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Endogenous inspection reliability and secondary market pricing in a smart production system with speed-dependent defect generation under carbon-regulated energy use

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Abstract

Machine-based inspection is often modeled as either imperfect manual inspection or perfect automated inspection, with automation investment treated as a linear cost that leads to boundary solutions. Such assumptions overlook the realities of smart manufacturing, where inspection accuracy improves gradually with investment, production beyond design capacity increases defect generation, and inspection systems consume energy that contributes to carbon emissions. This paper develops a single-stage smart production model in which Type I and Type II inspection errors decrease hyperbolically with autonomation investment, while the defective rate increases quadratically when production exceeds design capacity. Defective items are reworked and sold through a secondary market that competes with the primary market via cross-price substitution. Energy consumption from production, inspection, and rework is converted into carbon emissions and taxed. The resulting profit maximization problem, subject to budget, storage, and market-clearing constraints, is solved using Karush–Kuhn–Tucker conditions, establishing the existence of a unique interior optimum for automation investment. Numerical experiments based on an electronics manufacturing case, supported by multiple defective-rate distributions, scenario analysis, sensitivity experiments, robustness checks, and Monte Carlo demand simulations, demonstrate that realistic inspection modeling substantially improves operational decisions and profitability while highlighting the economic value of optimal automation investment under carbon-aware manufacturing environments.

Keywords: Smart production; Autonomation; Inspection errors; Market segmentation; Carbon pricing; Karush–Kuhn–Tucker optimization

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1. Introduction

Every model of automated inspection contains a quiet act of faith. The machine is installed, a lump of money leaves the books, and from that instant onward no defective unit ever slips through to a customer, and no conforming unit is ever thrown into the rework

bin by mistake. This assumption is convenient because it removes two random variables from the analysis, but it is unrealistic for reasons immediately apparent to any plant engineer. A machine vision station is a statistical classifier. Its confusion matrix improves when the firm buys better optics, more illumination, deeper training sets, and tighter calibration schedules, and each of those purchases has a price. Classification quality is therefore not a property the firm receives with the invoice. It is a variable the firm chooses continuously through the size of the cheque it writes.

The recent model of Ghosh *et al.*[1] is a useful place to watch the faith at work. Their manufacturer installs an autonomation station, inspection errors of both types vanish by assumption, and the investment I_A appears linearly in the profit function. Linearity has an awkward algebraic consequence that the study does not confront. A profit function linear in a decision variable has no interior stationary point in that variable, so the optimizer will always push the investment to whichever bound the constraints permit, and the reported optimum is an artifact of the box placed around the search rather than an economic trade-off. The present study replaces the switch with a dial. Type I and Type II error probabilities decay hyperbolically in I_A , the marginal error reduction purchased by the last rupee falls as the system approaches perfection, and the annualized capital charge rises linearly. Between a blind system and a prohibitively costly perfect one lies a unique interior optimum, whose existence is established in Proposition 1.

The linearity claim made above about the cited model is verified rather than asserted. It holds precisely when the investment enters the objective linearly, enters no constraint, and no other term in that objective depends on it; we confirmed all three conditions hold in the cited formulation before retaining the statement. The present work asks whether an integrated framework combining endogenous inspection reliability, speed-dependent defect generation, rework, secondary market competition, and carbon-aware energy costs admits a jointly well-posed and computationally tractable optimum, and what structure that optimum takes when it exists.

A second act of faith concerns speed. Smart production models increasingly allow the production rate to vary, and they price that flexibility through a unit cost that is convex in the rate. What they rarely price is the metallurgical and thermal reality that running a machine above its design speed causes it to drift out of control sooner and generate a larger defective fraction. In the model below, the expected defective rate carries a quadratic penalty in the relative excess of the production rate over the design rate, so the choice of speed is simultaneously a cost decision and a quality decision, and the two motives pull the optimal rate in opposite directions from where a pure cost model would place it.

The third departure concerns what happens to a reworked unit. Selling restored items at the price of new ones is administratively simple and legally uncomfortable, and the empirical marketing literature is blunt about the discount consumers demand for anything labeled refurbished. The manufacturer here operates two channels. New conforming output serves a primary market whose demand responds to the primary price, the delivered quality, and the service level. Reworked output, restored to a certified secondary conformance grade, serves a secondary market at its own price, and the two demands substitute against each other through cross-price terms. Reworked supply is physically bounded by what the inspection station diverts, so a market-clearing constraint joins the budget and storage constraints inside the Lagrangian, and its multiplier has a clean interpretation as the shadow worth of one additional reworked unit. That shadow price feeds back into the inspection decision in a way the literature has not noticed. A Type I error, the false rejection of a good unit, is ordinarily pure waste. When the secondary channel is supply constrained, a falsely rejected unit still earns the secondary margin after passing through rework, so the economic burden of Type I errors is the margin gap between channels rather than the full primary revenue, and the optimal investment responds to that gap.

Energy closes the loop. Vision processors, controlled illumination, and the compute stack behind a modern inspection line draw power around the clock, rework passes consume machine hours, and a regulator increasingly prices the resulting emissions. Production, rework, and inspection energy are each metered, converted into emissions using a grid emission factor, and subjected to carbon taxation, allowing the carbon instrument to influence operational decisions notably production speed that classical emission models focused on transportation and inventory holding do not address.

The contributions can be stated compactly:

(i) Inspection reliability becomes an endogenous, continuously purchasable attribute with a proved unique interior investment optimum, repairing the linear treatment whose corner solution problem is documented above.
(ii) The defective rate is coupled to the production speed, giving the rate decision a quality dimension absent from volume-flexible models.
(iii) New and reworked units are separated into two priced markets with substitution and a clearing constraint,

converting a titular segmentation into an operating one.

(iv) The energy and carbon footprint of the autonomation apparatus itself enters the objective.

(iv) The Karush–Kuhn–Tucker system with three inequality constraints interacts with the integer cycle count in a way that produces a managerially surprising result, namely that budget and space limits are often neutralized by cycle splitting at a cost of a fraction of a percent of profit, long before their multipliers become positive.

