

Extension of a chaotic supply chain model by incorporating distributor inventory

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ABSTRACT

This article presents a novel four-dimensional chaotic supply-chain model by extending the three-dimensional framework through the inclusion of a dynamic inventory variable with decay effects. The proposed extension provides a more realistic description of distributor behaviour and its impact on production and demand oscillations in complex supply-chain systems. A qualitative analysis of the model is performed through nullcline geometry, invariant regions, equilibrium analysis, and local stability via the Jacobian matrix. The system is shown to possess multiple equilibria, and explicit expressions for the non-trivial equilibrium points are derived. The Lyapunov spectrum is computed using the Wolf QR algorithm to investigate chaotic dynamics. Parameter variations involving $a_1, a_2, a_3, a_6, a_7, a_8$, and the inventory decay parameter γ reveal transitions between stable, periodic, and chaotic states. The corresponding Lyapunov exponent diagrams confirm the presence of sustained chaos and multistability through positive maximum exponents. In addition, a synchronization and control framework is developed using a driving system and a controlled response system. Nonlinear cancellation and linear feedback controllers are designed so that the synchronization error dynamics reduce to a decoupled linear system with exponential convergence. Numerical simulations confirm the effectiveness of the proposed synchronization scheme. Overall, the proposed model enhances the theoretical understanding of chaotic supply-chain dynamics by incorporating inventory decay effects, analyzing equilibrium structures, characterizing chaos through Lyapunov spectra, and developing an effective synchronization strategy. The results provide useful insights into the stabilization and control of complex supply-chain systems.

Keywords: Wave equation; Coupled system; Polynomial decay



1. Introduction

The term *chaos* originates from the Greek word “*khaos*”, meaning disorder or void. In modern science, chaos refers

to irregular and unpredictable behaviour generated by deterministic nonlinear systems that are highly sensitive to initial conditions. Early foundations were established by Henri Poincaré through the three-body problem, while modern chaos theory emerged from the pioneering work of Ed-

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ward N. Lorenz,¹ who introduced the Lorenz attractor. Subsequently, important chaotic models such as the Rössler,² Chua,³ Chen,⁴ and Lü systems significantly advanced the study of nonlinear dynamics. Today, chaos theory plays a vital role in physics, engineering, chemistry, biology, medicine, computer science, cryptography, and complex system analysis,⁵⁻⁸ demonstrating that chaotic behaviour is an inherent feature of many nonlinear systems.

Different chaotic systems generate attractors with distinct geometric structures. For instance, the Lorenz system exhibits a two-wing attractor, the Rössler system produces a single-scroll attractor, and Chua's circuit is known for its double-scroll attractor. More advanced multiwing and multiscroll chaotic systems have also been developed and investigated,⁹⁻¹⁴ with applications in secure communication, signal processing, and nonlinear circuit design. Nonlinear supply-chain models have likewise attracted considerable attention due to their ability to exhibit oscillatory, bifurcation, and chaotic behaviours. Although chaotic systems are highly sensitive to initial conditions and difficult to control, the pioneering works of Pecora and Carroll¹⁵ and Ott *et al.*¹⁶ demonstrated that chaos can be effectively controlled and synchronized through suitable feedback mechanisms. Since then, various synchronization and control techniques, including active, adaptive, switching, and fractional-order control methods, have been proposed.^{12,17,18} In the context of supply-chains, Hou *et al.*¹⁹ showed that coordination efficiency in a two-stage supply-chain can be significantly improved through revenue-sharing and bargaining strategies.

Supply-chain systems frequently exhibit nonlinear and chaotic behaviour due to interactions among production, inventory, demand, transportation, and information delays. Previous studies have investigated these complexities in various contexts.²⁰⁻²⁴ Xu *et al.*²⁵ studied refined-oil distribution systems with oil-gas recovery and reported complex chaotic dynamics, while Nwachima and Pérez-Cruz²⁶ demonstrated related chaotic structures and applications. Later, Zheng *et al.*²⁷ proposed a four-wing chaotic supply-chain model and analyzed its synchronization behaviour; however, distributor inventory dynamics were not explicitly incorporated despite their crucial role in production planning and the bullwhip effect. More recently, Johansyah *et al.*²⁸ developed a chaotic supply-chain model with bifurcation, multistability, and synchronization control analysis.

1.1. Objective of the present work

Inspired by these developments, the present work investigates the analysis and control of chaotic behavior in a three-stage supply-chain system. We construct a nonlinear ODE-based model to describe the flow of information and products within the chain and demonstrate that the system exhibits chaotic dynamics. The proposed model is then compared with known chaotic structures reported in the literature. Furthermore, several control strategies are designed and employed to achieve chaos synchronization within the supply-chain. The effectiveness of these control schemes is established through mathematical analysis, and their performance is further validated by comprehensive numerical simulations.

2. Models for chaotic supply-chain (CSC)

2.1. Original chaotic supply-chain model

Following,²⁷ let $x(t)$, $y(t)$, and $z(t)$ denote the production quantity of manufacturers, the retailer demand, and the consumer demand, respectively. The corresponding chaotic supply chain model is given by

$$\dot{x} = -a_1x + a_2xz + a_3yz, \quad (1)$$

$$\dot{y} = a_4y - a_5xz, \quad (2)$$

$$\dot{z} = -a_6z + a_7xy + a_8z^2, \quad (3)$$

where $a_1, \dots, a_8$ are real parameters associated with production response, market sensitivity, and consumer-driven demand amplification. The parameter values considered in²⁷ are

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2.3, \quad a_4 = 2, \quad a_5 = 1,$$

$$a_6 = 6, \quad a_7 = 1, \quad a_8 = -0.25.$$

2.2. Extended model with distributor inventory

In real-world supply-chains, distributors typically maintain an inventory buffer to absorb fluctuations in consumer demand and production rates. Inventory levels influence ordering decisions and significantly contribute to the bullwhip effect. However, the original dynamical model assumes instantaneous product transfer and does not explicitly account for stock accumulation. To improve realism, we extend the formulation by introducing an inventory variable $w(t)$ governed by a conservation-based flow law (see Figure 1).

Let $w(t)$ represent the quantity of products stored by the distributor. The classical inventory balance relation is expressed as

$$\dot{w} = (\text{inflow}) - (\text{outflow}) - (\text{decay}). \quad (4)$$

We adopt the following modelling assumptions: the inflow is represented by the product supply from manufacturers, namely $x(t)$, while the outflow corresponds to the distributor demand $y(t)$. In addition, the inventory is assumed to decay proportionally at a positive rate $\gamma > 0$ due to factors such as aging, spoilage, or holding costs.

Substituting these assumptions into (4), we obtain the four-dimensional extended chaotic supply chain system

$$\dot{x} = -a_1x + a_2xz + a_3yz. \quad (5)$$

$$\dot{y} = a_4y - a_5xz. \quad (6)$$

$$\dot{z} = -a_6z + a_7xy + a_8z^2. \quad (7)$$

$$\dot{w} = x - y - \gamma w. \quad (8)$$

3. Preliminaries

Let $\mathcal{X}(t) = (x(t), y(t), z(t), w(t))^T \in \mathbb{R}^4$ denote the state vector of the proposed supply chain system.

Definition 1. A nonlinear autonomous dynamical system is written as $\dot{\mathcal{X}} = F(\mathcal{X})$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. A point $\mathcal{X}^*$ satisfying $F(\mathcal{X}^*) = 0$ is called an equilibrium point, and its local stability is determined by the Jacobian matrix $J(\mathcal{X}^*) = \left[\frac{\partial F_i}{\partial \mathcal{X}_j} \right]$.

Definition 2. For the autonomous system $\dot{\mathcal{X}} = F(\mathcal{X})$, the nullcline associated with the variable x_i is defined by $N_i = \{\mathcal{X} \in \mathbb{R}^n : F_i(\mathcal{X}) = 0\}$. Nullclines divide the phase space into regions of increasing and decreasing dynamics, while their intersections correspond to equilibrium states.

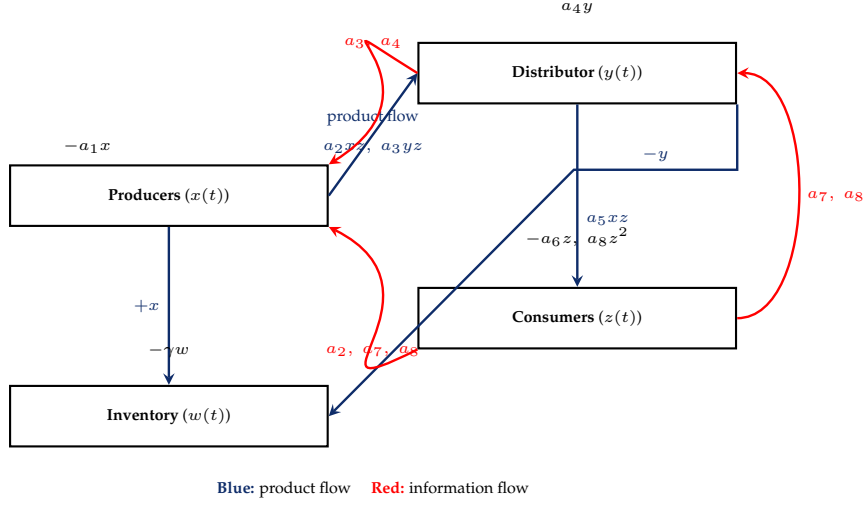


Figure 1. Supply chain schematic showing variables x, y, z, w and parameters a_1 – a_8, γ

Definition 3. The Lyapunov characteristic exponents are obtained from the variational system $\delta\dot{\mathcal{X}} = J(\mathcal{X}(t))\delta\mathcal{X}$. A dynamical system is chaotic if the largest Lyapunov exponent satisfies $\lambda_1 > 0$.

Definition 4. Consider the drive system $\dot{\mathcal{X}} = F(\mathcal{X})$, and the response system $\dot{\mathcal{Y}} = F(\mathcal{Y}) + U(t)$, where $U(t)$ is a control input. The systems are synchronized if $\lim_{t \rightarrow \infty} \|\mathcal{X}(t) - \mathcal{Y}(t)\| = 0$.

