gives:** Substitute the assumed expansion for  $U(\xi)$  into the reduced ODE obtained from the travelling wave transformation.
- (4) **Coefficient Matching:** Collect coefficients of identical powers of  $\phi(\xi)$  and set each to zero. This produces a system of algebraic equations involving the unknowns  $q_i$  and the wave parameter  $\omega$ .
- (5) **Solution Construction:** Solve the resulting algebraic system to compute the coefficients  $q_i$  and the wave parameters  $\omega$ , and thereby obtain the explicit travelling-wave solution  $U(\xi)$ . The auxiliary function  $\phi(\xi)$  is governed by a Riccati-type equation as established in Theorem 2, and its solutions determine the final form of the travelling wave profiles.

In summary, Theorem 1 identifies the maximum degree of the polynomial trial solution through the balancing mechanism, while Theorem 2 ensures that the auxiliary function  $\phi(\xi)$  satisfies a Riccati equation that dictates the polynomial structure of the reduced solution.

## 4. Solutions of the reducing equation

### 4.1. Solutions of the reducing equation for $\mathcal{D}(\mathcal{R})=2$

Consider the reducing equation

$$\phi'(\xi) = a\phi^2(\xi) + b\phi(\xi) + c, \quad a \neq 0, \quad (5)$$

whose solutions depend on the sign of the discriminant  $\Delta = b^2 - 4ac$ .

- **Case 1:**  $\Delta < 0$ . Then the solution is

$$\phi(\xi) = -\frac{b}{2a} + k \frac{\sqrt{4ac - b^2}}{2a} \tan\left(\frac{k}{2} \sqrt{4ac - b^2} \xi + A\right), \quad (6)$$

where  $A$  is an integration constant and  $k = \pm 1$ .

- **Case 2:**  $\Delta > 0$ . Then

$$\phi(\xi) = k \frac{\sqrt{b^2 - 4ac}}{2a} \frac{1 + \lambda e^{k\sqrt{b^2 - 4ac}\xi}}{1 - \lambda e^{k\sqrt{b^2 - 4ac}\xi}}, \quad (7)$$

where  $\lambda \neq 0$  is an integration constant.

- **Case 3:**  $\Delta = 0$  with  $b \neq 0$  and  $c \neq 0$ . The solution becomes

$$\phi(\xi) = -\frac{b}{2a} + \frac{1}{\beta - a\xi}, \quad (8)$$

where  $\beta$  is an integration constant.

- **Case 4:**  $b = 0$  and  $c = 0$ . The equation reduces to  $\phi' = a\phi^2$  and the solution is

$$\phi(\xi) = \frac{1}{\gamma - a\xi}, \quad (9)$$

where  $\gamma$  is an integration constant.

### 4.2. Solutions for $\mathcal{D}(\mathcal{R}) = 3$

For  $\mathcal{D}(\mathcal{R}) = 3$ , the reducing equation becomes

$$\phi'(\xi) = a\phi^3(\xi) + b\phi(\xi), \quad (10)$$

which yields two distinct cases:

- **Case 1:**  $b \neq 0$ . Then the solution is

$$\phi(\xi) = \sqrt{\frac{b}{\lambda_1 e^{-2b\xi} - a}}, \quad (11)$$

where  $\lambda_1$  is an integration constant.

- **Case 2:**  $b = 0$ . The equation reduces to  $\phi' = a\phi^3$ , giving

$$\phi(\xi) = \sqrt{\frac{1}{\lambda_2 - 2a\xi}}, \quad (12)$$

where  $\lambda_2$  is an integration constant.

### 4.3. Solutions for $\mathcal{D}(\mathcal{R}) = 4$

For  $\mathcal{D}(\mathcal{R}) = 4$ , the reducing equation takes the form

$$\phi'(\xi) = a\phi^4(\xi) + b\phi(\xi), \quad (13)$$

which leads to the following two classes of solutions:

- **Case 1:**  $b \neq 0$ . In this case, the solution is given by

$$\phi(\xi) = \sqrt[3]{\frac{b}{\lambda_1 e^{-3b\xi} - a}}, \quad (14)$$

where  $\lambda_1$  is an integration constant.

- **Case 2:**  $b = 0$ . When the linear term vanishes, the reducing equation becomes  $\phi' = a\phi^4$ , whose solution is

$$\phi(\xi) = \sqrt[3]{\frac{1}{\lambda_2 - 3a\xi}}, \quad (15)$$

where  $\lambda_2$  is an integration constant.

## 5. Solution of the sharma–tasso–olver equation using the homogeneous balance method

Consider the Sharma–Tasso–Olver (STO) equation

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \quad (16)$$

and assume a travelling wave solution of the form

$$u(x, t) = U(\xi), \quad \xi = x - \omega t.$$

Substituting this into Equation (16) gives

$$-\omega U'(\xi) + 3\alpha U^2(\xi)U'(\xi) + 3\alpha(U(\xi)U''(\xi) + (U'(\xi))^2) + \alpha U'''(\xi) = 0. \quad (17)$$

Applying the balancing principle from the preliminaries, the degree of  $\phi(\xi)$  in the reducing equation

and in the expansion of  $\mathcal{U}(\xi)$  satisfy

$$\mathfrak{D}(\mathcal{R}) = \mathfrak{D}(\mathcal{E}) + 1, \quad (18)$$

where  $\mathfrak{D}(\mathcal{R})$  denotes the degree of  $\phi(\xi)$  appearing in the reducing equation, and  $\mathfrak{D}(\mathcal{E})$  denotes the degree of  $\phi(\xi)$  appearing in the expansion of  $\mathcal{U}(\xi)$ .

### 5.1. Case 1: $\mathfrak{D}(\mathcal{R}) = 2, \mathfrak{D}(\mathcal{E}) = 1$

We assume the solution to be the form

$$\mathcal{U}(\xi) = q_0 + q_1 \phi(\xi), \quad (19)$$

together with the reducing equation

$$\phi'(\xi) = a \phi^2(\xi) + b \phi(\xi) + c, \quad (20)$$

where  $a \neq 0$ .

Substituting (81) and (20) into the reduced STO equation and equating coefficients of like powers of  $\phi(\xi)$  yields a system of algebraic equations for  $q_0, q_1$ , and  $\omega$ . To proceed, we compute the required derivatives:

$$\begin{aligned} \mathcal{U}'(\xi) &= q_1(a\phi^2 + b\phi + c), \\ \mathcal{U}'''(\xi) &= 6a^3q_1\phi^4 + 12a^2bq_1\phi^3 + (7ab^2q_1 + 8a^2cq_1)\phi^2 \\ &\quad + (b^3q_1 + 8abcq_1)\phi + (b^2 + 2ac)cq_1, \\ (\mathcal{U}'(\xi))^2 &= q_1^2(a^2\phi^4 + 2ab\phi^3 + (b^2 + 2ac)\phi^2 + 2bc\phi + c^2), \\ \mathcal{U}\mathcal{U}''(\xi) &= 2a^2q_1^2\phi^4 + (3abq_1^2 + 2a^2q_0q_1)\phi^3 \\ &\quad + [(b^2 + 2ac)q_1^2 + 3abq_0q_1]\phi^2 \\ &\quad + [(b^2 + 2ac)q_0q_1 + 3bcq_1^2]\phi + bcq_0q_1, \\ \mathcal{U}^2\mathcal{U}'(\xi) &= aq_1^3\phi^4 + (2aq_0q_1^2 + bq_1^3)\phi^3 \\ &\quad + (aq_1q_0^2 + 2bq_0q_1^2 + cq_1^3)\phi^2 \\ &\quad + (bq_1q_0^2 + 2cq_0q_1^2)\phi + cq_0^2q_1. \end{aligned}$$