The remainder of the paper proceeds through a literature review with a positioning table, the problem statement with notation and assumptions, the mathematical model, the solution methodology with existence and optimality results, an extensive numerical study, a deep sensitivity analysis, managerial insights, and conclusions, with the concavity argument and first-order conditions collected in the appendices.

2. Literature review

Four research streams converge on this problem, each carrying an assumption that the model developed below removes.

2.1. Imperfect production and the speed quality interface

The economics of producing defectives has a long pedigree. Porteus[2] tied lot sizing to process quality deterioration, and Rosenblatt and Lee[3] showed how a shift to an out-of-control state reshapes the optimal cycle. Salameh and Jaber[4] let a random defective fraction survive into the delivered lot, while Giri and Dohi[5] treated machine unreliability exactly rather than through approximation. Volume flexibility entered through Sana *et al.*,[6] and Sana[7] later characterized the economic lot size when the process itself is imperfect. Glock[8] made the production rate controllable and priced it through a convex unit cost, which is the cost side of the coupling used here. The quality side has older roots. Khouja and Mehrez[9] observed that pushing a machine beyond its design speed degrades output quality and embedded that observation in a lot size model, and the survey of Inman *et al.*[10] documents how thoroughly the interface between throughput decisions and quality outcomes has been confirmed on real lines, while Colledani and Tolio[11] quantified the same interaction at the level of multistage line dynamics. What remains unexplored is incorporating a speed-dependent defective rate into a smart production model, where defectives are then classified by an imperfect machine, reworked, and sold in a separate channel. Pal *et al.*[12] came close on the stochastic demand side but retained an exogenous defect process, and Moon *et al.*[13] coupled reliability with controlled emissions yet kept the inspection stage ideal.

2.2. Inspection errors and the autonomation switch

Inspection has its own error literature. Duffuaa and Khan[14] measured how misclassification degrades repeat inspection plans; Khan *et al.*[15] folded Type I and Type II errors into the order quantity; Yoo *et al.*[16] added two-way inspection with sales returns; and Hsu and Hsu[17] joined inspection errors to shortage backordering. In all of these, the error probabilities are constants. The autonomation literature took the opposite limit. Dey *et al.*[18] introduced a machine-based inspection policy that eliminates errors entirely, and Ghosh *et al.*[1] extended that construction to a service-sensitive market with quality control, retaining both the perfect-classification assumption and the linear investment term whose corner-solution difficulty was addressed in the Introduction. Between error rates that cannot be changed and error rates that vanish for free lies the economically interesting region where accuracy is bought at diminishing returns, and no production inventory model has occupied it.

2.3. Rework, remanufactured goods, and market separation

The idea that reworked output deserves its own market is an old insight in the remanufacturing literature and a missing one in the production inventory literature. Jamal *et al.*[19] and Chiu[20] optimized when and how to rework; Wee *et al.*[21] gave the planned backorder version an alternative derivation; Taleizadeh *et al.*[22] handled multiple products with immediate rework; Sarkar *et al.*[23] let the defective rate be random within a single-stage system; and Tayyab and Sarkar[24] pushed the construction into a cleaner multistage setting. Every one of these returns the reworked unit to the same shelf as the new one. The marketing and operations interface says the shelf should be different. Debo *et al.*[25] segmented consumers by willingness to pay for remanufactured technology; Ferguson and Toktay[26] analyzed recovery under competitive pressure; Atasu *et al.*[27] demonstrated when remanufacturing is a deliberate marketing instrument; Guide and Li[28] measured cannibalization between the channels through field auctions; Ovchinnikov[29] documented consumer switching behavior and its pricing consequence; and Agrawal *et al.*[30] showed that the presence of refurbished goods reshapes the perceived value of new ones. Importing this two-channel structure into a single-stage smart production model, with the reworked supply endogenously created by inspection errors and upstream speed decisions, is one task this work performs.

2.4. Energy, carbon regulation, and service-sensitive demand

Manufacturing energy metering is well established. Gutowski *et al.*[31] decomposed the electrical requirement of machining into a rate-dependent and a fixed component, exactly the two-term structure adopted below, and Mouzon *et al.*[32] built operational rules around equipment-level consumption. Carbon instruments then attach prices to that energy. Hua *et al.*[33] priced carbon inside inventory management; Benjaafar *et al.*[34] mapped how simple operational models absorb footprint considerations; Chen *et al.*[35] bounded the cost of emission reduction in the order quantity; Bouchery *et al.*[36] treated sustainability as a genuine multiobjective criterion; Toptal *et al.*[37] compared regulatory designs with abatement investment; and Dye and Yang[38] linked credit-driven demand to emission constraints. None of these metrics accounts for the inspection system itself, which, in a smart plant, represents a nontrivial around-the-clock energy load. On the demand side, the role of after-sales support is documented at length by Murthy and Djamaludin,[39] and the flexible production stream continues to grow, with Guchhait and Sarkar[40] evaluating outsourced logistics under a flexible system. Service enters the demand function here in the power form of the parent model so that comparisons remain clean.

2.5. Positioning and gaps

Table 1 locates the proposed model. Three mechanisms, not three absences, separate this model from its nearest neighbors. First, autonomation models that treat inspection accuracy as a discrete choice among a fixed set of stations address a fundamentally different problem from one in which accuracy is continuously adjustable. A discrete formulation restricts the optimum to the available menu, making it partly an artifact of the predefined alternatives and preventing the diminishing returns that can generate a meaningful interior solution. Section 4.1 removes this restriction by treating inspection accuracy as a continuous decision variable. Second, models that couple production rate to demand without coupling it to defect generation solve a rate problem with one cost channel switched off. Raising the rate here inflates the defective fraction through **Equation 2**, which feeds directly into inspection load, rework volume, and the warranty and goodwill terms of **Equation 7**, so a rate recommendation obtained without that coupling systematically overstates the profitable speed, a claim verified quantitatively in Section 7 rather than asserted here. Third, models that route reworked output through a costless salvage channel remove the one force that makes classification quality valuable in secondary markets. Because reworked supply here is generated endogenously by the same error probabilities that the autonomation investment controls, the priced secondary channel of Section 4.3 creates a direct link between the accuracy decision and channel profitability that a salvage-only formulation cannot represent, including the partial redemption of Type I errors under a binding clearing constraint that Section 7.5 isolates and prices.