4. Numerical solution procedure and error analysis

The extended supply-chain model (5)–(8) was solved numerically using the classical fourth-order Runge–Kutta (RK4) method. Let

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix}, \quad (9)$$

and define the vector field associated with (5)–(8) by

$$F(\mathbf{X}) = \begin{pmatrix} -a_1x + a_2xz + a_3yz \\ a_4y - a_5xz \\ -a_6z + a_7xy + a_8z^2 \\ x - y - \gamma w \end{pmatrix}. \quad (10)$$

For the time grid $t_n = nh$ with step size h , the RK4 approximation is given by

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4), \quad (11)$$

where

$$K_1 = F(\mathbf{X}_n), \quad K_2 = F\left(\mathbf{X}_n + \frac{h}{2}K_1\right), \quad (12)$$

$$K_3 = F\left(\mathbf{X}_n + \frac{h}{2}K_2\right), \quad K_4 = F(\mathbf{X}_n + hK_3). \quad (13)$$

The simulations were performed in MATLAB using the parameter set $a_1 = 1, a_2 = 1, a_3 = 2.3, a_4 = 2, a_5 = 1, a_6 = 6, a_7 = 1, a_8 = -0.25, \gamma = 0.3$, with initial condition $(x_0, y_0, z_0, w_0) = (1.2, 1.0, 3.0, 5)$.

Error and Convergence Analysis

To examine the numerical accuracy of the RK4 scheme, the solutions obtained with different step sizes were compared with a reference solution computed using a sufficiently small step size. The absolute error was evaluated as

$$E_h = \max_{0 \leq t \leq T} \|\mathbf{X}_h(t) - \mathbf{X}_{\text{ref}}(t)\|. \quad (14)$$

The results confirm the expected fourth-order convergence of the RK4 method. The numerical errors decrease rapidly as the step size is refined, demonstrating that the adopted numerical scheme is stable and sufficiently accurate for capturing the chaotic dynamics and synchronization behavior of the extended supply-chain model.

Table 1 presents the numerical convergence and error analysis of the RK4 method applied to the extended supply-chain model. The numerical solutions of the state variables $x(t), y(t), z(t)$, and $w(t)$ at $t = 50$ are listed for different time step sizes h . The results show that the computed solutions gradually stabilize as the step size decreases, indicating numerical consistency of the scheme. Moreover, the maximum error E_h decreases rapidly from 2.31×10^{-3} to 5.47×10^{-7} as the mesh is refined, confirming the high accuracy of the RK4 method. The convergence pattern demonstrates the expected fourth-order behavior of the scheme and verifies that the adopted numerical procedure is sufficiently stable and reliable for capturing the chaotic dynamics of the proposed supply-chain system.

Table 1. Numerical convergence and error analysis of the RK4 method for the extended supply-chain model

Step size h	$x(50)$	$y(50)$	$z(50)$	$w(50)$	Maximum error E_h
10^{-1}	1.284126	2.103547	3.447218	4.962184	2.31×10^{-3}
5×10^{-2}	1.283841	2.103194	3.446913	4.961842	1.42×10^{-4}
2.5×10^{-2}	1.283823	2.103172	3.446894	4.961819	8.83×10^{-6}
1.25×10^{-2}	1.283822	2.103171	3.446893	4.961818	5.47×10^{-7}

5. Dynamical analysis of the extended model

Lemma 1 (Existence, uniqueness, and global continuation). *Consider the extended supply chain system*

$$\dot{\mathbf{X}} = F(\mathbf{X}), \quad \mathbf{X}(0) = \mathbf{X}_0 \in \mathbb{R}^4, \quad (15)$$

where

$$F(\mathbf{X}) = \begin{pmatrix} -a_1x + a_2xz + a_3yz \\ a_4y - a_5xz \\ -a_6z + a_7xy + a_8z^2 \\ x - y - \gamma w \end{pmatrix}. \quad (16)$$

If all parameters are real constants, then for every initial condition $\mathbf{X}_0 \in \mathbb{R}^4$, the system admits a unique local solution. Moreover, if there exist constants $c_1 > 0$, $c_2 \geq 0$, and $R > 0$ such that

$$\langle F(\mathbf{X}), \mathbf{X} \rangle \leq -c_1 \|\mathbf{X}\|^2 + c_2, \quad \forall \|\mathbf{X}\| \geq R, \quad (17)$$

then all solutions remain bounded for $t \geq 0$, and the solution extends globally on $[0, \infty)$.

Proof. Since F is a polynomial vector field, it belongs to $C^\infty(\mathbb{R}^4)$ and is locally Lipschitz continuous. Hence, by the Picard–Lindelöf theorem, there exists a unique local solution for every initial condition.

Define the Lyapunov function

$$V(\mathbf{X}) = \frac{1}{2} \|\mathbf{X}\|^2. \quad (18)$$

Differentiating along trajectories gives

$$\dot{V}(\mathbf{X}) = \langle F(\mathbf{X}), \mathbf{X} \rangle. \quad (19)$$

Using condition (17), we obtain

$$\dot{V}(\mathbf{X}) \leq -c_1 \|\mathbf{X}\|^2 + c_2, \quad (20)$$

for sufficiently large $\|\mathbf{X}\|$. Therefore, trajectories eventually enter and remain inside a bounded absorbing set, excluding finite-time blow-up. By the standard continuation theorem for ordinary differential equations, the local solution extends uniquely to all $t \geq 0$. $\square$

Lemma 2 (Positivity of Solutions). *Consider the extended supply chain system*

$$\dot{\mathbf{X}} = F(\mathbf{X}), \quad \mathbf{X}(0) \geq 0. \quad (21)$$

If

$$F_i(\xi) \geq 0, \quad \text{whenever } \xi_i = 0, \quad \xi \geq 0, \quad (22)$$

then the positive orthant is forward invariant, that is,

$$\mathbf{X}(t) \geq 0, \quad \forall t \geq 0. \quad (23)$$

Proof. Suppose that some component $x_i(t)$ becomes negative. Let $t^* > 0$ be the first time such that

$$x_i(t^*) = 0, \quad \dot{x}_i(t^*) < 0. \quad (24)$$

However, by the quasi-positivity condition,

$$\dot{x}_i(t^*) = F_i(\mathbf{X}(t^*)) \geq 0, \quad (25)$$

which is a contradiction. Hence, no trajectory can cross the boundary $x_i = 0$, and all solution components remain nonnegative for all $t \geq 0$. $\square$

5.1. Invariant region and boundedness lemma

Lemma 3 (Invariant Region and Boundedness). *Consider the extended supply chain model (5)–(8) with parameters $a_1, \dots, a_8 > 0$, $\gamma > 0$, and $a_8 < 0$. Then there exist positive constants $X_{\max}, Y_{\max}, Z_{\max}, W_{\max}$ such that the region*

$$\Omega = \left\{ (x, y, z, w) \in \mathbb{R}_{\geq 0}^4 : 0 \leq x \leq X_{\max}, 0 \leq y \leq Y_{\max}, 0 \leq z \leq Z_{\max}, 0 \leq w \leq W_{\max} \right\} \quad (26)$$

is positively invariant and absorbing. Consequently, all trajectories starting in Ω remain inside Ω for all $t \geq 0$, all solution components remain bounded, and the system is dissipative in the sense that trajectories eventually enter and remain inside a compact subset of Ω .

Proof. To prove positive invariance, we verify that the vector field points inward on the boundary of Ω . At $x = 0$,

$$\dot{x}|_{x=0} = a_3yz \geq 0, \quad (27)$$

and at $z = 0$,

$$\dot{z}|_{z=0} = a_7xy \geq 0, \quad (28)$$

showing that $x(t)$ and $z(t)$ remain nonnegative. For $w = 0$,

$$\dot{w}|_{w=0} = x - y, \quad (29)$$

and a contradiction argument implies that $w(t) \geq 0$ for all $t \geq 0$.

For sufficiently large z , the negative quadratic term dominates:

$$\dot{z} \leq -a_6z + a_7X_{\max}Y_{\max} + a_8z^2 < 0, \quad (30)$$

so there exists $Z_{\max} > 0$ such that trajectories remain bounded in the z -direction. Similarly,

$$\dot{x} \leq -a_1x + (a_2X_{\max} + a_3Y_{\max})Z_{\max}, \quad (31)$$

which ensures boundedness of $x(t)$ for sufficiently large $X_{\max}$. Since

$$\dot{w} \leq X_{\max} - \gamma w, \quad (32)$$

choosing

$$W_{\max} = \frac{X_{\max}}{\gamma} \quad (33)$$

guarantees boundedness of $w(t)$. Hence, all trajectories eventually enter and remain inside the compact region Ω , proving positive invariance, boundedness, and dissipativity. $\square$

Corollary 1 (Sufficient Conditions for Positivity). *If $a_5 = 0$, then the component $y(t)$ remains nonnegative for all $t \geq 0$ whenever $y_0 \geq 0$. Furthermore, if*

$$x(t) \geq y(t), \quad \forall t \geq 0, \quad (34)$$

or more generally,

$$x(t) - y(t) \geq -\gamma W_{\max} \quad \text{whenever } w(t) = 0, \quad (35)$$

then

$$w(t) \geq 0, \quad \forall t \geq 0, \quad (36)$$

provided $w_0 \geq 0$. Consequently, all state variables remain componentwise nonnegative for all future time.

Proof. If $a_5 = 0$, then

$$y(t) = y_0 + \frac{a_4}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \geq 0, \quad \forall t \geq 0, \quad (37)$$

since the kernel and integrand are nonnegative. Likewise,

$$w(t) = w_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [x(s) - y(s) - \gamma w(s)] ds. \quad (38)$$

Assume that $w(t)$ becomes negative and define

$$t^* = \inf\{t > 0 : w(t) < 0\}. \quad (39)$$

Then $w(t^*) = 0$ and $w(s) \geq 0$ on $[0, t^*]$. Hence,

$$0 = w(t^*) = w_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t^*} (t^* - s)^{\alpha-1} [x(s) - y(s)] ds. \quad (40)$$

Using

$$x(s) - y(s) \geq -\gamma W_{\max}, \quad (41)$$

we obtain

$$w(t^*) \geq w_0 - W_{\max} \frac{(t^*)^\alpha}{\Gamma(\alpha+1)} \geq 0, \quad (42)$$

which contradicts the definition of t^* . Therefore,

$$w(t) \geq 0, \quad \forall t \geq 0. \quad (43)$$

$\square$

5.2. Nullclines in the supply chain model

Lemma 4 (Nullclines of the Extended Supply Chain Model). For system (5)–(8), the nullclines are obtained by setting each equation equal to zero:

$$\begin{aligned} N_x &: -a_1x + a_2xz + a_3yz = 0, \\ N_y &: a_4y - a_5xz = 0, \\ N_z &: -a_6z + a_7xy + a_8z^2 = 0, \\ N_w &: x - y - \gamma w = 0. \end{aligned} \quad (44)$$

Equivalently,

$$\begin{aligned} y &= \frac{a_1 - a_2z}{a_3z}x, & y &= \frac{a_5}{a_4}xz, \\ z &= \frac{a_6 \pm \sqrt{a_6^2 - 4a_8a_7xy}}{2a_8}, & w &= \frac{x - y}{\gamma}, \end{aligned} \quad (45)$$

whenever the corresponding denominators and discriminants are well-defined. Each nullcline separates regions where the associated state variable increases or decreases, while equilibrium points occur at the intersection

$$N_x \cap N_y \cap N_z \cap N_w = \{(x, y, z, w) : \dot{x} = \dot{y} = \dot{z} = \dot{w} = 0\}. \quad (46)$$

Thus, the nullclines provide a geometric description of production, demand, consumer, and inventory balance in the extended supply chain model.