Substituting these expressions into the reduced STO Equation (17):

$$\text{Coefficient of } \phi^4: 6\alpha a^3q_1 + 9\alpha a^2q_1^2 + 3\alpha aq_1^3 = 0, \quad (21)$$

$$\begin{aligned} \text{Coefficient of } \phi^3: 12\alpha a^2bq_1 + 3\alpha(5abq_1^2 + 2a^2q_0q_1 \\ + 2aq_0q_1 + bq_1^2) = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} \text{Coefficient of } \phi^2: \alpha(7ab^2q_1 + 8a^2cq_1) \\ + 3\alpha(2(b^2 + 2ac)q_1^2 + 3abq_0q_1 + cq_1^3 \\ + aq_1q_0^2 + 2bq_0q_1^2) - \omega bq_1 = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} \text{Coefficient of } \phi^1: \alpha(b^3q_1 + 8abcq_1) \\ + 3\alpha[(b^2 + 2ac)q_0q_1 + 3bcq_1^2 \\ + bq_1q_0^2 + 2cq_0q_1^2] - \omega bq_1 = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \text{Coefficient of } \phi^0: \alpha cq_1(b^2 + 2ac) + 3\alpha(bcq_0q_1 \\ + cq_0^2q_1 + c^2q_1^2) - \omega cq_1 = 0. \end{aligned} \quad (27)$$

Solving this system yields two distinct sets of admissible parameters:

#### • Set 1:

$$q_1 = -2a, \quad q_0 = -b, \quad \omega = \alpha(b^2 - 4ac).$$

#### • Set 2:

$$q_1 = -a, \quad \omega = \alpha(b^2 - ac + 3q_0^2 + 3bq_0).$$

Since Set 2 introduces more dependencies, we restrict attention to **Set 1**.

#### 5.1.1. Final solutions for set 1

From Set 1, the travelling-wave solution is given by

$$\begin{aligned} \mathcal{U}(\xi) &= -b - 2a\phi(\xi), \quad \text{or equivalently} \\ u(x, t) &= -b - 2a\phi(x - \omega t). \end{aligned} \quad (28)$$

Depending on the discriminant  $\Delta = b^2 - 4ac$ , different functional forms of  $\phi(\xi)$  (and hence  $u(x, t)$ ) arise.

#### 5.1.2. Case 1: $b^2 - 4ac < 0$ (trigonometric-type solution)

When the discriminant is negative, the corresponding auxiliary function is

$$\phi(\xi) = \frac{k}{2a} \sqrt{4ac - b^2} \tan\left(\frac{k}{2} \sqrt{4ac - b^2} \xi + A\right), \quad k = \pm 1,$$

where  $A$  is an integration constant.

Substituting this into (28), we obtain the travelling-wave solution

$$\begin{aligned} u(x, t) &= -b - k \sqrt{4ac - b^2} \tan\left(\frac{k}{2} \sqrt{4ac - b^2} \right. \\ &\quad \left. (x - \omega t) + A\right), \end{aligned} \quad (29)$$

This solution exhibits vertical asymptotes whenever the argument of the tangent function satisfies

$$\frac{k}{2} \sqrt{4ac - b^2} (x - \omega t) + A = \frac{(2n + 1)\pi}{2}, \quad n \in \mathbb{Z},$$

which correspond to the singular points of the solution profile.

To illustrate the behavior of the travelling-wave solution of the STO equation in Case 1, Figure 1 presents both the 3D surface plot and the corresponding line plots for the parameter values  $a = b = k = \alpha = 1$ ,  $A = 0$ , and  $c = 10$ . For the line plot, three different time levels  $t = -3, 0$ , and  $3$  are selected. From the line plot in Figure 1, it is evident that oscillations with sharp spikes occur at certain spatial locations, while for  $t = -3$  and  $t = 3$ , the oscillations appear smoother but still exhibit localized irregularities. This behavior indicates that the solution is highly sensitive to both temporal and spatial variations, and that instabilities tend to develop at specific focal points.

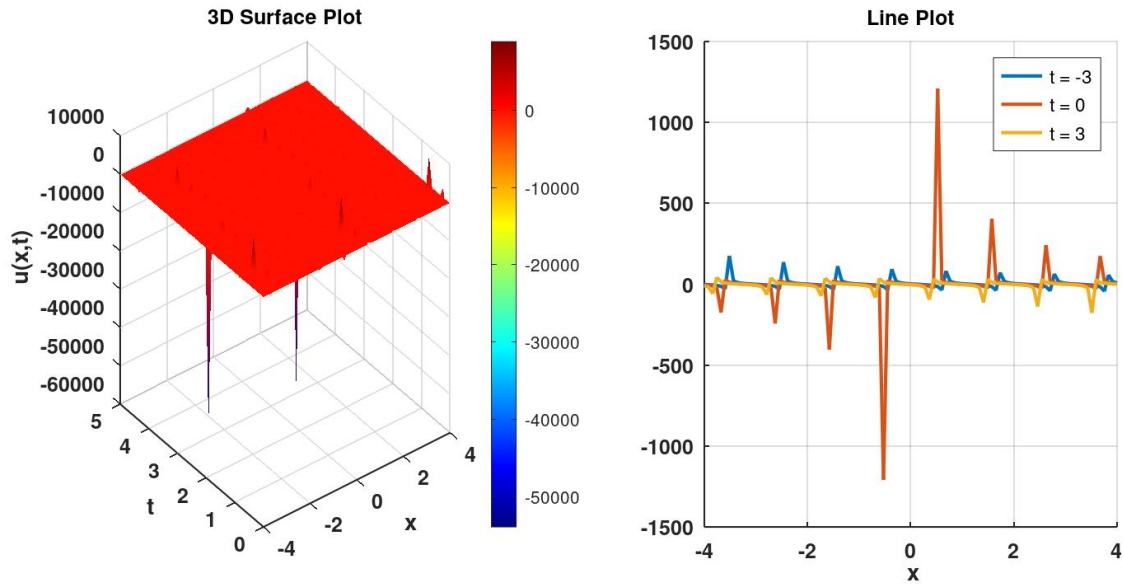
#### 5.1.3. Case 2: $b^2 - 4ac > 0$ :

$$u(x, t) = -b - k \sqrt{b^2 - 4ac} \frac{1 + \lambda e^{k\sqrt{b^2 - 4ac}(x - \omega t)}}{1 - \lambda e^{k\sqrt{b^2 - 4ac}(x - \omega t)}}, \quad (30)$$

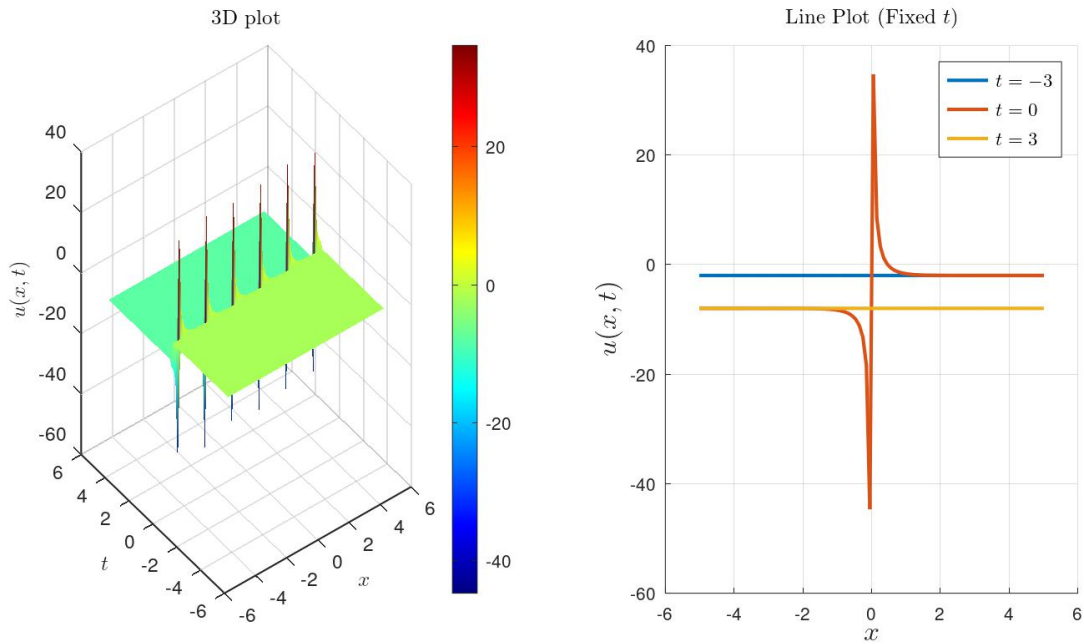
where  $k = \pm 1$  and  $\lambda$  is an integration constant. The solution has a vertical asymptote at:

$$\xi = \frac{-\ln(\lambda)}{k\sqrt{b^2 - 4ac}}. \quad (31)$$

The travelling-wave solution for Case 2 ( $b^2 - 4ac > 0$ ) exhibits a hyperbolic-type structure dominated by moving singular fronts. As shown in Figure 2, the 3D surface plot reveals sharp vertical spikes corresponding to the poles generated by the exponential term in the auxiliary function. Away from these singular regions, the solution rapidly approaches a nearly constant background state. The line plots at  $t = -3, 0, 3$  confirm this behaviour: at  $t = 0$ , the solution develops a strong singularity near  $x = 0$  with large positive and negative amplitudes, whereas at  $t = \pm 3$  the profile becomes smoother and tends toward a constant level. These results demonstrate that the Case 2 solution is highly sensitive to both spatial position and wave speed, producing localized blow-up behaviour at intermediate times while remaining stable elsewhere.



**Figure 1.** 3D surface plot (left) and line plots (right) of the solution  $u(x,t)$  at different times for Case 1, i.e.,  $u(x,t) = -k\sqrt{4ac - b^2} \tan\left(\frac{k}{2}\sqrt{4ac - b^2}(x - \omega t) + A\right)$ , for  $c = 10, b = 1, a = 1, k = 1, \alpha = \lambda = 2, A = 0$ , and  $\lambda = 1$ , when  $b^2 - 4ac < 0$ . The line plots at representative times  $t = -3, 0, 3$  illustrate how the singular structure moves across space, with smoother oscillations away from the poles and strong localized deviations near the singularities.



**Figure 2.** 3D surface plot (left) and line plots (right) of the solution  $u(x,t)$  at different times for Case 1, i.e.,  $u(x,t) = -b - k\sqrt{b^2 - 4ac} \frac{1 + \lambda e^{k\sqrt{b^2 - 4ac}(x - \omega t)}}{1 - \lambda e^{k\sqrt{b^2 - 4ac}(x - \omega t)}}$  for  $c = 4, b = 5, a = 1, k = -1, \alpha = \lambda = 2, A = 0$ , and  $\lambda = 1$ , when  $b^2 - 4ac > 0$ . The line plots at fixed times  $t = -3, 0, 3$  reveal how these singular fronts propagate in the  $x$ -direction with speed  $\omega = \alpha(b^2 - 4ac)$ , while the background solution approaches a constant state away from the singular region.

#### 5.1.4. Case 3: $b \neq 0, c \neq 0, b^2 - 4ac = 0$

In this case, the auxiliary equation reduces to a rational form and the corresponding travelling-wave solution becomes

$$u(x,t) = \frac{2}{(x - \omega t) - \gamma}, \quad (32)$$

where  $\gamma$  is an integration constant.

The solution exhibits a vertical asymptote at  $\xi = x - \omega t = \gamma$ . Since  $b^2 - 4ac = 0$  implies  $\omega = 0$ , the wave does not propagate, and the profile represents a **standing singular wave**.

#### 5.1.5. Case 4: $b = 0, c = 0$

When the linear and constant terms vanish, the reducing equation simplifies to  $\phi' = a\phi^2$ , and the resulting solution

takes the form

$$u(x, t) = \frac{2}{(x - \omega t) - \beta}, \quad (33)$$

where  $\beta$  is an integration constant.

As in Case 3, this solution has a vertical asymptote at  $\xi = \beta$ , and again  $\omega = 0$ , indicating that the wave is a **standing singular wave**.

## 5.2. The case $\mathfrak{D}(\mathcal{R}) = 3$ and its consequence $\mathfrak{D}(\mathcal{E}) = 2$

Substituting the expansion of  $\mathcal{U}(\xi)$  in powers of  $\phi(\xi)$ ,

$$\mathcal{U}(\xi) = q_0 + q_1\phi(\xi) + q_2\phi^2(\xi),$$

together with the reducing equation

$$\phi'(\xi) = a\phi^3(\xi) + b\phi(\xi),$$

We compute the derivatives  $\mathcal{U}'(\xi)$ ,  $\mathcal{U}^2\mathcal{U}'(\xi)$ ,  $\mathcal{U}\mathcal{U}''(\xi)$ ,  $(\mathcal{U}'(\xi))^2$ , and  $\mathcal{U}'''(\xi)$  in powers of  $\phi(\xi)$ . The resulting expressions are:

$$\mathcal{U}'(\xi) = 2aq_2\phi^4 + aq_1\phi^3 + 2bq_2\phi^2 + bq_1\phi,$$

$$\begin{aligned} \mathcal{U}'''(\xi) = & 48a^3q_2\phi^8 + 15a^3q_1\phi^7 + 96a^2bq_2\phi^6 + 27a^2bq_1\phi^5 \\ & + 56ab^2q_2\phi^4 + 13ab^2q_1\phi^3 + 8b^3q_2\phi^2 + b^3q_1\phi, \end{aligned}$$

$$\begin{aligned} (\mathcal{U}'(\xi))^2 = & 4a^2q_2^2\phi^8 + 4a^2q_1q_2\phi^7 + (8abq_2^2 + a^2q_1^2)\phi^6 \\ & + 8abq_1q_2\phi^5 + (2abq_1^2 + 4b^2q_2^2)\phi^4 + 4b^2q_1q_2\phi^3 \\ & + b^2q_1^2\phi^2, \end{aligned}$$

$$\begin{aligned} \mathcal{U}\mathcal{U}''(\xi) = & 8a^2q_2^2\phi^8 + 11a^2q_1q_2\phi^7 \\ & + (12abq_2^2 + 3a^2q_1^2 + 8a^2q_0q_2)\phi^6 \\ & + (16abq_1q_2 + 3a^2q_0q_1)\phi^5 \\ & + (12abq_0q_2 + 4abq_1^2 + 4b^2q_2^2)\phi^4 \\ & + (5b^2q_1q_2 + 4abq_0q_1)\phi^3 \\ & + (4b^2q_0q_2 + b^2q_1^2)\phi^2 + b^2q_0q_1\phi, \end{aligned}$$

$$\begin{aligned} \mathcal{U}^2\mathcal{U}'(\xi) = & 2aq_2^3\phi^8 + 3aq_1q_2^2\phi^7 \\ & + (2bq_2^3 + 4aq_0q_2^2 + 2aq_1^2q_2)\phi^6 \\ & + (4bq_1q_2^2 + 4aq_0q_1q_2 + 2aq_1^3)\phi^5 \\ & + (2aq_2q_0^2 + 2aq_1^2q_0 + bq_2^3 + 4bq_0q_2^2 + 2bq_1^2q_2)\phi^4 \\ & + (2bq_1^2q_2 + 4bq_0q_1q_2 + aq_0^2q_1)\phi^3 \\ & + (2bq_0q_1q_2 + bq_1^3 + 2bq_0^2q_2)\phi^2 + 2bq_0q_1^2\phi + bq_1q_0^2. \end{aligned}$$