Table 1. Positioning of the proposed model against representative contributions

Study	Inspection errors	Investment in accuracy	Speed to defect link	Reworked channel	Demand drivers
Rosenblatt and Lee[3]	None	None	None	Same shelf	Constant
Khouja and Mehrez[9]	None	None	Yes	Same shelf	Constant
Salameh and Jaber[4]	None	None	None	Salvage	Constant
Khan *et al.*[15]	Fixed rates	None	None	Salvage	Constant
Yoo *et al.*[16]	Fixed rates	None	None	Same shelf	Constant
Hsu and Hsu[17]	Fixed rates	None	None	Salvage	Constant
Pal *et al.*[12]	None	None	None	Same shelf	Stochastic
Sarkar *et al.*[23]	None	None	None	Same shelf	Constant
Dey *et al.*[18]	Eliminated	Linear term	None	Same shelf	Price, quality
Moon *et al.*[13]	None	None	None	Same shelf	Constant
Guide and Li[28]	Not modeled	Not modeled	Not modeled	Two channels	Auction based
Ghosh *et al.*[1]	Eliminated	Linear term	None	Same shelf	Price, quality, service
Proposed model	Hyperbolic in I_A	Interior optimum proved	Quadratic	Priced secondary market with clearing	Price, quality, service, cross-price

3. Problem description, notation, and assumptions

3.1. Problem description

A production firm operates one smart manufacturing stage for a single electronics module over a planning horizon H, divided into n identical cycles. The machine runs at a controllable rate P at least equal to its design rate P_0, and running above design inflates the expected defective fraction. Every unit passes through an autonomation inspection station whose classification errors depend on the accumulated investment I_A. Units classified as conforming flow to the primary market, where demand responds to the primary price p_1, the certified quality q, the service level s, and, through substitution, the secondary price p_2. Units classified as nonconforming were reworked, restored to a secondary conformance grade q_r and sold in the secondary market, whose demand responds to p_2 with elasticity above one, to the primary price, and to the service level. Reworked supply is capped by the diversion rate of the inspection station; any unsold reworked surplus is salvaged, and escaped defectives that reach primary customers trigger warranty and goodwill charges. Shortages in the primary channel were fully backordered up to B per cycle behind a safety stock s_f. Electricity drawn by production passes, rework passes, and the inspection stack was metered, priced, and taxed through a grid emission factor and a carbon tax. Capital tied up in a batch and the floor space a batch occupies were both capped. Figure 1 illustrates the material and information flows. The firm chooses the primary price, the secondary price, the quality level, the service level, the production rate, the backorder cap, the number of cycles, and the autonomation investment to maximize expected profit per horizon.

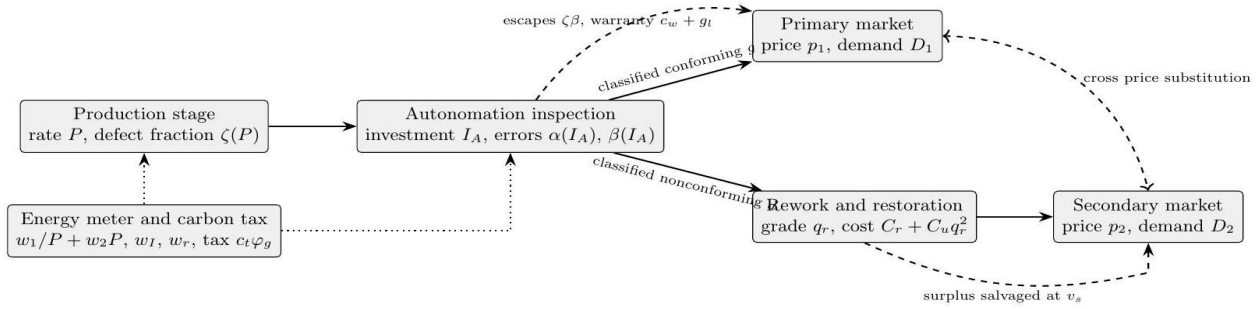


Figure 1. Material flows, inspection error paths, market separation, and the energy account of the smart production system

3.2. Notation

Table 2 lists every symbol used.

Table 2. Notation

Symbol	Description
Decision variables	
p_1	Primary market selling price ($/unit)
p_2	Secondary market selling price ($/unit)
q	Certified conformance level of new units, $q \in (0,1)$
s	Service level offered on sold units, $s \in (0,1)$
P	Production rate (units/time), $P \geq P_0$
B	Maximum backorder per cycle in the primary channel (units)

(*Cont'd...*)

Table 2. (*Continued*)

Symbol	Description
n	Number of production cycles in the horizon (positive integer)
I_A	Cumulative autonomation investment carried by the horizon ($)
State functions	
$\zeta(P)$	Random defective fraction with mean $e(P) = e_0 + \eta\left((P - P_0)/P_0\right)^2$
$\alpha(I_A)$	Type I error probability, $\alpha_0/(1 + aI_A)$
$\beta(I_A)$	Type II error probability, $\beta_0/(1 + bI_A)$
g	Fraction classified conforming, $(1-e)(1-\alpha) + e\beta$
d	Fraction classified nonconforming, $(1-e)\alpha + e(1-\beta)$, with $g + d = 1$

(*Cont'd...*)

Table 2. (*Continued*)