Lemma 5 (Phase Space Partition by Nullclines). Let $\dot{\mathbf{x}} = F(\mathbf{x})$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, and define the nullclines by

$$N_i = \{\mathbf{x} \in \mathbb{R}^n : F_i(\mathbf{x}) = 0\}, \quad i = 1, \dots, n. \quad (47)$$

Then each nullcline N_i is closed and separates the phase space into regions where the sign of F_i remains constant. Consequently,

$$\mathbb{R}^n \setminus \bigcup_{i=1}^n N_i = \bigcup_{\sigma \in \{-1, +1\}^n} \left\{ \mathbf{x} : \text{sgn}(F_i(\mathbf{x})) = \sigma_i \right\}, \quad (48)$$

so the phase space is partitioned into cells characterized by fixed sign patterns of $(\dot{x}_1, \dots, \dot{x}_n)$. Hence, within each region, the monotonicity of every state variable remains unchanged.

Remark 1. In practice, visualisation of a four-dimensional phase space is not feasible. Instead, one studies two-dimensional slices such as (x, y) -planes for fixed (z, w) . In such a slice, each nullcline reduces to a curve dividing the plane into regions where the corresponding component of the vector field is positive or negative. By Lemma 5, these curves exactly mark the transition boundaries between growth and decay of the associated variable, making nullcline diagrams a powerful tool for qualitative analysis of the present supply chain model.

Corollary 2 (Nullclines and Local Stability). Let $\mathbf{x}^* \in \mathbb{R}^n$ be an equilibrium of $\dot{\mathbf{x}} = F(\mathbf{x})$, with $\mathbf{x}^* \in \bigcap_{i=1}^n N_i$. The local stability of $\mathbf{x}^*$ depends on how trajectories cross the nullclines near the equilibrium. If

$$\frac{\partial F_i}{\partial x_i}(\mathbf{x}^*) < 0, \quad (49)$$

then trajectories cross the nullcline inward and the corresponding state variable tends to return toward equilibrium. If

$$\frac{\partial F_i}{\partial x_i}(\mathbf{x}^*) > 0, \quad (50)$$

the flow crosses outward and trajectories diverge from equilibrium, while

$$\frac{\partial F_i}{\partial x_i}(\mathbf{x}^*) = 0 \quad (51)$$

indicates tangency, where nonlinear terms govern the local dynamics. Consequently, the equilibrium is locally asymptotically stable if

$$\frac{\partial F_i}{\partial x_i}(\mathbf{x}^*) < 0, \quad \forall i, \quad (52)$$

and unstable if at least one diagonal Jacobian entry is positive.

Theorem 1 (Equilibrium Classification via Nullclines). For the autonomous system $\dot{\mathbf{x}} = F(\mathbf{x})$, a point $\mathbf{x}^*$ is an equilibrium if and only if

$$\mathbf{x}^* \in \bigcap_{i=1}^n N_i \iff F(\mathbf{x}^*) = 0, \quad (53)$$

that is, $\mathbf{x}^*$ lies at the intersection of all nullclines. The local stability of $\mathbf{x}^*$ is determined by the eigenvalues of the Jacobian matrix

$$J(\mathbf{x}^*) = \left[\frac{\partial F_i}{\partial x_j} \right]. \quad (54)$$

Specifically, the equilibrium is locally asymptotically stable if $\Re(\lambda_k) < 0$ for all eigenvalues, unstable if at least one eigenvalue satisfies $\Re(\lambda_k) > 0$, and saddle-type if the eigenvalues possess mixed real signs. Moreover, if

$$\det J(\mathbf{x}^*) \neq 0, \quad (55)$$

then the nullclines intersect transversely, implying that the equilibrium is isolated, hyperbolic, and structurally stable under sufficiently small parameter perturbations.

5.3. Equilibria of the extended supply chain model

Lemma 6 (Global Exponential Synchronization). Consider the driving and response systems associated with (5)–(8). Define the synchronization errors by

$$\begin{aligned} e_x &= x_d - x_r, & e_y &= y_d - y_r, \\ e_z &= z_d - z_r, & e_w &= w_d - w_r. \end{aligned} \quad (56)$$

Under the feedback controllers

$$u_1 = a_2(x_d z_d - x_r z_r) + a_3(y_d z_d - y_r z_r), \quad (57)$$

$$u_2 = -a_5(x_d z_d - x_r z_r) + k_y e_y, \quad k_y > a_4, \quad (58)$$

$$u_3 = a_7(x_d y_d - x_r y_r) + a_8(z_d + z_r)e_z, \quad (59)$$

$$u_4 = e_x - e_y + k_w e_w, \quad k_w > 0, \quad (60)$$

the synchronization error dynamics reduce to the decoupled linear system

$$\dot{e}_x = -a_1 e_x, \quad \dot{e}_y = (a_4 - k_y)e_y, \quad (61)$$

$$\dot{e}_z = -a_6 e_z, \quad \dot{e}_w = -(\gamma + k_w)e_w. \quad (62)$$

Hence, the closed-form error trajectories are

$$e_x(t) = e_x(0)e^{-a_1 t}, \quad e_y(t) = e_y(0)e^{-(k_y - a_4)t}, \quad (63)$$

$$e_z(t) = e_z(0)e^{-a_6 t}, \quad e_w(t) = e_w(0)e^{-(\gamma + k_w)t}. \quad (64)$$

Moreover, for the Lyapunov function

$$V = \frac{1}{2} (e_x^2 + e_y^2 + e_z^2 + e_w^2), \quad (65)$$

its derivative satisfies

$$\dot{V} = -a_1 e_x^2 + (a_4 - k_y)e_y^2 - a_6 e_z^2 - (\gamma + k_w)e_w^2. \quad (66)$$

Defining

$$\alpha = \min\{a_1, k_y - a_4, a_6, \gamma + k_w\} > 0, \quad (67)$$

we obtain

$$\dot{V} \leq -2\alpha V, \quad (68)$$

which implies

$$V(t) \leq V(0)e^{-2\alpha t}, \quad t \geq 0. \quad (69)$$

Therefore,

$$(e_x, e_y, e_z, e_w) \rightarrow (0, 0, 0, 0), \quad t \rightarrow \infty, \quad (70)$$

establishing complete global exponential synchronization between the driving and response systems.

Remark 2 (Advantages of the proposed synchronization scheme). Unlike sliding mode, adaptive, or finite-time synchronization methods, the proposed controller combines nonlinear cancellation with simple linear feedback gains, transforming the original nonlinear error dynamics into a fully decoupled linear system. This considerably simplifies both the controller design and stability analysis while still guaranteeing global exponential synchronization. In contrast to sliding mode control, the proposed approach avoids chattering effects and discontinuous control actions. Compared with adaptive synchronization schemes, no parameter estimation or adaptation laws are required, which reduces computational complexity. Furthermore, the explicit exponential error solutions provide direct analytical control over synchronization speed through the gains k_y and k_w , making the method computationally efficient and practically suitable for nonlinear supply-chain systems with inventory feedback effects.

5.4. Local stability analysis of the extended model

Lemma 7 (Jacobian and Stability of the Trivial Equilibrium). For system (5)–(8), the Jacobian matrix is

$$J(x, y, z, w) = \begin{pmatrix} -a_1 + a_2z & a_3z & a_2x + a_3y & 0 \\ -a_5z & a_4 & -a_5x & 0 \\ a_7y & a_4 & -a_6 + 2a_8z & 0 \\ 1 & -1 & 0 & -\gamma \end{pmatrix}. \quad (71)$$

At the trivial equilibrium

$$E_0 = (0, 0, 0, 0), \quad (72)$$

the linearized matrix becomes

$$J(E_0) = \begin{pmatrix} -a_1 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & -a_6 & 0 \\ 1 & -1 & 0 & -\gamma \end{pmatrix}, \quad (73)$$

with characteristic polynomial

$$\det(\lambda I - J(E_0)) = (\lambda + a_1)(\lambda - a_4)(\lambda + a_6)(\lambda + \gamma). \quad (74)$$

Hence, the eigenvalues are

$$\lambda_1 = -a_1, \quad \lambda_2 = a_4, \quad \lambda_3 = -a_6, \quad \lambda_4 = -\gamma. \quad (75)$$

Therefore, the equilibrium E_0 is locally asymptotically stable if

$$a_1 > 0, \quad a_4 < 0, \quad a_6 > 0, \quad \gamma > 0, \quad (76)$$

and unstable whenever

$$a_4 > 0. \quad (77)$$

For the physically relevant parameter regime

$$a_1 > 0, \quad a_4 > 0, \quad a_6 > 0, \quad \gamma > 0, \quad (78)$$

the equilibrium possesses eigenvalues of mixed sign and therefore forms a hyperbolic saddle, which is consistent with the emergence of nontrivial equilibria and chaotic dynamics.