Substituting the expansion into Equation (2) yields the polynomial relation

$$\sum_{n=0}^8 C_n \phi^n(\xi) = 0, \quad (34)$$

where the coefficients  $C_n$  are given by

$$C_8 = 6\alpha a q_2^3 + 36\alpha a^2 q_2^2 + 48\alpha a^3 q_2, \quad (35)$$

$$C_7 = 9\alpha a q_1 q_2^2 + 45\alpha a^2 q_1 q_2 + 15\alpha a^3 q_1 + 96\alpha a^2 b q_2, \quad (36)$$

$$\begin{aligned} C_6 = & 3\alpha(2bq_2^3 + 4aq_1^2q_2 + 4aq_0q_2^2 + 20abq_2^2 + 4a^2q_1^2 \\ & + 8a^2q_0q_2) + 96\alpha a^2 b q_1, \end{aligned} \quad (37)$$

$$\begin{aligned} C_5 = & 3\alpha(4bq_1q_2^2 + 4aq_0q_1q_2 + aq_1^3 + 24abq_1q_2 \\ & + 3a^2q_0q_1) + 27\alpha a^2 b q_1, \end{aligned} \quad (38)$$

$$\begin{aligned} C_4 = & -2\omega a q_2 + 3\alpha(2aq_2q_0^2 + 2aq_1^2q_0 + bq_1q_2^2 + 2bq_0q_2^2 \\ & + 2bq_1^2q_2 + 12abq_0q_2 + 6abq_1^2 + 8b^2q_2^2) + 56\alpha ab^2q_2, \end{aligned} \quad (39)$$

$$\begin{aligned} C_3 = & -a\omega q_1 + 3\alpha(2bq_1^2q_2 + 4bq_0q_1q_2 + aq_0^2q_1 + 9bq_1q_2 \\ & + 4abq_0q_1) + 13\alpha ab^2q_1, \end{aligned} \quad (40)$$

$$\begin{aligned} C_2 = & -2\omega q_1 + 3\alpha(2bq_0q_1q_2 + bq_1^3 + 2bq_2q_0^2 + 4b^2q_2q_0 \\ & + 2b^2q_1^2) + 8\alpha b^3q_2, \end{aligned} \quad (41)$$

$$C_1 = -b\omega q_1 + 3\alpha(2bq_0q_1^2 + b^2q_1^2) + \alpha b^3q_1, \quad (42)$$

$$C_0 = 3\alpha b q_1 q_0^2. \quad (43)$$

By solving the above system of equations, we obtain two possible sets of values for  $(q_0, q_1, q_2)$  when  $\omega = 4\alpha b^2$ :

- **Set 1:**  $(q_0, q_1, q_2) = (-2b, 0, -4a)$ , Hence,

$$\begin{aligned} \mathcal{U}(\xi) = & q_0 + q_1\phi(\xi) + q_2\phi^2(\xi) \\ = & -2b - 4a\left(\frac{b}{\lambda_1 e^{-2b\xi} - a}\right) \end{aligned}$$

It is evident that the solution attains a vertical asymptote at

$$\xi = \frac{\ln(\lambda_1)}{2b}, \quad \lambda_1 > 0, \quad (44)$$

The limiting behaviour of  $\mathcal{U}(\xi)$  near the asymptote is

$$\begin{aligned} \lim_{\xi \rightarrow \left(\frac{\ln(\lambda_1)}{2b}\right)^+} \mathcal{U}(\xi) = & +\infty, \\ \lim_{\xi \rightarrow \left(\frac{\ln(\lambda_1)}{2b}\right)^-} \mathcal{U}(\xi) = & -\infty. \end{aligned} \quad (45)$$

Hence, the explicit travelling-wave solution in  $(x, t)$  variables is

$$u(x, t) = 2b \left( \frac{1 + \lambda_1 e^{-2b(x - \omega t)}}{1 - \lambda_1 e^{-2b(x - \omega t)}} \right). \quad (46)$$

Figure 3 illustrates the singular travelling-wave solution obtained from the Riccati-based homogeneous balance formulation. The 3D surface plot shows the emergence of a blow-up singularity where the denominator of  $u(x, t)$  vanishes, producing a sharp singular ridge that moves linearly in time according to  $x_{\text{sing}}(t) = \omega t + \ln(\lambda_1)/(2b)$ . The corresponding line plots at  $t = -3, 0$ , and  $3$  demonstrate that the singularity shifts along the  $x$ -axis with constant speed  $\omega$ , characteristic of a travelling-wave solution. Despite the presence of the singularity, the surrounding profile remains nearly unchanged, further confirming the wave's coherent propagation.

**For  $b = 0$ , the solution reduces to**

$$\mathcal{U}(\xi) = -4a\phi^2(\xi). \quad (47)$$

Using the reducing equation solution for  $b = 0$ ,  $\phi(\xi)$  satisfies

$$\phi(\xi) = \sqrt{\frac{1}{\lambda_2 - 2a\xi}},$$

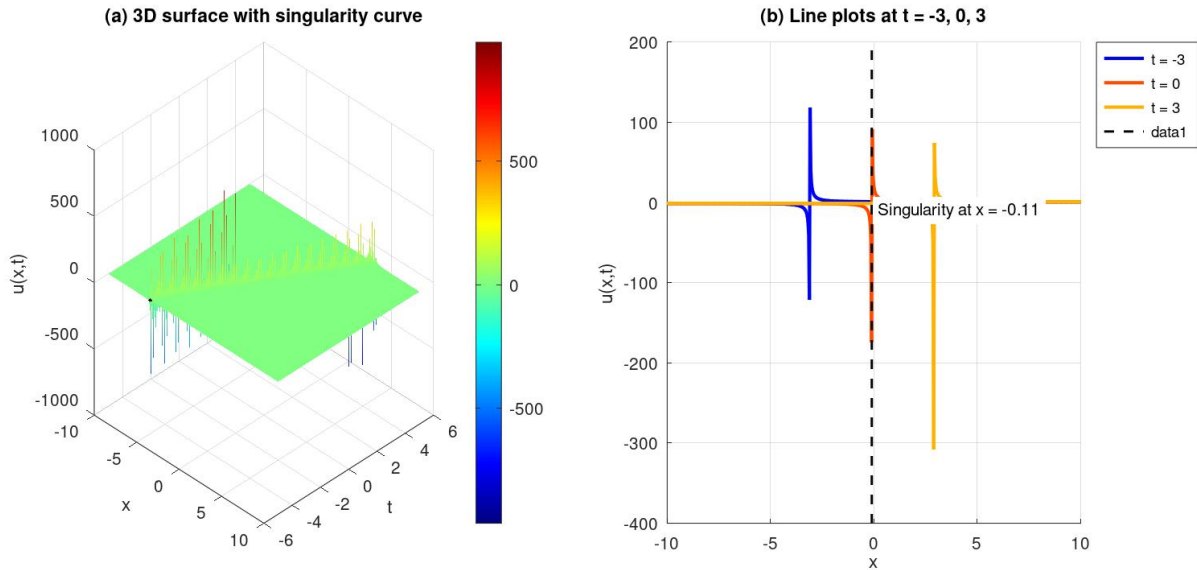
and hence

$$\mathcal{U}(\xi) = -\frac{4a}{\lambda_2 - 2a\xi}. \quad (48)$$

After rearranging, the expression simplifies to

$$\mathcal{U}(\xi) = \frac{2}{\xi - \mu}, \quad (49)$$

where  $\mu$  is an integration constant determined by  $\lambda_2$ .