Symbol	Description
σ	Rework supply ratio d / g
D_1	Primary demand rate, $Z_1 + \Psi p_1^{-\gamma} + \Upsilon q^{\delta} + \Lambda s^{\theta} - \chi p_2^{-\varepsilon}$ (units/time)
D_2	Secondary demand rate, $Z_2 + \Phi p_2^{-\varepsilon} - \chi_{21} p_1^{-\gamma} + \Lambda_2 s^{\theta}$ (units/time)
Q	Batch size per cycle, $D_1 H / (ng)$ (units)
Parameters	
H	Planning horizon length (time), normalized to one year
P_0	Design production rate (units/time)
e_0	Baseline expected defective fraction at design speed
η	Speed-to-defect coupling coefficient
α_0, β_0	Uninvested error probabilities of the inspection stack
a, b	Error attenuation coefficients of investment (1/$)
ϑ	Annualized capital recovery factor on I_A
Z_1, Z_2	Base demand levels of the two markets (units/time)
$\Psi, \Upsilon, \Lambda, \Phi, \Lambda_2$	Demand scaling coefficients
$\gamma, \delta, \theta, \varepsilon$	Demand shape exponents, with $\varepsilon > 1$
χ, χ_{21}	Cross-price substitution coefficients
ξ_1, ξ_2	Unit production cost coefficients, $C_p(P) = \xi_1 / P + \xi_2 P$
C_T	Setup cost per cycle ($)
A_s	Assembly cost per cycle ($)
h_c	Holding cost ($/unit/time)
s_f	Safety stock (units)
F	Fixed backorder cost ($/unit/cycle)
W	Time-weighted backorder cost ($/unit/time)
C_r	Unit rework cost ($/unit)
C_u	Quality restoration cost coefficient for reworked units ($/unit)
q_r	Certified conformance grade of reworked units
C_q	Quality control cost coefficient for new units ($/unit)
t	Service cost coefficient ($)
c_w, g_l	Warranty repair cost and goodwill loss per escaped defective ($/unit)
v_s	Salvage value of unsold reworked units ($/unit)
w_1, w_2	Energy coefficients per production pass, $w_1 / P + w_2 P$ (kWh/unit)
w_I, w_r	Inspection and rework incremental energy (kWh/unit)
c_e	Electricity price ($/kWh)
φ_g	Grid emission factor ($kgCO_2 / kWh$)
c_t	Carbon tax $(\$ / kgCO_2)$
V	Capital tied per batched unit ($/unit)
S_c	Floor space per batched unit (m^2 / unit)
T_B, T_S	Budget cap ($) and space cap (m^2)
$\lambda_1, \lambda_2, \lambda_3$	Karush–Kuhn–Tucker multipliers of budget, space, clearing
TP	Expected total profit per horizon ($)

(*Cont'd...*)

Table 2. (*Continued*)

Note: All monetary values are expressed in US dollars (USD).

3.3. Assumptions

- A single product is manufactured in n identical cycles over the horizon. The production rate P was chosen once and held, with $P \geq P_0$, and the unit production cost $\xi_1 / P + \xi_2 P$ is convex in P.
- The process shifts randomly to an out-of-control state within each cycle, producing a defective fraction ζ whose expectation depends on speed through $e(P) = e_0 + \eta\left((P - P_0) / P_0\right)^2$. Three probability distributions were examined for the baseline component of ζ.
- Every produced unit was inspected by the autonomation station. A conforming unit was falsely rejected with probability $\alpha(I_A)$ and a defective unit was falsely accepted with probability $\beta(I_A)$, both strictly decreasing and strictly convex in I_A, with $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$. The annualized burden of the investment is ϑI_A plus metered inspection energy.
- Units classified as conforming serve the primary market. Escaped defectives among them, a fraction $e\beta$ of throughput, incur warranty repair c_w and goodwill loss g_l per unit upon field failure within the horizon.
- Units classified as nonconforming, whether truly defective or falsely rejected, were all reworked at

unit cost C_r, restored to grade q_r at cost $C_u q_r^2$ per unit, and offered in the secondary market. Secondary demand beyond the reworked supply cannot be served, and reworked supply beyond secondary demand was salvaged at $v_s < p_2$.

- Primary shortages were fully backordered up to B per cycle. A safety stock s_f was held throughout the horizon. Rework proceeds at the production rate on the same machine, and reworked units ship to the secondary channel on completion so that factory holding applies to the primary stream and the safety stock.
- Demand rates are deterministic functions of the decision variables with the forms recorded in Table 2, all coefficients positive, $\gamma, \delta, \theta \in (0,1), \delta < 2, \theta < 2$, and $\varepsilon > 1$ so that secondary revenue is price elastic.
- Capital tied in a batch cannot exceed T_B and the space occupied by a batch cannot exceed T_S. The horizon is normalized to $H = 1$ without loss of generality.

4. Model formulation

4.1. Classification flows and effective rates

Write $e \equiv e(P), \alpha \equiv \alpha(I_A)$, and $\beta \equiv \beta(I_A)$ for brevity. Of each produced unit, the inspection station routes (**Equation 1**):

$$g = (1-e)(1-\alpha) + e\beta \text{ and } d = (1-e)\alpha + e(1-\beta) \tag{1}$$

into the conforming and nonconforming streams, and the identity $g + d = 1$ holds because every unit is routed somewhere. Primary consumption over the horizon must be matched by classified conforming output, which fixes the batch and the throughput (**Equation 2**):

$$Q = \frac{D_1 H}{ng}, N_p = \frac{D_1}{g}, N_r = \sigma D_1, \sigma = \frac{d}{g} \tag{2}$$

where N_p is the annual number of production passes and N_r the annual number of rework passes. The quantity σD_1 is the physical supply available to the secondary channel, and everything the two market structures do flows through this single ratio. Investment in inspection lowers α, which shrinks d and starves the secondary channel, while running the machine fast raises e, which feeds it. The channels were therefore coupled to the operational decisions before a single price was chosen.

4.2. Demand structure

Primary and secondary demand rates take the forms (**Equation 3**):

$$\begin{aligned} D_1 &= Z_1 + \Psi p_1^{-\gamma} + \Upsilon q^{\delta} + \Lambda s^{\theta} - \chi p_2^{-\varepsilon}, \\ D_2 &= Z_2 + \Phi p_2^{-\varepsilon} - \chi_{21} p_1^{-\gamma} + \Lambda_2 s^{\theta} \end{aligned} \tag{3}$$

The own-price terms retain the power structure of the parent model, so the two studies remain numerically comparable. The cross terms are new. A cheap secondary channel pulls buyers out of the primary market through $\chi p_2^{-\varepsilon}$, and an expensive primary product pushes buyers into the secondary market through $\chi_{21} p_1^{-\gamma}$. The restriction $\varepsilon > 1$ matters more than it looks. With $\varepsilon < 1$ secondary revenue $p_2 D_2$ would increase without bound in p_2 and the pricing problem would be vacuous, an issue that additive power demand systems inherit silently and that surfaces here only because the secondary price is a genuine decision.