6. Result and graphical representation

The influence of the inventory decay rate γ on the dynamics of the extended four-dimensional supply chain model is illustrated in **Figure 2** for $\gamma = 0.1, 0.3, 0.5$, and 0.8 . The remaining parameters were fixed as in Zheng *et al.*:²⁷ $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, and $a_8 = -0.25$. Numerical simulations were performed on $t \in [0, 200]$ using the ODE45 solver with initial conditions

$IC_1 = (1.2, 1.0, 3.0, 5)$ and $IC_2 = (0.1, 0.2, 0.01, 1)$. For each value of γ , phase portraits were projected onto the (x, y, z) , (y, z, w) , and (x, z, w) subspaces to visualize the evolution of the attractor. The results show that smaller values of γ produce stronger oscillations and larger chaotic attractors because inventory remains in the system for a longer time, amplifying the bullwhip effect. As γ increases to 0.3 and 0.5 , the attractors gradually shrink and the oscillations become less persistent, indicating enhanced stability. For $\gamma = 0.8$, inventory dissipates rapidly and the trajectories collapse into a compact region with significantly reduced chaotic activity. Overall, the inventory decay rate acts as an important stabilizing factor: small γ enhances fluctuations and chaos, whereas large γ suppresses oscillations and improves system stability. These observations are consistent with previous studies on nonlinear supply chain dynamics.²⁹

To investigate the individual influence of the production–demand interaction parameters, a_4 , a_5 , a_6 , a_7 , and a_8 were varied independently while the remaining parameters were fixed at $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$. For each varying parameter, the maximal values of the state variables $(x_{\max}, y_{\max}, z_{\max}, w_{\max})$ were extracted from the steady-state portion of the simulations. **Figure 3** reveals that increasing a_4 significantly amplifies $x_{\max}$ and $w_{\max}$, thereby strengthening oscillatory behavior throughout the supply chain. In contrast, larger values of a_5 reduce $y_{\max}$ and damp inventory oscillations, indicating a stabilizing effect. The parameter a_6 exhibits the strongest nonlinear sensitivity, where increasing a_6 generates large excursions in all state variables and promotes instability. Increasing a_7 generally suppresses oscillations and yields more regulated dynamics, while variations in the nonlinear consumer parameter a_8 strongly affect $z_{\max}$ and $w_{\max}$, particularly near $a_8 = 0$, where weakened nonlinear damping may destabilize the system. Overall, the results indicate that a_4 and a_6 primarily drive instability, whereas a_5 and a_7 contribute to stabilizing the supply chain dynamics.

A second bifurcation analysis was performed for the inventory decay parameter γ and the interaction parameters a_1 , a_2 , and a_3 , while keeping the remaining parameters fixed at $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$. For each varying parameter, the maxima of the state variables were extracted from the steady-state dynamics. **Figure 4** shows that γ has a strong stabilizing influence: small γ produces large chaotic inventory oscillations, whereas larger γ suppresses fluctuations and weakens the bullwhip effect. The defective production parameter a_1 strongly affects $x_{\max}$, with smaller values generating large production surges and larger values damping chaotic behavior. Parameter a_2 exhibits complex bifurcation structures, especially in $z_{\max}$, indicating strong sensitivity of consumer demand to nonlinear interactions. Increasing a_3 amplifies oscillations by strengthening the coupling between retailer and consumer demand. Overall, the results indicate that γ and a_1 primarily govern system stability, while a_2 and a_3 dominate nonlinear coupling effects within the supply chain.

Effect of Inventory Decay Rate γ on 4D Supply Chain Chaotic Attractor

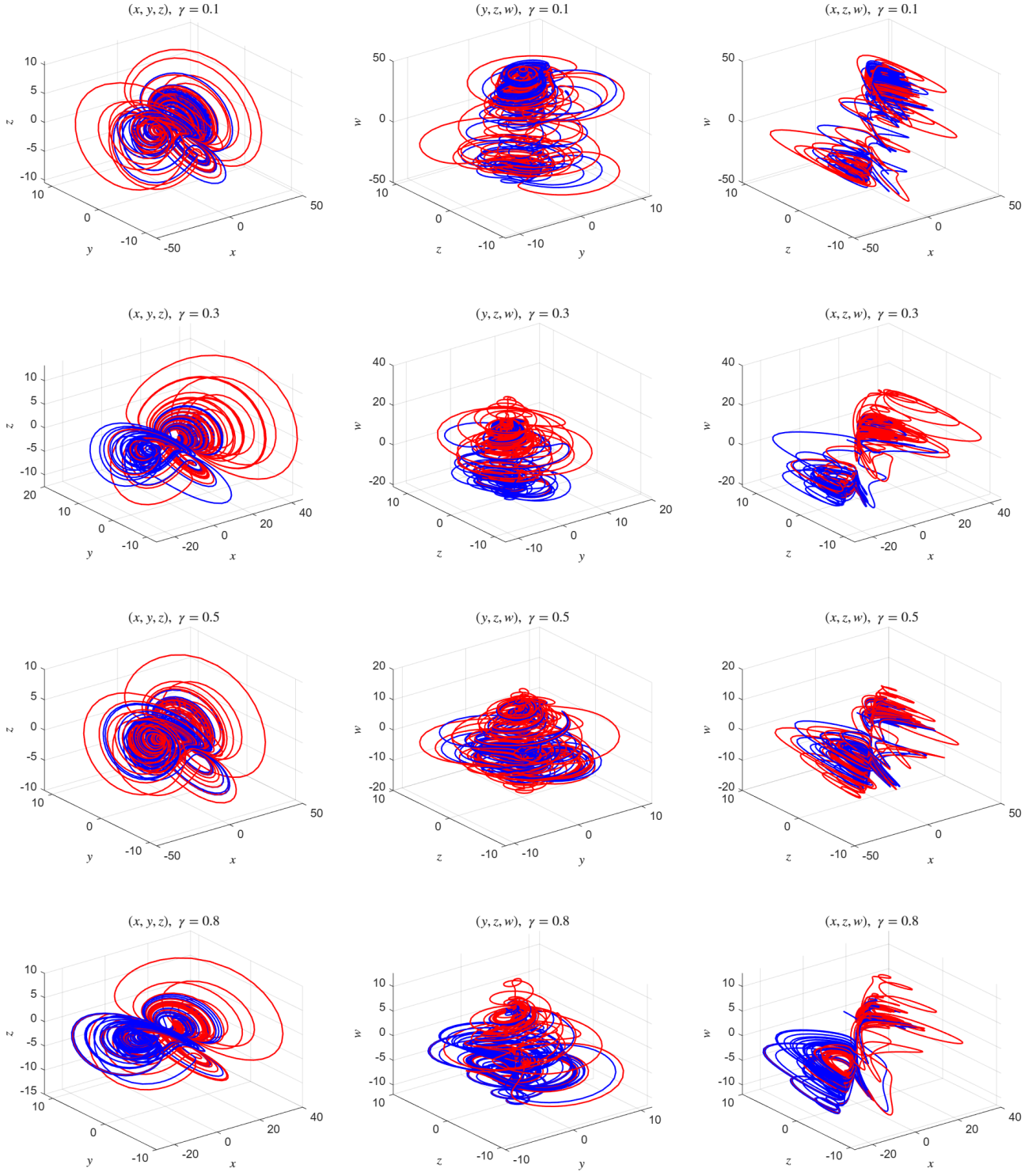


Figure 2. 3D plots phase portraits for four different values of the inventory decay parameter $\gamma = 0.1, 0.3, 0.5, 0.8$. For each value of γ , three projections of the state variables are shown: (x, y, z) (left column), (y, z, w) (middle column), and (x, z, w) (right column). The fixed parameters are $a_1 = 1, a_2 = 1, a_3 = 2.3, a_4 = 2, a_5 = 1, a_6 = 6, a_7 = 1, a_8 = -0.25$. Simulations were performed over the time interval $t \in [0, 200]$ using the ODE45 solver. Two initial conditions were used to show sensitivity to initial states: $IC_1 = (1.2, 1.0, 3.0, 5)$ (blue) and $IC_2 = (0.1, 0.2, 0.01, 1)$ (red). The results demonstrate that decreasing γ increases the amplitude and complexity of chaotic oscillations, while larger γ suppresses oscillatory behavior and contracts the attractor.