**Figure 3.** 3D surface plot (left) and line plots (right) of the solution  $u(x, t)$  at different times for Set 1, i.e.,  $u(x, t) = 2b \left( \frac{1 + \lambda_1 e^{-2b(x-\omega t)}}{1 - \lambda_1 e^{-2b(x-\omega t)}} \right)$  for the parameter choice  $b = 0.5$ ,  $\lambda_1 = 0.9$ , and wave speed  $\omega = \alpha(2b)^2 = 1$  with  $\alpha = 1$ . The line plots at  $t = -3, 0, 3$  illustrating the propagation of the singular travelling wave. At  $t = 0$  the singularity occurs at  $x_{\text{sing}}(0) = \frac{\ln(\lambda_1)}{2b} \approx 0$ , which is indicated by the vertical dashed line. As time evolves, the peak shifts horizontally with speed  $\omega$ , confirming the travelling-wave structure, while the waveform retains its overall shape.

The solution clearly exhibits a vertical asymptote at  $\xi = \mu$ . Its limiting behaviour near the asymptote is given by

$$\lim_{\xi \rightarrow \mu^+} \mathcal{U}(\xi) = +\infty, \quad \lim_{\xi \rightarrow \mu^-} \mathcal{U}(\xi) = -\infty. \quad (50)$$

Transforming back to the physical variables  $(x, t)$ , we obtain

$$u(x, t) = u(x) = \frac{2}{x - \mu}. \quad (51)$$

Since  $b = 0$  implies  $\omega = 0$ , this case corresponds to a **standing wave** rather than a travelling wave.

- **Set 2:**  $(q_0, q_1, q_2) = (0, 0, -2a)$ .

For  $b \neq 0$ , the solution takes the form

$$\mathcal{U}(\xi) = -2a \phi^2(\xi). \quad (52)$$

Using the corresponding solution of the reducing equation, this can be rewritten as

$$\mathcal{U}(\xi) = \frac{2b}{1 - \lambda_0 e^{-2b\xi}}, \quad (53)$$

where  $\lambda_0$  is an integration constant.

It is evident that the solution possesses a vertical asymptote at

$$\xi = \frac{\ln(\lambda_0)}{2b}. \quad (54)$$

The limiting behaviour near the asymptote is

$$\begin{aligned} \lim_{\xi \rightarrow \left(\frac{\ln(\lambda_0)}{2b}\right)^+} \mathcal{U}(\xi) &= +\infty, \\ \lim_{\xi \rightarrow \left(\frac{\ln(\lambda_0)}{2b}\right)^-} \mathcal{U}(\xi) &= -\infty. \end{aligned} \quad (55)$$

Transforming back to  $(x, t)$  via  $\xi = x - \omega t$ , the explicit solution becomes

$$u(x, t) = \frac{2b}{1 - \lambda_0 e^{-2b(x-\omega t)}}. \quad (56)$$

Figure 4 shows the explicit travelling-wave solution for  $b = 1$  and  $\lambda_0 = 0.5$ . A vertical singularity forms where  $1 - \lambda_0 e^{-2b(x-\omega t)} = 0$ , located at  $x_{\text{sing}} = \ln(\lambda_0)/(2b) \approx -0.35$ . The 3D surface displays this moving blow-up clearly. The line profiles at  $t = 0, 1, 2$  demonstrate that the singularity propagates horizontally with speed  $\omega$ , while the rest of the waveform remains essentially unchanged, confirming the travelling-wave character of the solution.

When  $b = 0$ , the solution reduces to

$$\mathcal{U}(\xi) = -2a \phi^2(\xi),$$

which can be written as

$$\mathcal{U}(\xi) = \frac{1}{\xi - \lambda},$$

where  $\lambda$  is an integration constant.

It is clear that the solution has a vertical asymptote at  $\xi = \lambda$ . The limiting behavior around the asymptote is

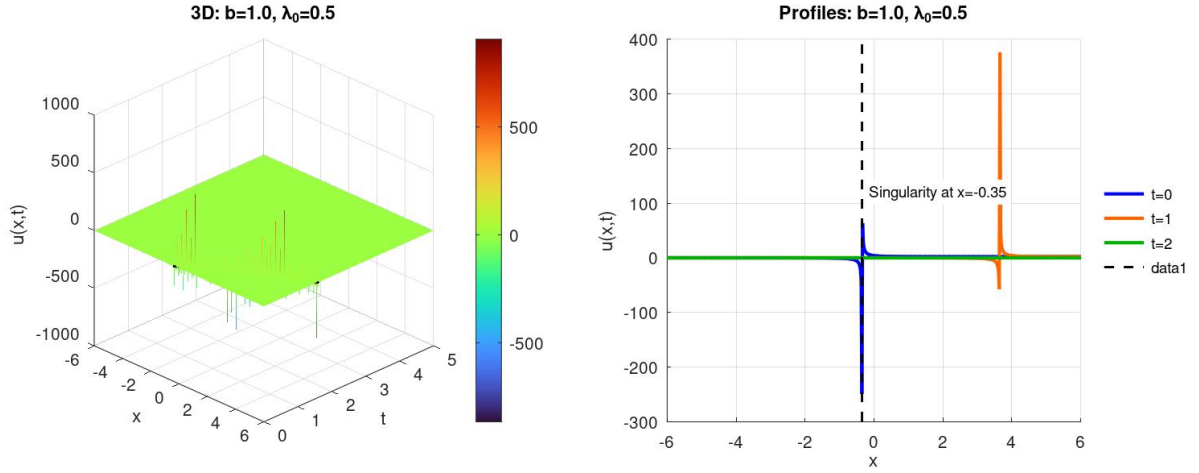
$$\lim_{\xi \rightarrow \lambda^+} \mathcal{U}(\xi) = +\infty, \quad \lim_{\xi \rightarrow \lambda^-} \mathcal{U}(\xi) = -\infty.$$

Thus, in terms of  $x$ ,

$$u(x, t) = u(x) = \frac{1}{x - \lambda}.$$

It should be noted that the case  $b = 0$  corresponds to  $\omega = 0$ , which implies the formation of **standing waves**. In contrast, for  $b \neq 0$ , the parameter  $\omega$  represents a forward traveling wave.





**Figure 4.** 3D surface plots (left) and line profiles at  $t = 0, 1, 2$  (right) for the solution  $u(x, t) = \frac{2b}{1 - \lambda_0 e^{-2b(x - \omega t)}}$ , for the parameter choice  $b = 1$  and  $\lambda_0 = 0.5$ . The three-dimensional surface plot (left) clearly shows the formation of a blow-up singularity along the curve where the denominator of the solution vanishes. This corresponds to the analytic singularity  $x_{\text{sing}} = \frac{\ln(\lambda_0)}{2b} \approx -0.35$ , which appears as a sharp ridge on the surface and remains aligned with the travelling-wave coordinate  $x - \omega t$ . The line profiles (right) at  $t = 0, 1, 2$  confirm the propagation of this singular structure: at  $t = 0$  the singularity occurs at  $x \simeq -0.35$ , while for later times the peak shifts to the right with wave speed  $\omega$ , consistent with the analytical expression  $x_{\text{sing}}(t) = \omega t + \ln(\lambda_0)/(2b)$ . Apart from this singular point, the waveform maintains its overall shape, demonstrating a rigid travelling-wave motion governed by the exponential term in the denominator.