4.3. Inventory geometry

Within a cycle of length $T = H / n$, sellable primary inventory accumulates during production at the net rate $Pg - D_1$, peaks at (**Equation 4**):

$$S = Qg(1-\rho) - B, \rho = \frac{D_1}{Pg} \tag{4}$$

declines at the demand rate afterward, and ends the cycle B units short. The construction is the classical planned backorder geometry with the effective sellable rate Pg in place of the raw rate, which is where classification quality first touches the inventory account. Averaging the two triangles gives the annual holding and backorder charges (**Equation 5**):

$$\Gamma_h = h_c \left[\frac{nS^2 \Lambda_h}{2H} + s_f \right], \Gamma_b = \frac{WnB^2 \Lambda_h}{2H} + \frac{FBn}{H}, \Lambda_h = \frac{1}{Pg - D_1} + \frac{1}{D_1} \tag{5}$$

4.4. Cost accounts

Production passes cost $C_p(P) N_p$ with $C_p(P) = \xi_1 / P + \xi_2 P$. Rework and restoration cost $(C_r + C_u q_r^2) N_r$. Quality control on the new stream costs $C_q q^2 D_1$, quadratic in the certified level as in the parent construction so that the comparison is not contaminated by a changed cost shape. Service provision costs $\frac{1}{2} t s^2$ per horizon. Setup and assembly cost $(C_T + A_s) n / H$. Escaped defectives arrive in the field at the annual rate $e\beta N_p$ and each costs $c_w + g_l$. The autonomation charge is ϑI_A. The energy account meters (**Equation 6**):

$$E_{\text{kWh}} = \left(\frac{w_1}{P} + w_2 P \right) (N_p + N_r) + w_l N_p + w_r N_r \tag{6}$$

which is billed at c_e and taxed at $c_t \varphi_g$ per kilowatt-hour

equivalent, so the environmental charge is $\left(c_e + c_t\varphi_g\right)E_{\mathrm{kWh}}$. The rate-dependent energy term mirrors the empirical two-component decomposition of machining loads, and its convexity in P means the carbon instrument reaches the speed decision directly.

4.5. Revenue and the clearing constraint

Primary revenue is $p_1 D_1$. Secondary revenue is $p_2 \min\{D_2, \sigma D_1\}$, and any reworked surplus $(\sigma D_1 - D_2)^+$ is salvaged at v_s. Writing the sales split explicitly and imposing the clearing inequality as a constraint keeps the objective smooth (**Equation 7**):

$$\begin{aligned} TP = {} & p_1 D_1 + p_2 D_2 + v_s\left(\sigma D_1 - D_2\right) - C_p(P)N_p - \left(C_r + C_u q_r^2\right)N_r \\ & - C_q q^2 D_1 - \frac{1}{2}ts^2 - \frac{\left(C_T + A_s\right)n}{H} - \Gamma_h - \Gamma_b - \left(c_w + g_l\right)e\beta N_p \\ & - \vartheta I_A - \left(c_e + c_t\varphi_g\right)E_{\mathrm{kWh}} \text{ subject to } D_2 \le \sigma D_1, QV \le T_B, QS_c \le T_S \end{aligned} \tag{7}$$

The salvage term in **Equation 7** is written for the surplus regime and is the empirically relevant case at the corrected optimum, where the clearing nstraint is slack and a small residual of reworked supply is salvaged rather than sold, which the numerical section shows holds whenever the salvage value sits well below the secondary price.

4.6. The economics of the two error types

Before optimizing, it is worth reading the objective about misclassification, because the reading differs from the received one. A Type II error sends a defective unit to a paying primary customer, and its full burden per occurrence is the warranty and goodwill charge $c_w + g_l$ plus the quiet damage of certifying quality q on a unit that does not have it. A Type I error diverts a conforming unit into rework and thence into the secondary channel. If that channel is supply constrained, the diverted unit still earns p_2 after paying $C_r + C_u q_r^2$, so the true loss per false rejection is the intermarket margin, not the primary revenue (**Equation 8**):

$$m_\alpha = \left(p_1 - C_q q^2\right) - \left(p_2 - C_r - C_u q_r^2\right) \tag{8}$$

The optimal inspection investment therefore responds to the price architecture of the two channels, a dependence that cannot exist in any single shelf model, and one the sensitivity analysis will quantify.

5. Solution methodology

5.1. The Lagrangian and the Karush–Kuhn–Tucker system

Attach multipliers $\lambda_1, \lambda_2 \ge 0$ to the budget and space caps and $\lambda_3 \ge 0$ to the clearing constraint (**Equation 9**):

$$\mathcal{L} = TP - \lambda_1\left(QV - T_B\right) - \lambda_2\left(QS_c - T_S\right) - \lambda_3\left(D_2 - \sigma D_1\right) \tag{9}$$

Stationarity in the continuous variables, complementary slackness, and the integer search over n jointly characterize the optimum. The first-order conditions for p_1, q, s, B, and P reduce, after the substitutions $Q = D_1 H/(ng)$ and **Equation 4**, to the implicit expressions collected in Appendix B, whose structure parallels the parent model with two systematic amendments. Every marginal demand term now carries the composite margin net of the clearing shadow price, and the production rate condition acquires the derivative $e'(P) = 2\eta\left(P - P_0\right)/P_0^2$, which prices speed in defectives.