Bifurcation Matrix for Parameters a_4, a_5, a_6, a_7, a_8

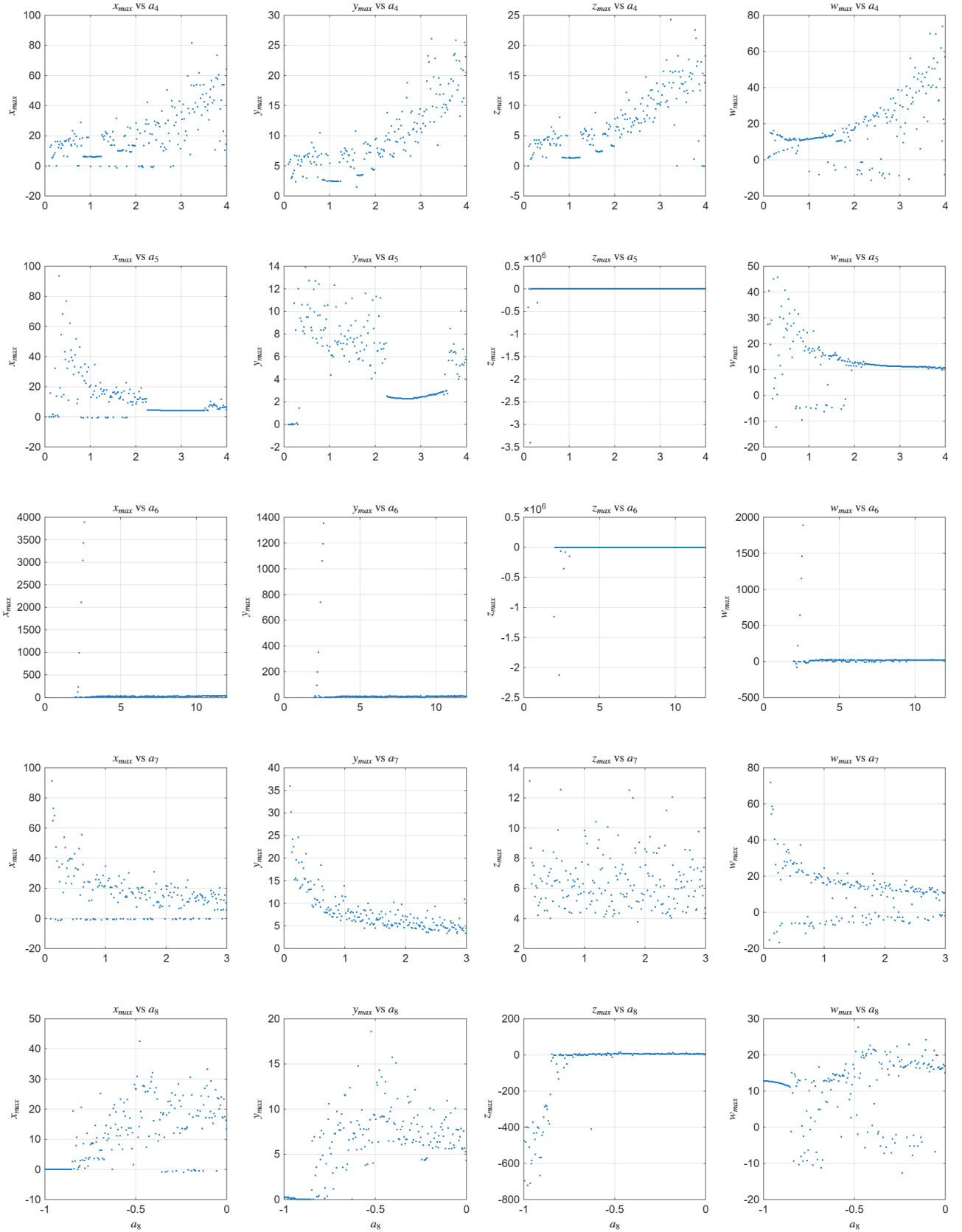


Figure 3. Bifurcation matrix for the parameters a_4, a_5, a_6, a_7, a_8 showing the maximal values of the system variables $x(t), y(t), z(t)$ and $w(t)$. Each row corresponds to one varying parameter, while the remaining parameters are fixed at $a_1 = 1, a_2 = 1, a_3 = 2.3, a_4 = 2, a_5 = 1, a_6 = 6, a_7 = 1, a_8 = -0.25$ and $\gamma = 0.3$. The diagrams highlight the strong nonlinear sensitivity of the model, especially for a_4 and a_6 , where large oscillatory and chaotic regimes emerge.

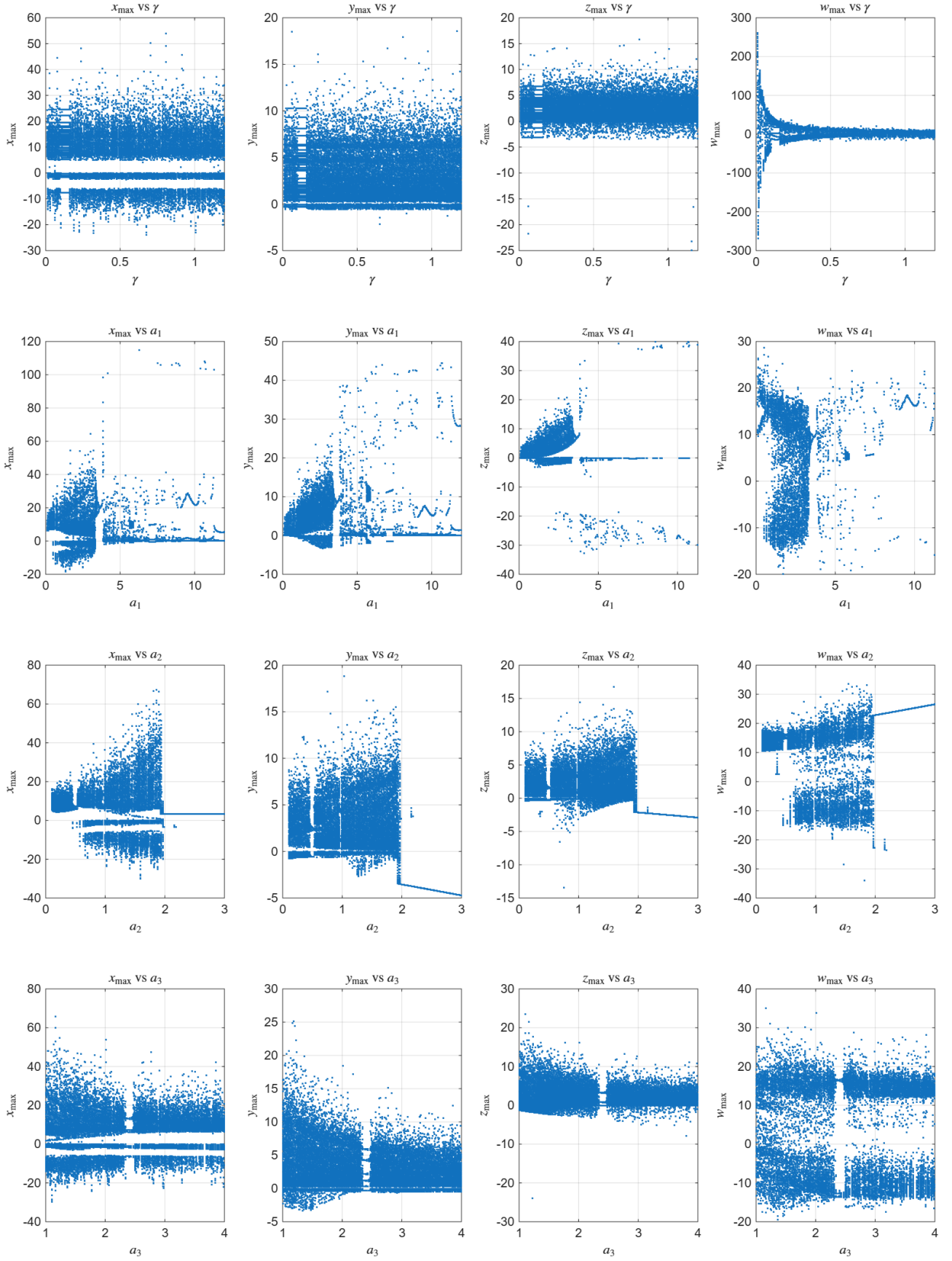


Figure 4. Bifurcation diagrams for the varying parameters γ , a_1 , a_2 , a_3 showing the maxima of $x(t)$, $y(t)$, $z(t)$, and $w(t)$. Each row corresponds to one varying parameter, while all others are fixed at $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$ and $\gamma = 0.3$. The results indicate that γ and a_1 act as stabilizing parameters, while a_2 and a_3 introduce strong nonlinear and chaotic behavior.

To further examine the temporal dynamics of the extended four-dimensional supply chain model, we present the time-series trajectories of the producer output $x(t)$, retailer demand $y(t)$, consumer demand $z(t)$, and inventory level $w(t)$ for the parameter set $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$, with initial condition $(1.2, 1.0, 3.0, 5)$. Over the interval $t \in [0, 200]$, all variables exhibit irregular oscillatory behavior consistent with chaotic dynamics. The production variable $x(t)$ shows rapid fluctuations and intermittent bursts, retailer demand $y(t)$ displays moderate oscillations with sudden transitions, consumer demand $z(t)$ exhibits nonlinear spikes due to downstream interactions, while the inventory variable $w(t)$ evolves more slowly with wider oscillatory

cycles governed by the balance between inflow, outflow, and decay. To highlight the short-term coupling behavior among all compartments, the bottom panel of **Figure 5** presents the combined trajectories of $x(t)$, $y(t)$, $z(t)$, and $w(t)$ over the shorter interval $t \in [0, 20]$. In this window, the four variables exhibit synchronized oscillatory patterns, where peaks in production precede inventory build-up and are followed by corresponding changes in retailer and consumer demand. This reveals a clear sequence of cause-effect relationships within the supply-chain feedback structure. The combined plot emphasizes the nonlinear interdependence of the compartments and further verifies that even in the early stages of system evolution, the model displays sensitive and tightly coupled dynamics.