### 5.3. $\mathfrak{D}(\mathcal{R}) = 4$ , $\mathfrak{D}(\mathcal{E}) = 3$

We consider the polynomial expansion

$$\mathcal{U}(\xi) = q_0 + q_1\phi(\xi) + q_2\phi^2(\xi) + q_3\phi^3(\xi), \quad (57)$$

where the auxiliary function satisfies the reducing equation

$$\phi'(\xi) = a\phi^4(\xi) + b\phi(\xi). \quad (58)$$

Substituting (57) and (58) and computing  $\mathcal{U}'(\xi)$ ,  $\mathcal{U}\mathcal{U}''(\xi)$ ,  $\mathcal{U}^2\mathcal{U}'(\xi)$ ,  $\mathcal{U}'''(\xi)$  and  $(\mathcal{U}'(\xi))^2$  in powers of  $\phi(\xi)$ , we obtain:

$$\begin{aligned} \mathcal{U}'(\xi) &= 3aq_3\phi^6 + 2aq_2\phi^5 + aq_1\phi^4 + 3bq_3\phi^3 \\ &\quad + 2bq_2\phi^2 + bq_1\phi. \end{aligned} \quad (59)$$

$$\begin{aligned} \mathcal{U}\mathcal{U}''(\xi) &= 18a^2q_3^2\phi^{12} + 28a^2q_2q_3\phi^{11} \\ &\quad + (22a^2q_1q_3 + 10a^2q_2^2)\phi^{10} \\ &\quad + (27abq_3^2 + 18a^2q_0q_3 + 14a^2q_1q_2)\phi^9 \\ &\quad + (41abq_2q_3 + 4a^2q_1^2 + 10a^2q_0q_2)\phi^8 \\ &\quad + (4a^2q_0q_1 + 32abq_1q_3 + 14abq_2^2)\phi^7 \\ &\quad + (9b^2q_3^2 + 19abq_1q_2 + 27abq_0q_3)\phi^6 \\ &\quad + (13b^2q_2q_3 + 5abq_1^2 + 14abq_0q_2)\phi^5 \\ &\quad + (10b^2q_1q_3 + 4b^2q_2^2 + 5abq_0q_1)\phi^4 \\ &\quad + (5b^2q_1q_2 + 9b^2q_0q_3)\phi^3 + (4b^2q_0q_2 + b^2q_1^2)\phi^2 \\ &\quad + b^2q_1q_0\phi. \end{aligned} \quad (60)$$

$$\begin{aligned} \mathcal{U}^2\mathcal{U}'(\xi) &= 3aq_3^3\phi^{12} + 8aq_2q_3\phi^{11} \\ &\quad + (12aq_1q_2q_3 + 6aq_0q_3^2 + 2aq_2^2 + 3bq_3^2)\phi^{10} \\ &\quad + (5aq_1^2q_3 + 5aq_1q_2^2 + 10aq_0q_2q_3 + 8bq_2q_3^2)\phi^9 \\ &\quad + (4aq_1^2q_2 + 4aq_0q_2^2 + 7bq_1q_3^2 + 7bq_2^2q_3 \\ &\quad + 8aq_0q_1q_2)\phi^8 \\ &\quad + (2bq_2^3 + aq_1^3 + 6bq_0q_3^2 + 3aq_0^2q_3 + 12bq_1q_2q_3 \\ &\quad + 6aq_0q_1q_2)\phi^7 \\ &\quad + (10bq_0q_2q_3 + 4bq_1q_2q_3 + 5bq_1^2q_3 + bq_1q_2^2 \\ &\quad + 2aq_0q_1^2 + 2aq_0^2q_2)\phi^6 \\ &\quad + (8bq_0q_1q_3 + 4bq_1^2q_2 + 4bq_0q_2^2 + aq_0^2q_1)\phi^5 \\ &\quad + (6bq_0q_1q_2 + 3bq_3q_0^2 + bq_1^3)\phi^4 \\ &\quad + (2bq_0q_1^2 + 2bq_2q_0^2)\phi^3 + bq_1q_0^2\phi. \end{aligned} \quad (61)$$

$$\begin{aligned} \mathcal{U}'''(\xi) &= 162a^3q_3\phi^{12} + 80a^3q_2\phi^{11} + 28a^3q_1\phi^{10} \\ &\quad + 324a^2bq_3\phi^9 + 150a^2bq_2\phi^8 + 48a^2bq_1\phi^7 \\ &\quad + 189ab^2q_3\phi^6 + 78ab^2q_2\phi^5 + 21ab^2q_1\phi^4 \\ &\quad + 27b^3q_3\phi^3 + 8b^3q_2\phi^2 + b^3q_1\phi. \end{aligned} \quad (62)$$

$$\begin{aligned} (\mathcal{U}'(\xi))^2 &= 9a^2q_3^2\phi^{12} + 12a^2q_2q_3\phi^{11} \\ &\quad + (6a^2q_1q_3 + 4a^2q_2^2)\phi^{10} \\ &\quad + (18abq_3^2 + 4a^2q_1q_2 + a^2q_1^2)\phi^9 \\ &\quad + (24abq_2q_3 + a^2q_1^2)\phi^8 \\ &\quad + (12abq_1q_3 + 8abq_2^2)\phi^7 \\ &\quad + (8abq_1q_2 + 9b^2q_3^2)\phi^6 \\ &\quad + (12b^2q_2q_3 + 2abq_1^2)\phi^5 \\ &\quad + (6b^2q_1q_3 + 4b^2q_2^2)\phi^4 \\ &\quad + 4b^2q_1q_2\phi^3 + b^2q_1^2\phi^2. \end{aligned} \quad (63)$$

Substituting the above expressions into Equation (2) yields the reduced form

$$\sum_{n=0}^{12} C_n \phi^n(\xi) = 0, \quad (64)$$

where the coefficients  $C_n$  are given by:

$$C_{12} = 162\alpha a^3 q_3 + 81\alpha a^2 q_3^2 + 9\alpha a q_3^3, \quad (65)$$

$$C_{11} = 80\alpha a^3 q_2 + 24\alpha a q_2 q_3^2 + 120\alpha a^2 q_2 q_3, \quad (66)$$

$$C_{10} = 28\alpha a^3 q_1 + 3\alpha(7a q_3 q_2^2 + 7a q_1 q_3^2) + 3\alpha(28a^2 q_1 q_3 + 14a^2 q_2^2), \quad (67)$$

$$C_9 = 324\alpha a^2 b q_3 + 3\alpha(12a q_1 q_2 q_3 + 6a q_0 q_3^2 + 2a q_2^3 + 3b q_3^2) + 3\alpha(45ab q_3^2 + 18a^2 q_0 q_3 + 4a^2 q_1 q_2 + 14a^2 q_1 q_3), \quad (68)$$

$$C_8 = 150\alpha a^2 b q_2 + 3\alpha(5a q_1^2 q_3 + 5a q_1 q_2^2 + 10a q_0 q_2 q_3 + 8b q_2 q_3^2) + 3\alpha(65ab q_2 q_3 + 5a^2 q_1^2 + 10a^2 q_0 q_2), \quad (69)$$

$$C_7 = 48\alpha a^2 b q_1 + 3\alpha(4a q_1^2 q_2 + 4a q_0 q_2^2 + 7b q_2^2 q_3 + 7b q_1 q_3^2 + 8a q_0 q_1 q_3) + 3\alpha(4a^2 q_0 q_1 + 44ab q_1 q_3 + 22ab q_2^2), \quad (70)$$

$$C_6 = 189\alpha a b^2 q_3 + 3\alpha(2b q_2^3 + a q_1^3 + 6b q_0 q_2^2 + 3a q_0^2 q_3 + 12b q_1 q_2 q_3 + 6a q_0 q_1 q_2) + 3\alpha(9b^2 q_3^2 + 19ab q_1 q_2 + 27ab q_0 q_3) - 3a q_3 \omega, \quad (71)$$

$$C_5 = 78\alpha a b^2 q_2 + 3\alpha(10b q_0 q_2 q_3 + 4b q_1 q_2 q_3 + 5b q_1^2 q_3 + b q_1 q_2^2 + 2a q_0 q_1^2 + 2a q_0^2 q_2) + 3\alpha(13b^2 q_2 q_3 + 5ab q_1^2 + 14ab q_0 q_2 + 12b^2 q_2 q_3) - 2a q_2 \omega, \quad (72)$$