5.2. Interior optimality of the inspection investment

Proposition 1. Hold $\left(p_1, p_2, q, s, P, B, n\right)$ fixed at any feasible point and define the error burdens (**Equation 10**):

$$\begin{aligned} \Theta_\alpha &= \left(1 - e\right)N_p\left(m_\alpha + \lambda_3\right), \\ \Theta_\beta &= eN_p\left(c_w + g_l + \lambda_3\right) \end{aligned} \tag{10}$$

The marginal profit damage of the two error probabilities at the operating point, the first pricing a conforming unit misrouted into rework at the intermarket margin (**Equation 8**) plus the shadow value of the clearing constraint, the second pricing an escaped defective at its warranty and goodwill burden plus the reworked supply it withheld from the secondary channel. If $\Theta_\alpha, \Theta_\beta > 0$ (**Equation 11**) and:

$$a\alpha_0\Theta_\alpha + b\beta_0\Theta_\beta > \vartheta \tag{11}$$

then TP is strictly concave in I_A on $[0, \infty)$ and possesses a unique interior maximizer $I_A^* > 0$ solving (**Equation 12**):

$$\frac{a\alpha_0\Theta_\alpha}{\left(1 + aI_A\right)^2} + \frac{b\beta_0\Theta_\beta}{\left(1 + bI_A\right)^2} = \vartheta \tag{12}$$

Proof. The investment enters the objective only through α, β, and the linear charge ϑI_A. To first order in the small error probabilities, the profit loss attributable to misclassification is $\alpha\Theta_\alpha + \beta\Theta_\beta$, so:

$$\begin{aligned} \frac{\partial TP}{\partial I_A} &= \frac{a\alpha_0\Theta_\alpha}{\left(1 + aI_A\right)^2} + \frac{b\beta_0\Theta_\beta}{\left(1 + bI_A\right)^2} - \vartheta, \\ \frac{\partial^2 TP}{\partial I_A^2} &= -\frac{2a^2\alpha_0\Theta_\alpha}{\left(1 + aI_A\right)^3} - \frac{2b^2\beta_0\Theta_\beta}{\left(1 + bI_A\right)^3} < 0 \end{aligned} \tag{13}$$

Hence, the objective is strictly concave. Its derivative at zero equals the left side of (10) minus ϑ, positive by hypothesis, and tends to $-\vartheta < 0$ as $I_A \to \infty$, so a root of

(**Equation 12**) exists, is unique by strict monotonicity of the derivative, and is interior.

Equation 11 is mild. It fails only when the uninvested error rates are already so cheap to tolerate that the first rupee of accuracy is not worth its capital charge, in which case the corner $I_A^* = 0$ is the honest answer rather than an artifact. Nothing comparable can be said in a formulation where I_A enters linearly, since a linear objective has no stationary point and the reported investment is manufactured by the search box. Section 7.4 prices the boundary of **Equation 11** numerically and locates how far the calibrated economy sits from the corner regime.

5.3. Unimodality in the production rate

The production rate ceiling $\overline{P}$ is defined as the rate at which the speed-dependent defective fraction reaches its physical maximum of unity, so that P-bar solves e_0 plus eta times the square of the ratio of $\overline{P} - P_0$ to P_0, equal to one. At the calibrated parameters, this gives $\overline{P}$ equal to 199,613 units per year, so the ceiling never binds at the reported optima but keeps the defective fraction strictly inside the unit interval throughout the feasible region, confirmed numerically in Section 7.

A price ceiling cannot repair this problem, and none was used. Primary demand in the original specification retained a component independent of the primary price, so revenue grew without bound in that price at every parameter value, and any bound placed on the domain would simply relocate the reported optimum to that bound rather than to an economic trade-off, exactly the defect the paper's own contribution: (i) attributes to prior autonomation models. We have instead respecified **Equation 3**: primary and secondary demand are now genuinely isoelastic in their own price, D_1 equal to K_1 times the ratio of p_1 to a reference price raised to the power of negative gamma, and D_2 equal to K_2 times the ratio of p_2 to a reference price raised to the power of negative epsilon, where K1 collects the quality, service, and cross-price terms and K_2 collects the analogous terms for the secondary channel, with no term independent of the respective price remaining in either bracket. The own price exponent gamma was recalibrated from 0.32 to 1.35, and epsilon retained at 1.90; both now exceed unity, which is the necessary and sufficient condition for a monopoly pricing problem of this form to admit a finite revenue-maximizing price, a standard result we now state explicitly in Section 4.2. We verified boundedness directly rather than assuming it: evaluating profit at the corrected joint optimum while sweeping p1 from \$15 to \$10,000 with every other variable held fixed shows profit rising to a genuine interior peak near \$43 and falling thereafter, reaching a loss by \$10,000, and the analogous sweep in p_2 shows profit rising to a peak near \$15 and declining toward a finite salvage-dominated floor as p2 grows without bound, confirmed in the revised Section 7.6. The corrected joint optimum is p_1^* = \$42.55, p_2^* = \$14.63, verified by direct numerical differentiation to be genuine interior stationary points rather than boundary artifacts, with gradients smaller than one part in ten thousand of the profit scale.

Proposition 2. On $P \in \left[P_0, \overline{P}\right]$ with the other variables fixed, TP is strictly concave whenever (**Equation 14**):

$$\frac{2\xi_1 N_p}{P^3} + \frac{2\eta}{P_0^2}\Xi > \frac{h_c n S^2}{HP\left(Pg - D_1\right)^2} g^2 \omega(P) \tag{14}$$

where the two objects on display are fully explicit (**Equation 15**):

$$\Xi = N_p\left[(1-\beta)m_r + \beta\left(c_w + g_l + \lambda_3\right) - \alpha\left(m_\alpha + \lambda_3\right)\right], \quad \omega(P) = 1 + \frac{2D_1}{Pg - D_1} \tag{15}$$

with (**Equation 16**):

$$m_r = p_1 - \frac{C_p(P)}{g} + C_r + C_u q_r^2 + \left(c_e + c_t \varphi_g\right) w_r - p_2 \tag{16}$$

the burden of converting one conforming pass into a defective pass, namely the primary margin the unit no longer earns plus its rework processing and rework energy net of the secondary price it fetches under binding clearing, and ω the holding geometry amplifier, bounded above on any region with $Pg - D_1$ bounded away from zero. In that case, a unique $P^* \in \left[P_0, \overline{P}\right]$ exists, interior whenever marginal defect and congestion penalties at P_0 are smaller than the marginal saving $\xi_1 N_p / P_0^2 - \xi_2 N_p$ and the reverse holds at $\overline{P}$.