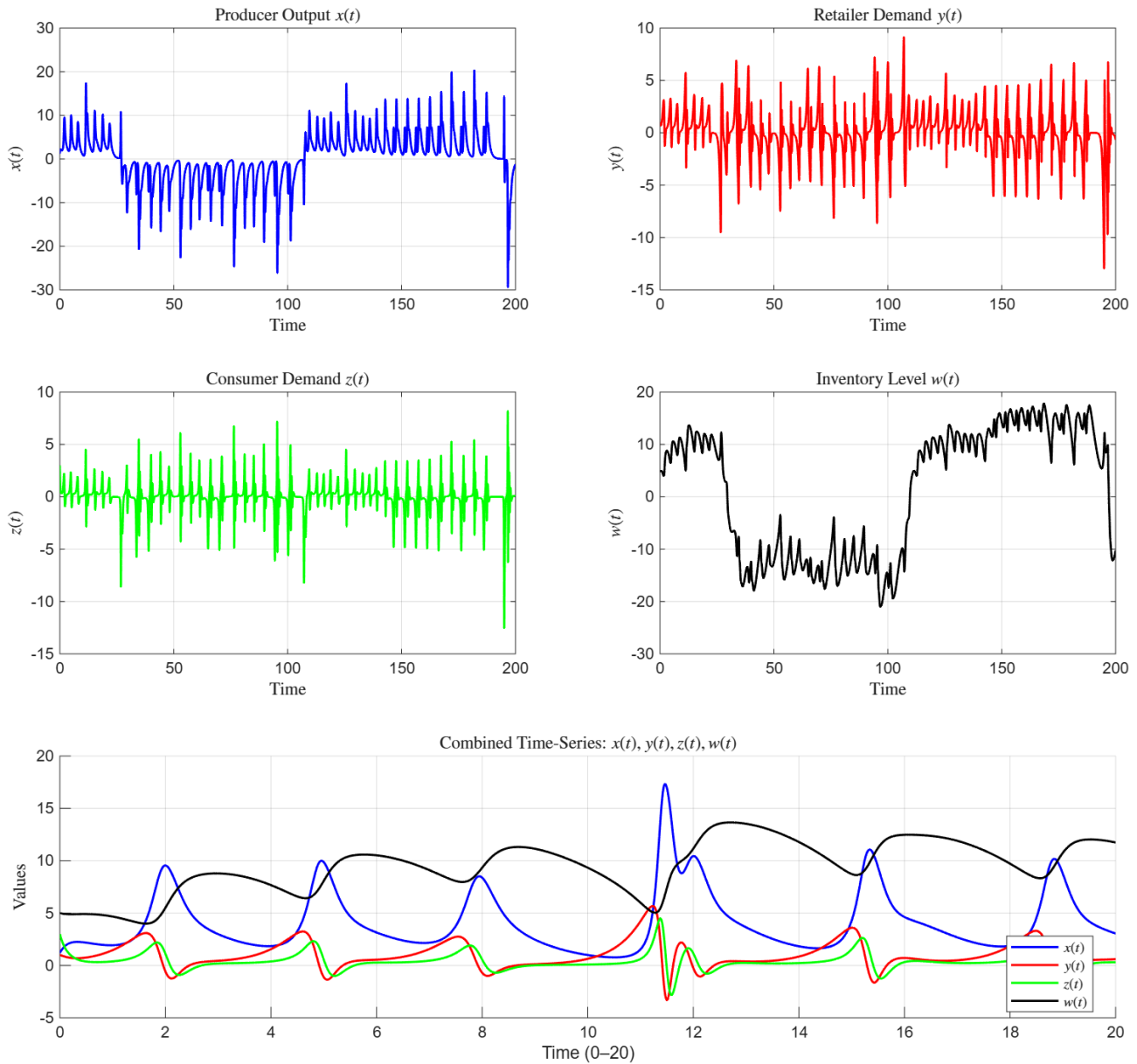


Figure 5. Time-series dynamics of the four state variables of the extended supply-chain model. The first four panels (top two rows) show the individual trajectories of producer output $x(t)$, retailer demand $y(t)$, consumer demand $z(t)$, and inventory level $w(t)$ over the interval $t \in [0, 200]$. The bottom panel displays the combined short-term evolution of all variables over $t \in [0, 20]$, highlighting their relative oscillatory behavior and coupling. All simulations were performed using fixed parameter values $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$, with initial condition $(x(0), y(0), z(0), w(0)) = (1.2, 1.0, 3.0, 5)$.

To examine the geometric structure of the nonlinear interactions in the extended supply chain model (5)–(8), the nullclines were computed for the parameter set $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$. Using the nullcline relations obtained in Lemma 4, the (x, y) -nullclines for fixed consumer demand z are $N_x : y = \frac{x(a_1 - a_2 z)}{a_3 z}$ and $N_y : y = \frac{a_5}{a_4} x z$, which correspond to the stationary states of producer output and retailer

demand, respectively. **Figure 6** presents several representative (x, y) slices for $z \in \{0, 0.05, 0.5941, 0.5, 1\}$ together with the critical values of z where the nullclines become parallel or coincide. These values satisfy $\frac{a_1 - a_2 z}{a_3 z} = \frac{a_5}{a_4} z$, which yields the quadratic condition $\frac{a_5 a_3}{a_4} z^2 + a_2 z - a_1 = 0$. The resulting roots identify consumer-demand levels where the nullcline geometry changes significantly, indicating possible multistability and nonlinear transitions in the supply chain dynamics.

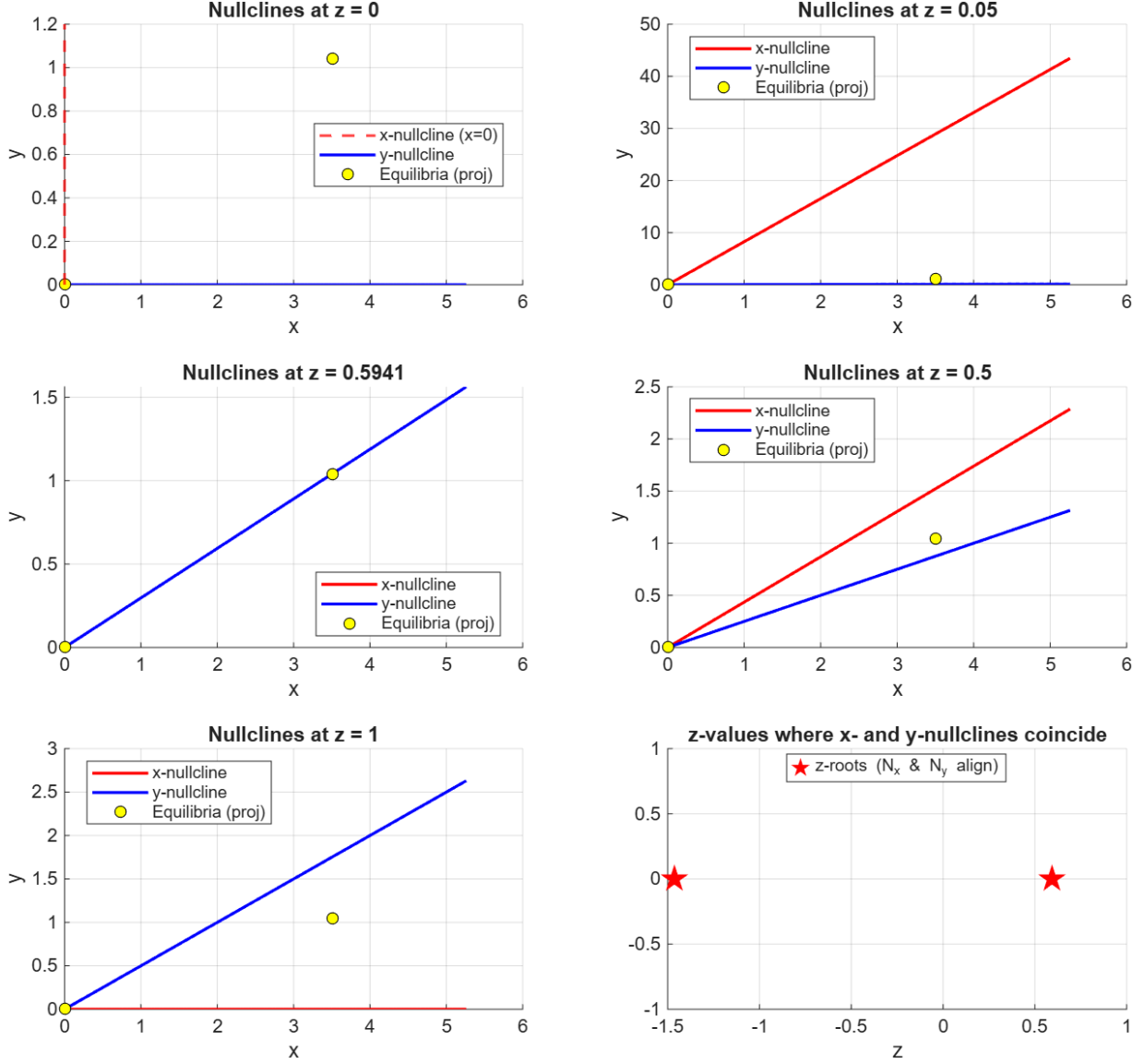


Figure 6. Nullcline analysis of the extended supply-chain model for five fixed z -values and one compatibility panel. Parameters are fixed at $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, $\gamma = 0.3$. The first five subplots show the analytical x -nullclines (red) and y -nullclines (blue) in the (x, y) -plane for different values of z , while yellow markers indicate projections of numerically computed equilibria. The final panel shows the special z -values solving the compatibility condition $\frac{a_5 a_3}{a_4} z^2 + a_2 z - a_1 = 0$, representing points where N_x and N_y become parallel or coincide.

Figure 7 illustrates the time histories of the synchronization errors $e_x(t)$, $e_y(t)$, $e_z(t)$, and $e_w(t)$ for the extended four-dimensional supply-chain system under the proposed control laws. As predicted by the analytical error system, all error components decay exponentially to zero within a short time interval. The error $e_x(t)$ exhibits smooth, monotonic convergence from its initial negative value toward zero, in agreement with the linear error dynamics $\dot{e}_x = -a_1 e_x$. The component $e_y(t)$ converges extremely rapidly due to the stabilizing gain $k_y > a_4$, consistent with the theoretical equation $\dot{e}_y = (a_4 - k_y)e_y$. Likewise, $e_z(t)$ decays exponentially in accordance with $\dot{e}_z = -a_6 e_z$. The fourth error variable $e_w(t)$ also approaches zero as prescribed by the linear relation $\dot{e}_w = -(\gamma + k_w)e_w$. Overall, the numerical simulations are in perfect agreement with the theoretical

analysis: the cancellation of nonlinear mismatch terms, combined with the introduction of appropriate linear stabilizing gains, guarantees exponential convergence of all synchronization errors. Consequently, the driving and response systems achieve fast and complete synchronization under the proposed control strategy. The nontrivial equilibria satisfy $x^2 = \frac{a_4}{a_5 a_7}(a_6 - a_8 z)$, $y = \frac{a_5}{a_4} x z$, $w = \frac{x - y}{\gamma}$, forming two equilibrium branches, while the trivial equilibrium remains unstable in the chaotic regime. **Figures 8 and 9** show that increasing a_7 enlarges the equilibrium branches, variations in a_8 strongly alter the manifold curvature near $a_8 = 0$, and larger γ compresses the equilibrium surface in the w -direction, indicating reduced overshoot and improved stability. Overall, the results demonstrate how nonlinear coupling and inventory decay shape the equilibrium geometry and long-term dynamics of the supply-chain system.