$$C_4 = 21\alpha a b^2 q_1 + 3\alpha(8b q_0 q_1 q_3 + 4b q_1^2 q_2 + 4b q_0 q_2^2 + a q_0^2 q_1) + 3\alpha(16b^2 q_1 q_3 + 8b^2 q_2^2 + 5ab q_0 q_1) - \omega a q_1, \quad (73)$$

$$C_3 = 27\alpha b^3 q_3 + 3\alpha(6b q_0 q_1 q_2 + 3b q_0^2 q_3 + b q_1^3) + 3\alpha(9b^2 q_1 q_2 + 9b^2 q_0 q_3) - 3\omega b q_3, \quad (74)$$

$$C_2 = 8\alpha b^3 q_2 + 3\alpha(2b q_0 q_1^2 + 2b q_2 q_0^2) + 3\alpha(4b^2 q_0 q_2 + 2b^2 q_1^2) - 2\omega b q_2, \quad (75)$$

$$C_1 = \alpha b^3 q_1 + 3\alpha b q_1 q_0^2 + 3\alpha b^2 q_0 q_1 - \omega b q_1. \quad (76)$$

Solving the system of Equations (58)–(76), we obtain two admissible sets of coefficients:

$$(q_0, q_1, q_2, q_3) = (-3b, 0, 0, -6a), \quad (77)$$

$$(q_0, q_1, q_2, q_3) = (0, 0, 0, -3a). \quad (78)$$

For both sets, the corresponding wave speed is

$$\omega = 9\alpha b^2, \quad (79)$$

which shows that the travelling-wave velocity is independent of the particular choice of the coefficient set  $(q_0, q_1, q_2, q_3)$  and depends solely on the system parameters.

### 5.3.1. Case I: $(q_0, q_1, q_2, q_3) = (-3b, 0, 0, -6a)$

Substituting the values of  $q_0, q_1, q_2$ , and  $q_3$  into the expansion

$$\mathcal{U}(\xi) = q_0 + q_1 \phi(\xi) + q_2 \phi^2(\xi) + q_3 \phi^3(\xi), \quad (80)$$

and using the auxiliary Riccati-type solution for  $b \neq 0$ , we obtain the reduced form

$$\mathcal{U}(\xi) = 3b \left( \frac{1 + \lambda_1 e^{-3b\xi}}{1 - \lambda_1 e^{-3b\xi}} \right), \quad (81)$$

where  $\lambda_1$  is a nonzero integration constant.

The solution possesses a vertical asymptote at

$$\xi = \frac{\ln(\lambda_1)}{3b}, \quad \lambda_1 > 0. \quad (82)$$

The limiting behavior near the singularity is given by

$$\lim_{\xi \rightarrow \left(\frac{\ln \lambda_1}{3b}\right)^+} \mathcal{U}(\xi) = +\infty, \quad \lim_{\xi \rightarrow \left(\frac{\ln \lambda_1}{3b}\right)^-} \mathcal{U}(\xi) = -\infty. \quad (83)$$

Consequently, the corresponding travelling-wave solution in the original variables is

$$u(x, t) = 3b \left( \frac{1 + \lambda_1 e^{-3b(x-\omega t)}}{1 - \lambda_1 e^{-3b(x-\omega t)}} \right), \quad (84)$$

where  $\omega = 9\alpha b^2$  as obtained earlier. The analytical solution was evaluated for the parameters  $b = 1.5$ ,  $\lambda_1 = 0.5$ , and  $\omega = 2$ , with the corresponding plots presented in Figure 5. The 3D surface illustrates the full spatiotemporal evolution of  $u(x, t)$  and reveals strong nonlinear behaviour, particularly in regions where the denominator of the solution approaches zero. These locations produce sharp peaks and troughs that characterize the intrinsic singular structure of the solution.

To assess the temporal influence more clearly, line profiles were plotted for fixed times  $t = -4, 0, 4$ . For  $t = -4$ , the solution remains nearly constant with no pronounced gradients. At  $t = 0$ , the solution develops a sharp singularity near  $x \approx 0$ , where the amplitude diverges to large positive or negative values, depending on the spatial direction of approach. By  $t = 4$ , the profile again relaxes to an almost constant state but with opposite polarity compared with  $t = -4$ . These observations demonstrate that the STO travelling-wave solution exhibits strong spatiotemporal sensitivity, with localized nonlinear structures emerging and propagating according to the wave speed determined by the parameter  $b$ .

### 5.3.2. Case I: $b = 0$

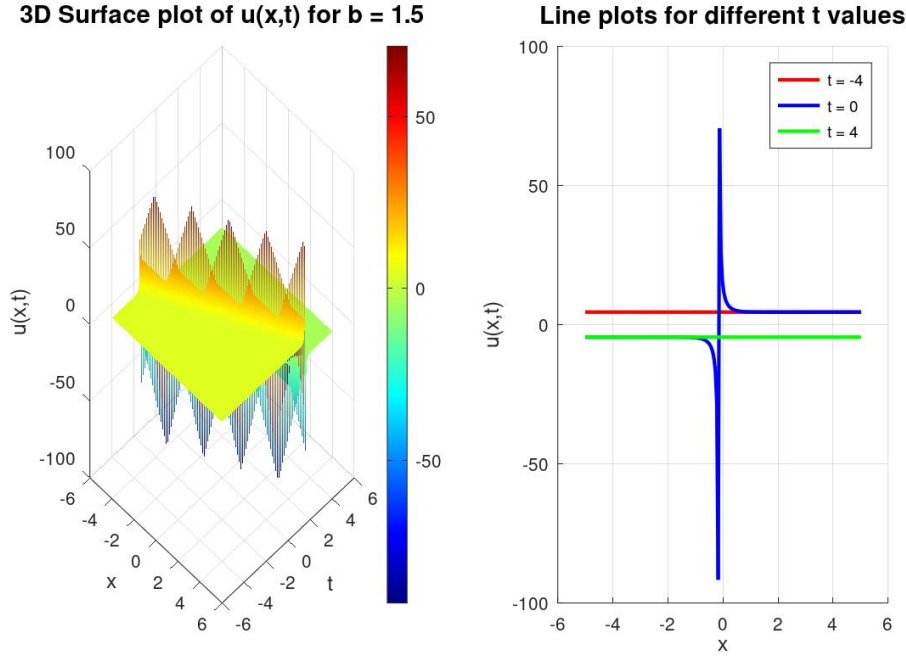
In this case, we have

$$\mathcal{U}(\xi) = -6a \phi^3(\xi). \quad (85)$$

Using the reduced form of  $\phi(\xi)$ , this becomes

$$\mathcal{U}(\xi) = -\frac{4a}{\lambda_2 - 2a\xi} = \frac{2}{\xi - \mu}, \quad (86)$$

where  $\mu = \lambda_2/(2a)$  is a constant. This coincides with the repeated rational solution already obtained earlier.



**Figure 5.** 3D surface plot (left) and line profiles at  $t = -4, 0, 4$  (right) of the analytical solution  $u(x, t) = 3b \frac{1 + \lambda_1 e^{-3b(x - \omega t)}}{1 - \lambda_1 e^{-3b(x - \omega t)}}$  for the parameter set  $b = 1.5$ ,  $\lambda_1 = 0.5$ , and  $\omega = 2$ . The 3D surface highlights the pronounced nonlinear and singular behaviour arising when the denominator approaches zero. The line plots show how the singular feature shifts with time, confirming the travelling-wave nature of location.