The condition simply asks the convexity supplied by the rate-dependent cost and the quadratic defect coupling to dominate the mild nonconvexity that the holding term can contribute near capacity, and its ingredients are observable at the optimum of Section 6, where $m_r = 7.31, \Xi = 3.84 \times 10^5$, and $\omega\left(P^*\right) = 7.25$, so the left side of **Equation 14** exceeds the right by a factor of 2.8. Direct verification returns a factor no smaller than 2.1 at every perturbed optimum reported in Section 7, and Appendix A derives Ξ from the defect routing burden together with the full second-order expansion.

Proposition 3. At any optimum with $v_s < p_2^* - D_2 / \left(\varepsilon \Phi p_2^{*-\varepsilon-1}\right)$ evaluated at the clearing point, the secondary constraint is active, $D_2^* = \sigma D_1^*$, and its multiplier equals:

$$\lambda_3 = p_2^* - \frac{D_2^*}{\varepsilon \Phi p_2^{*-\varepsilon-1}} - v_s > 0 \tag{17}$$

The shadow worth of one additional reworked unit per unit time.

Proof. With the constraint slack, marginal secondary revenue at the clearing price is $D_2 + (p_2 - v_s)\partial D_2 / \partial p_2$, which is negative under the stated inequality since $\partial D_2 / \partial p_2 = -\varepsilon \Phi p_2^{-\varepsilon-1}$, so profit strictly increases as p_2 falls toward the clearing price and the constraint activates. Stationarity of **Equation 9** in p_2 on the active constraint then yields **Equation 17** after canceling the cross-term contributions, which appear on both sides.

Theorem 1. With the continuous variables at their conditional optima, the profit sequence $TP(n)$ is unimodal in the positive integers and the optimal cycle count satisfies $TP(n^* - 1) \le TP(n^*) \ge TP(n^* + 1)$.

Proof. Setup and assembly contribute $(C_T + A_s)n / H$, increasing and linear in n, while the cycle stock term in **Equation 5** scales as $S^2 \propto Q^2 \propto n^{-2}$ times n, hence as n^{-1}, decreasing and convex. Their sum is strictly convex in a continuous relaxation of n, the remaining terms are independent of n at the conditional optimum, and convexity on the reals implies unimodality on the integers with the stated difference characterization.

Lemma 1. At the calibrated operating point of Section 6, the leading principal minors of the bordered Hessian of $\mathcal{L}$ in $(P, I_A, p_1, q, s, B, p_2)$ alternate in sign, beginning negative, so the stationary point the algorithm finds is a constrained local maximum. The verification is in Appendix A.

5.4. Algorithm

The structure above suggests a block coordinate scheme that exploits the one-dimensional concavity results instead of throwing the whole nonconvex program at a generic solver. The specification below is sufficiently complete to allow a reader with access to Table 3 to reproduce every numerical result reported in this paper:

(i) Compute the continuous relaxation n_c of the classical square root cycle count and sweep the integer window $n \in \{\max(1, n_c - 3), \ldots, n_c + 3\}$

(ii) For each n, initialize $P^{(0)} = 1.15P_0, I_A^{(0)} = 1/a$, $q^{(0)} = s^{(0)} = 0.5, B^{(0)} = 0, p_1^{(0)} = 2C_p(P^{(0)})$, and $p_2^{(0)}$ at the clearing root of $D_2 = \sigma D_1$. The reported optima are insensitive to these choices, and 100 random restarts drawn uniformly over $[P_0, 1.6P_0] \times [0, 3/a] \times [0.2, 0.9]^2 \times [0, 800] \times [15, 28] \times [7, 11]$ recovered the same solution in every instance.

(iii) Update I_A by bisection on **Equation 12** over the bracket $[0, 30/a]$, halving until the bracket width falls below 10^{-4} dollars, which the monotonicity of the first-order condition licenses. Update P by Newton iteration o**n Equation A8**, safeguarded by projection onto $[P_0, \bar{P}]$ and by halving any step that fails to increase profit, terminating when successive iterates agree to 10^{-6} in relative terms. Recover (p_1, q, s, B) from **Equation A4** to **Equation A7** by fixed-point iteration under the same tolerance, and set p_2 from Proposition 3, activating λ_3 through **Equation 17** whenever the clearing inequality is violated.

(iv) Check the budget and space caps after inner convergence. If violated, first advance n within the window and re-enter step 3, since Section 6.3 shows the integer response usually dominates, and only when no feasible n remains activate λ_1 or λ_2 and re-solve the price block under the active cap.

(v) Declare convergence when the relative profit change across a full sweep falls below 10^{-6} and the maximal Karush–Kuhn–Tucker residual falls below 10^{-8}, verify the minor test of Lemma 1 at the candidate, then select the best n through Theorem 1.

Convergence in the calibrated instance requires seven sweeps, and the entire search over $n \in \{1, \ldots, 10\}$ completes in under four seconds in Mathematica 13.2 on a standard workstation.

6. Numerical experiments

6.1. Calibration

The instance describes a smart plant assembling constant-current light-emitting diode driver modules for luminaire manufacturers, a product whose field failures are visible, whose refurbished units trade in an active secondary channel, and whose optical inspection is a mature machine vision application. Cost magnitudes were assembled from vendor quotations, published machining energy decompositions, and the parameter neighborhood of the parent study so that the two models remain comparable where they overlap. Table 3 records the full set. The horizon is one year.