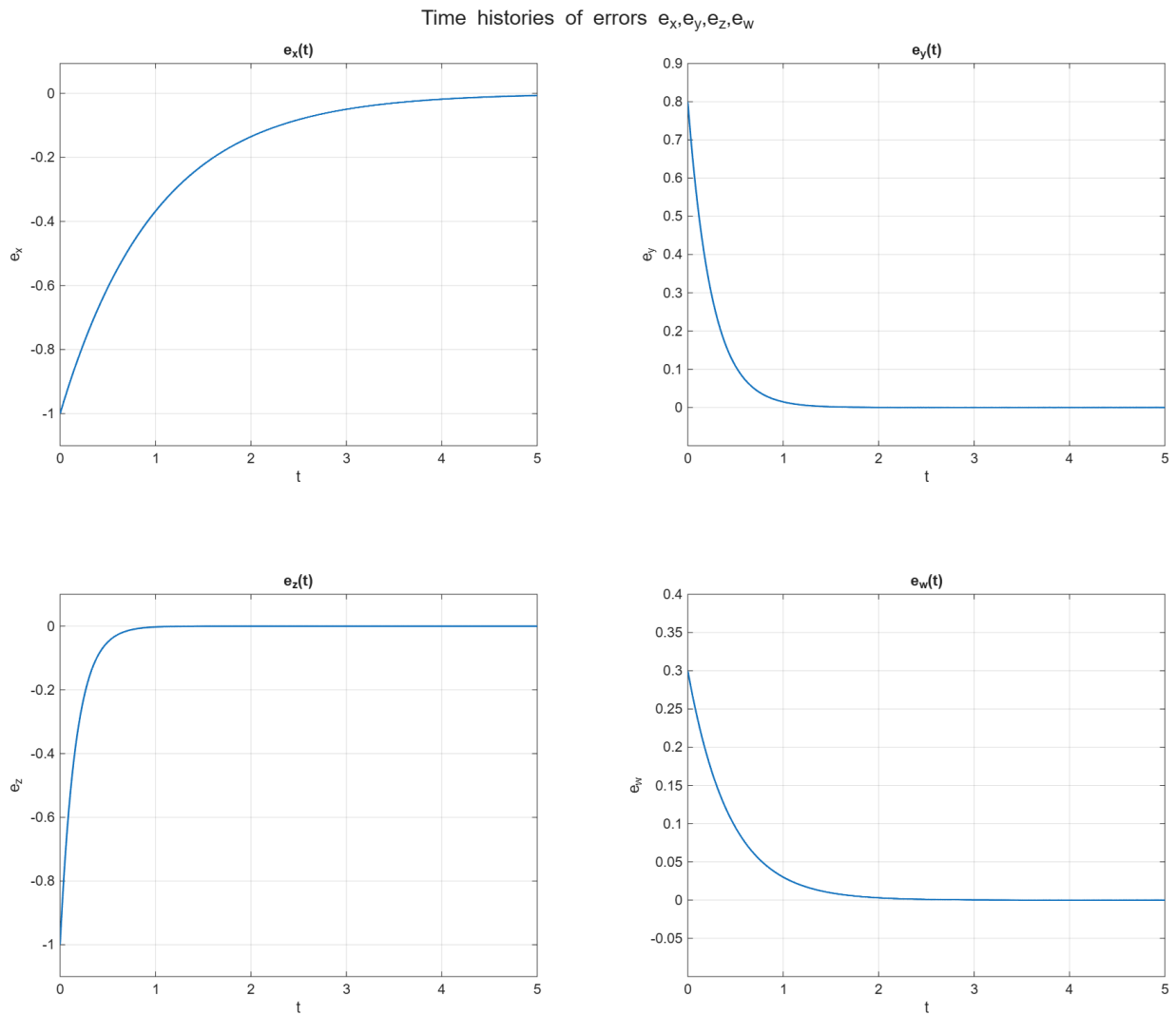


Figure 7. Time histories of the synchronization errors $e_x(t)$, $e_y(t)$, $e_z(t)$, and $e_w(t)$ for the extended four-dimensional chaotic system under the proposed control laws. All error components converge exponentially to zero, confirming successful synchronization between the driving and receiving systems.

Figure 10 illustrates the influence of the parameters a_7 , a_8 , and γ on the equilibrium manifold of the extended supply-chain system. The results show that increasing a_7 enlarges the equilibrium branches, indicating stronger nonlinear coupling between production and consumer demand. Variations in a_8 significantly modify the curvature of the

equilibrium surface, particularly near $a_8 = 0$, where the branches diverge sharply and indicate potential instability. In contrast, increasing the inventory decay rate γ compresses the equilibrium surface in the w -direction, confirming that faster inventory turnover reduces overshoot and improves system stability.

Lyapunov exponents quantify the average exponential divergence or convergence of nearby trajectories, where $LE_1 > 0$ indicates chaos, $LE_1 = 0$ quasi-periodicity, and $LE_1 < 0$ asymptotic stability. **Figure 11** presents the Lyapunov exponent spectra of the extended supply-chain system under variations of a_1 , a_3 , γ , and a_8 , while the remaining parameters were fixed at $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, and $\gamma = 0.3$. The exponents were computed using the Wolf QR algorithm. Increasing a_1 preserves positive LE_1 values, indicating persistent chaos despite stronger production damping.

Increasing a_3 gradually reduces LE_1 , reflecting stabilization of the production–consumer interaction. Larger values of γ weaken memory effects and slightly decrease LE_1 , confirming the stabilizing role of inventory decay. In contrast, as a_8 approaches zero, nonlinear damping weakens and LE_1 increases, leading to stronger chaotic activity. Overall, the Lyapunov spectra confirm that the extended supply-chain model exhibits robust chaotic dynamics whose intensity strongly depends on nonlinear coupling and inventory decay effects.

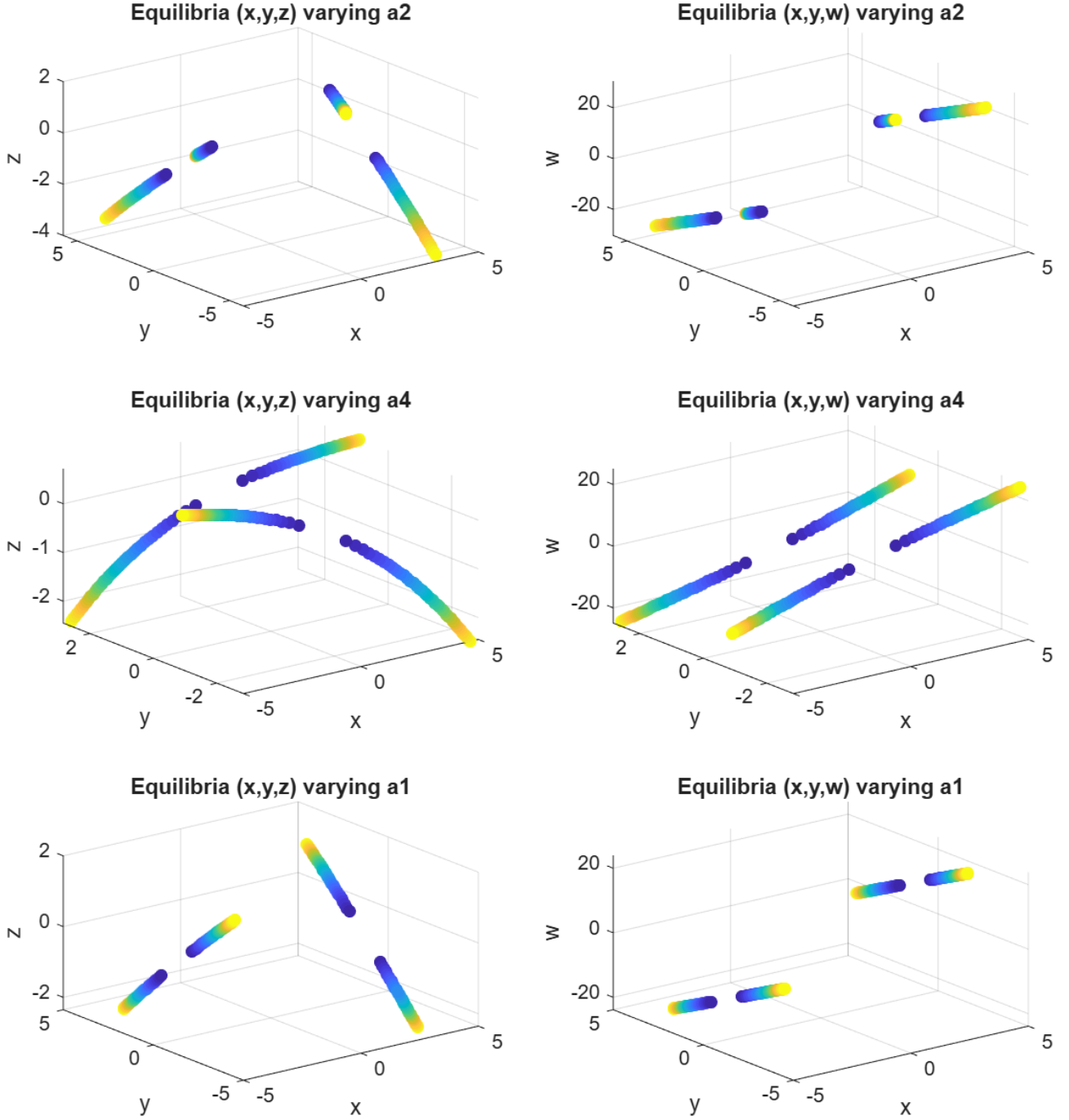


Figure 8. Equilibrium locations in (x, y, z) (left column) and (x, y, w) (right column) as parameters a_2 (top row), a_4 (middle row), and a_1 (bottom row) are varied. All other parameters are fixed at $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \gamma) = (1, 1, 2.3, 2, 1, 6, 1, -0.25, 0.3)$. The colour gradient represents the parameter value. Increasing a_2 or a_4 stretches the equilibrium curve outward along x and y , while varying a_1 shifts both positive and negative branches toward the origin, consistent with its role as a production damping coefficient. The marker colors represent the value of the varied parameter, with blue indicating the minimum parameter value and yellow indicating the maximum parameter value.

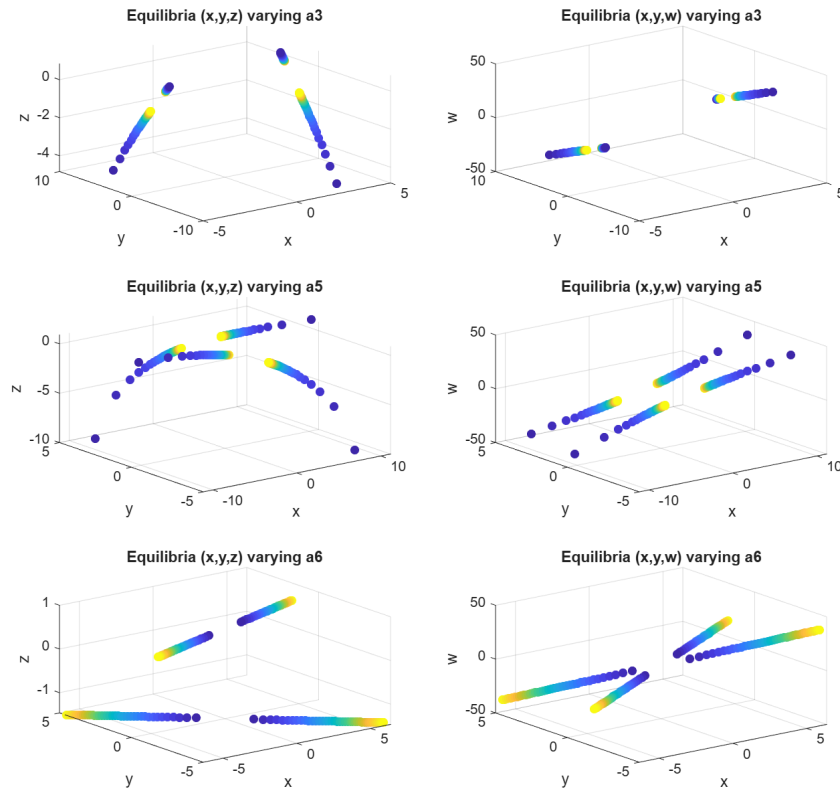


Figure 9. Equilibrium branches in (x, y, z) (left) and (x, y, w) (right) when varying a_3 (top row), a_5 (middle row), and a_6 (bottom row) for fixed parameters as baseline values. The marker colors represent the value of the varied parameter, with blue indicating the minimum parameter value and yellow indicating the maximum parameter value.