### 5.3.3. Case II: $(q_0, q_1, q_2, q_3) = (0, 0, 0, -3a)$

For  $b \neq 0$ , we obtain from the general form of  $\phi(\xi)$  the solution

$$\mathcal{U}(\xi) = -3a \phi^3(\xi) = \frac{3b}{1 - \lambda_0 e^{-3b\xi}}. \quad (87)$$

Vertical Asymptote. The singularity occurs when the denominator in (87) vanishes:

$$1 - \lambda_0 e^{-3b\xi} = 0 \implies \xi = \frac{\ln(\lambda_0)}{3b}. \quad (88)$$

Hence, the limiting behaviour is

$$\begin{aligned} \xi \rightarrow \left(\frac{\ln \lambda_0}{3b}\right)^+ &\implies \mathcal{U}(\xi) \rightarrow +\infty, \\ \xi \rightarrow \left(\frac{\ln \lambda_0}{3b}\right)^- &\implies \mathcal{U}(\xi) \rightarrow -\infty. \end{aligned}$$

Traveling Wave Solution. Since  $\xi = x - \omega t$ , the corresponding traveling-wave solution is

$$u(x, t) = \frac{3b}{1 - \lambda_0 e^{-3b(x - \omega t)}}. \quad (89)$$

Case  $b = 0$  (Degenerate Limit). If  $b = 0$ , then the expression (87) reduces to the rational form

$$\mathcal{U}(\xi) = -3a \phi^3(\xi) = \frac{1}{\xi - \lambda}, \quad (90)$$

which is the same solution as in Case 5.3.2.

## 6. Generalized solution framework via homogeneous balance and riccati equation

We begin with the balancing relation

$$\mathfrak{D}(\mathcal{R}) = \mathfrak{D}(\mathcal{E}) \left( \frac{s + q - 1}{r - sp} \right) + 1, \quad (91)$$

where the quantity

$$\mathcal{K} = \mathfrak{D}(\mathcal{R}) \quad (92)$$

denotes the degree of the auxiliary function  $\phi(\xi)$ ,

$$j = \mathfrak{D}(\mathcal{E}), \quad (93)$$

and

$$m = \frac{s + q - 1}{r - sp}. \quad (94)$$

Using (91)–(94), the balancing condition can be rewritten in the compact form

$$\mathcal{K} = m j + 1. \quad (95)$$

Accordingly, the travelling-wave solution is expanded as the finite polynomial

$$V(\xi) = \sum_{i=0}^j q_i \phi^i(\xi), \quad (96)$$

where the auxiliary function  $\phi(\xi)$  satisfies the reducing equation

$$\phi'(\xi) = a \phi^{\mathcal{K}}(\xi) + b \phi(\xi). \quad (97)$$

Moreover, due to the algebraic structure imposed by the Riccati-type Equation (97), the coefficients in the expansion (96) satisfy

$$q_i = 0 \quad \text{whenever} \quad (\mathcal{K} - 1) \nmid i, \quad i = 0, 1, \dots, j. \quad (98)$$

Relation (98) ensures that only those powers of  $\phi(\xi)$  consistent with the Riccati-induced degree structure appear in the final solution.

### 6.1. Generalized traveling wave solutions

Based on the above framework, the family of travelling-wave solutions can be expressed as

$$\mathcal{U}(\xi) = \mathfrak{D}(\mathcal{E}) b \left( \frac{1 + \lambda_1 e^{-\mathfrak{D}(\mathcal{E})b\xi}}{1 - \lambda_1 e^{-\mathfrak{D}(\mathcal{E})b\xi}} \right). \quad (99)$$

The corresponding wave speed is given by

$$\omega = \alpha(\mathfrak{D}(\mathcal{E})b)^2. \quad (100)$$

### Asymptotic behaviour

The solution (99) has a vertical asymptote located at

$$\xi = \frac{\ln(\lambda_1)}{\mathfrak{D}(\mathcal{E})b}. \quad (101)$$

As  $\xi$  approaches the asymptote, the solution satisfies

$$\lim_{\xi \rightarrow \left(\frac{\ln(\lambda_1)}{\mathfrak{D}(\mathcal{E})b}\right)^+} \mathcal{U}(\xi) = +\infty, \quad (102)$$

$$\lim_{\xi \rightarrow \left(\frac{\ln(\lambda_1)}{\mathfrak{D}(\mathcal{E})b}\right)^-} \mathcal{U}(\xi) = -\infty. \quad (103)$$

Thus, the travelling-wave solution in the original variables  $(x, t)$  becomes

$$u(x, t) = \mathfrak{D}(\mathcal{E})b \left( \frac{1 + \lambda_1 e^{-\mathfrak{D}(\mathcal{E})b(x-\omega t)}}{1 - \lambda_1 e^{-\mathfrak{D}(\mathcal{E})b(x-\omega t)}} \right). \quad (104)$$

An equivalent representation is

$$\mathcal{U}(\xi) = \frac{\mathfrak{D}(\mathcal{E})b}{1 - \lambda_0 e^{-\mathfrak{D}(\mathcal{E})b\xi}}, \quad (105)$$

with wave speed identical to (100):

$$\omega = \alpha(\mathfrak{D}(\mathcal{E})b)^2. \quad (106)$$

This general formulation shows that the homogeneous balance method, combined with Riccati equations of various polynomial degrees, provides a unified mechanism for constructing explicit travelling-wave solutions. Multiple sub-cases arise naturally depending on the balancing condition and the chosen degree of the Riccati polynomial.

**Remark.** The parameter values used in the graphical illustrations were selected after performing several exploratory numerical experiments. We tested multiple combinations of amplitudes, coefficients, and wave speeds reported in related STO literature, and retained those sets that produced the clearest and most representative travelling-wave profiles. The chosen parameters therefore do not correspond to any special physical regime; rather, they are selected to highlight the qualitative behaviour, singular structures, and propagation features of the solutions in a visually transparent manner.

## 7. Conclusion

The Sharma–Tasso–Olver (STO) equation has been examined in this manuscript using the travelling-wave method combined with the recently introduced Homogeneous Balance Method (HBM). This hybrid framework enables the systematic reduction of the nonlinear partial differential equation to a tractable ordinary differential form that admits exact analytical solutions. In our formulation, the HBM is coupled with an extended Riccati equation approach, allowing multiple parameter configurations to be incorporated naturally. This flexibility makes it possible to classify several sub-cases under each balancing condition, reflecting the diverse nonlinear characteristics inherent in the STO equation. Through successive application of the balancing principle, three primary cases were identified, and for each, some sub-cases were analyzed based on the degree of the Riccati polynomial and the structural features of the reduced system. Explicit solutions were obtained for all cases and later expressed in generalized form, yielding a complete family of analytical travelling-wave solutions for each balancing scenario. To illustrate the behavior of these solutions, graphical representations—both 3D surfaces and line profiles—were provided for representative parameter

values, offering clear visual insight into the dynamics governed by the STO equation under the proposed HBM–Extended Riccati framework.

Overall, this methodology expands the applicability of classical travelling-wave analysis and demonstrates the effectiveness of the extended Riccati-type HBM in solving nonlinear equations with multi-parameter interactions.

**Summary of the paper.** In this work, we develop a systematic analytical framework for deriving travelling-wave solutions of the Sharma–Tasso–Olver (STO) equation by combining the homogeneous balance method with a Riccati-type auxiliary equation. The proposed approach allows us to obtain multiple classes of exact solutions, including rational, singular, and exponential-type profiles, and provides a clear interpretation of how the balancing mechanism determines solution structure. Numerical visualizations are used to illustrate the behaviour of the resulting wave forms and to highlight the influence of key parameters. The method also offers a unified viewpoint for understanding why certain trial equations succeed or fail, and how the degree-consistency condition guides the construction of valid solutions. Overall, the developed framework contributes a flexible tool for analyzing nonlinear dispersive models and can be extended to more general or higher-order STO-type systems.

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## Conflict of interest

The authors declare no conflicts of interest.

## Author contributions

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