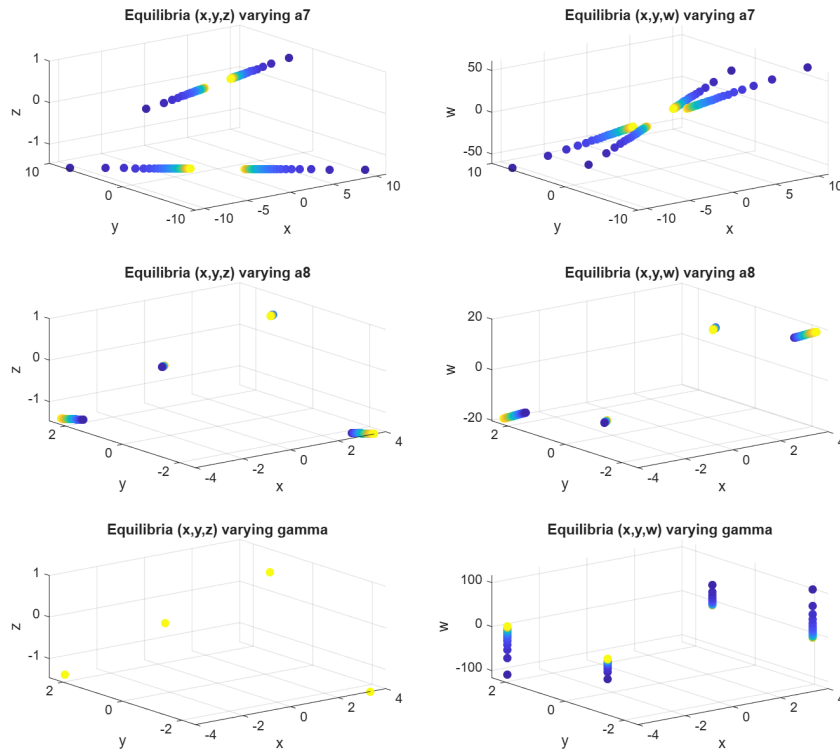


Figure 10. Equilibrium manifolds in (x, y, z) (left column) and (x, y, w) (right column) as the parameters a_7 (top row), a_8 (middle row), and γ (bottom row) are varied. All other parameters are fixed at the baseline chaotic operating regime $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \gamma) = (1, 1, 2.3, 2, 1, 6, 1, -0.25, 0.3)$. The color gradient encodes the parameter value. Increasing a_7 (production-consumer coupling) stretches the equilibrium branches outward, illustrating the strengthening of nonlinear feedback. The parameter $a_8 < 0$ governs quadratic consumer effects: as a_8 approaches 0, the equilibrium manifold bends sharply, reducing the set of feasible consumer states. The inventory decay parameter γ compresses the manifold vertically in the w -direction, showing that faster inventory dissipation stabilizes equilibrium by reducing long-term accumulation. The marker colors represent the value of the varied parameter, with blue indicating the minimum parameter value and yellow indicating the maximum parameter value.

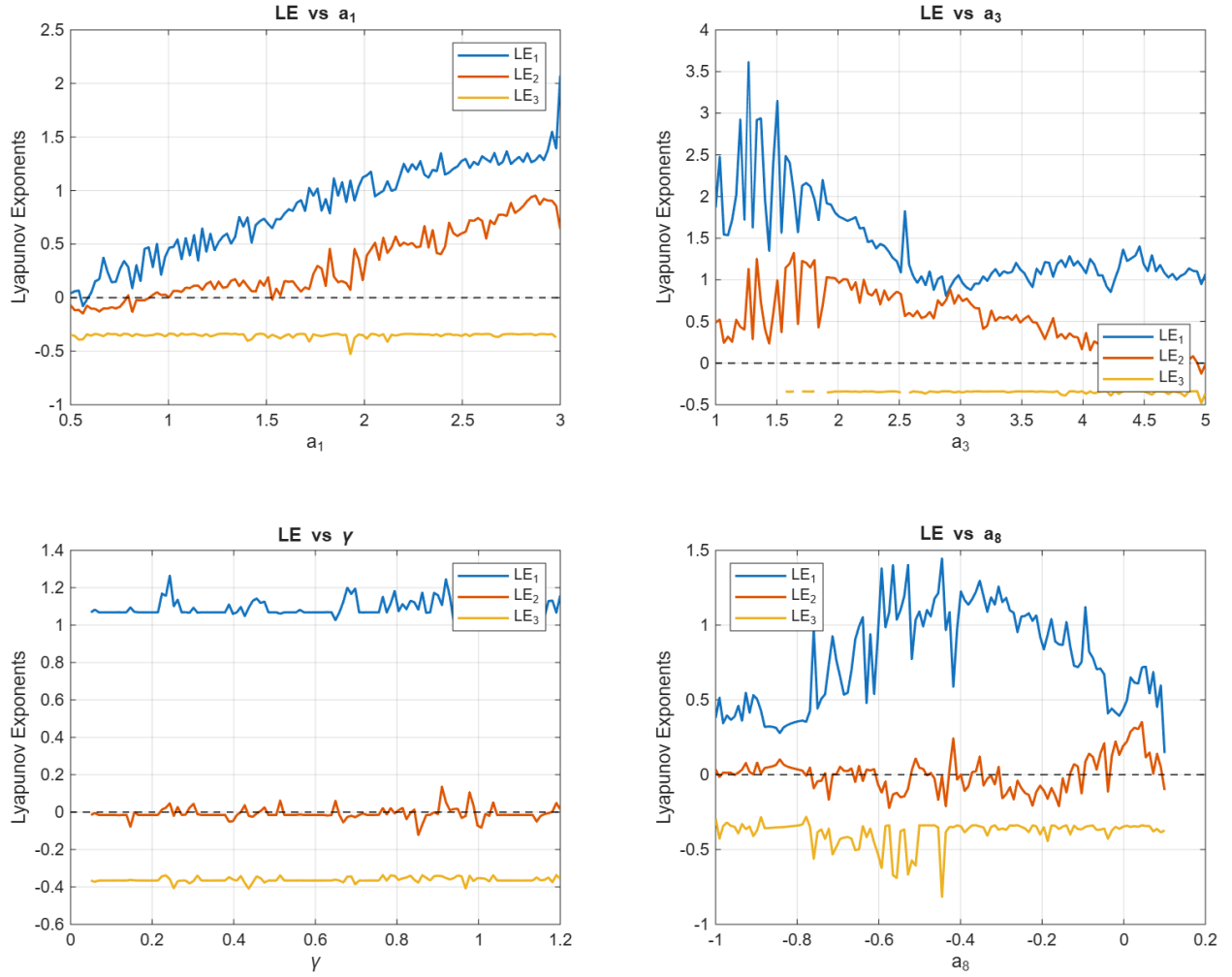


Figure 11. Lyapunov exponent (LE) diagrams of the extended four-dimensional supply-chain model under parameter variations. Each subplot displays the three largest Lyapunov exponents LE_1 , LE_2 , and LE_3 computed using the Wolf QR algorithm for a simulation time of $T_{\max} = 80$, time step $\Delta t = 0.05$, and baseline parameters $a_1 = 1$, $a_2 = 1$, $a_3 = 2.3$, $a_4 = 2$, $a_5 = 1$, $a_6 = 6$, $a_7 = 1$, $a_8 = -0.25$, $\gamma = 0.3$. One parameter is varied in each subplot while all others remain fixed. A positive largest Lyapunov exponent $LE_1 > 0$ indicates chaotic behaviour, while $LE_1 < 0$ indicates periodic or stable dynamics. The second and third exponents (LE_2 , LE_3) describe transversal stability and the contraction of volumes in phase space.

7. Conclusion

This work proposed and analyzed a new four-dimensional chaotic supply-chain model incorporating inventory as an additional state variable. The inclusion of inventory dynamics extends the three-dimensional model of Zheng *et al.*²⁷ and provides a more realistic description of interactions among producers, retailers, consumers, and inventory flow. A qualitative analysis was carried out by deriving the equilibrium conditions, identifying both the trivial and nontrivial equilibria, and obtaining the full Jacobian matrix of the system. Stability analysis showed that the trivial equilibrium is unstable in the physically relevant parameter regime, while the nontrivial equilibrium depends sensitively on production, demand, and inventory-decay parameters. To further investigate the nonlinear dynamics, Lyapunov exponent diagrams were computed for several key parameters, revealing transitions between stable, periodic, and chaotic regimes. The results confirmed the presence of chaos through positive largest Lyapunov exponents and highlighted important bifurcation structures. In addition, a synchronization framework was developed by constructing a controlled response system and deriving a decoupled linear error system. Appropriate nonlinear feedback controllers ensured

exponential convergence of all synchronization errors, which was verified numerically. Overall, the proposed model exhibits rich nonlinear phenomena, including multistability, bifurcation, chaos, and synchronization. The study demonstrates the significant influence of inventory policies and nonlinear feedback mechanisms on supply-chain stability and dynamics. The proposed framework may also serve as a basis for future research on optimal control, fractional-order extensions, delay effects, and data-driven parameter estimation in complex logistic networks.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

Author contributions

This is a single authored article.

Availability of data

None.

AI tools statement

AI-based language tools were used only for grammar checking, language refinement, and improving the readability of the manuscript. All mathematical modeling, theoretical analysis, numerical simulations, interpretations, and scientific conclusions were developed and verified entirely by the authors.

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