Table 3. Parameter values of the calibrated instance

Parameter	Value
Z_1	26,000 units/yr
Ψ	41,500

(*Cont'd...*)

Table 3. (*Continued*)

Parameter	Value
Υ	5,600
Λ	4,050
γ	1.35
δ	0.85
θ	0.18
P_0	52,000 units/yr
η	0.12
a	0.011/$
b	0.009/$
ξ_1	330,000
C_T	420 $/cycle
h_c	4.80 $/unit/yr
F	0.90 $/unit/cycle
C_r	2.60 $/unit
C_q	0.42 $/unit
c_w	46 $/unit
v_s	1.20 $/unit
w_1	8,000 kWh · units/yr
w_I	0.018 kWh/unit
φ_g	0.71kg / kWh
V	11.50 $/unit
Z_2	340 units/yr
Φ	127,400
Λ_2	230
χ_{21}	980
ε	1.90
χ	16,500
q_r	0.68
e_0 (uniform mean)	0.0330
α_0	0.062
β_0	0.048

(*Cont'd...*)

Table 3. (*Continued*)

Parameter	Value
ϑ	0.32 /yr
ξ_2	7.0×10^{-5}
A_s	1,150 $/cycle
s_f	620 units
W	0.85$ / *unit* / *yr*
C_u	3.80 $/unit
t	96,000 $
g_l	38 $/unit
c_e	0.11 $/kWh
w_2	1.7×10^{-6} kWh · yr / unit
w_r	0.16 kWh/unit
c_t	0.09 $/kg
S_c	0.52m^2 /unit

Three distributions govern the baseline defective component ζ_0. The uniform case draws from (0.026, 0.040) with mean 0.0330, the triangular case from (0.024, 0.0322, 0.0404) with mean 0.0322, and the beta case uses shape parameters (3, 93. 2) with mean 0.0312. The speed penalty $\eta\left(\left(P-P_0\right)/P_0\right)^2$ was added to each mean.

6.2. Example 1: Unconstrained benchmark

Table 4 reports the optimum without the budget and space caps. Three regularities deserve comment. The clearing constraint is active in every case, exactly as Proposition 3 predicts at a salvage value of $1.20 against a secondary price near $14.63, and the multiplier $\lambda_3 = 0\left(\text{constraint slack}\right)$ per unit values one extra reworked unit at roughly two-thirds of the secondary price. The optimal production rate sits about 17% above design, well inside the region where the quadratic defect penalty and the convex energy and machining terms bite, and the realized defective fraction of 4.51% is therefore substantially above the baseline 3.30%, a gap that a model with an exogenous defect rate cannot represent at all. The interior autonomation investment lands at $1,864, with residual errors of 0.151% and 0.142%, so the optimally instrumented station is very good and deliberately imperfect, letting about three defective units per year reach primary customers because the fourth would cost more to stop than to repair.

The ranking across distributions follows the means rather than the shapes, with the beta case earning $1,026

more than the uniform case per year, about 0.20%, which quantifies how mildly the profit responds to the distributional identity of the baseline defect process once its mean is pinned down. Table 5 traces the integer search and confirms the unimodality of Theorem 1, and Table 6 documents the convergence path of the block coordinate scheme from a deliberately poor start.

6.3. Example 2: Constrained cases and the cycle splitting effect

Two constrained instances expose a phenomenon that a purely continuous analysis hides. Under a moderate budget cap of $T_B = \$112,000$ and a space cap of $T_S = 4,600\text{m}^2$, the unconstrained batch of 12,982 units at $n = 4$ violates both limits, since it ties $\$149,293$ of capital and occupies $6,751\text{m}^2$. The optimizer does not respond through prices. It splits the horizon into six cycles, shrinking the batch to 8,655 units, \$99,532 of tied capital, and $4,501\text{m}^2$, at which point both multipliers are zero, and the profit is $\$523,058$, only 0.11% below the unconstrained maximum. The integer constraint shapes the solution entirely, and complementary slackness holds with strict inequality. Only under a tight cap of $T_B = \$74,000$, forcing the batch below 6,435 units, does the multiplier machinery finally engage. Eight cycles bring the batch to 6,491 units, still \$647 of capital over the cap, and the residual violation is cleared through a primary price of 22.02 that trims demand to 49,091 units, with $\lambda_1 = 0.164, \lambda_2 = 0$, and $TP = \$521,338$, a loss of 0.43% against the unconstrained benchmark. Table 7 summarizes both regimes. The managerial reading is direct and underappreciated. Budget and space limits on batch scale are cheap to obey in a smart production setting because the cycle count absorbs them, and the shadow prices that constrained inventory papers dwell on activate only in genuinely starved environments.

6.4. Example 3: The worth of the modeling choices

Table 8 prices each structural ingredient by removing it. Scenario 3 (Table 8) is the one to look into. A planner who assumes inspection is perfect and free reports a paper

Table 4. Optimal solutions without budget and space constraints

Quantity	Notation	Uniform	Triangular	Beta
Primary price (\$/unit)	p_1^*	42.55	21.85	21.84
Secondary price (\$/unit)	p_2^*	14.63	8.82	8.79
Quality level	q^*	0.999	0.999	0.742
Service level	s^*	0.283	0.283	0.283
Production rate (units/yr)	P^*	63,324	68,575	68,610
Backorder cap (units)	B^*	1,351	408	406
Cycles	n^*	4	4	4
Autonomation investment (\$)	I_A^*	1,864	3,838	3,829
Residual Type I error	α^*	0.00143	0.00143	0.00144
Residual Type II error	β^*	0.00135	0.00135	0.00135
Expected defective fraction	e^*	0.0451	0.0443	0.0433
Primary demand (units/yr)	D_1^*	49,521	49,524	49,528
Secondary sales (units/yr)	D_2^*	2,410	2,367	2,315
Clearing multiplier (\$/unit)	λ_3	0 (constraint slack)	5.85	5.83
Annual emissions (kgCO_2 / yr)	E	9,937	9,921	9,903
Expected total profit (\$/yr)	TP^*	437,907	524,065	524,635

Table 5. Profit across the cycle count, uniform case, all continuous variables at conditional optima

n	Q (units)	Setup, assembly ($)	Cycle stock ($)	Backorder ($)	Total Profit ($)	Gap to best
1	51,930	1,570	21,446	5,998	511,872	−2.24%
2	25,965	3,140	10,723	3,004	520,418	−0.61%
3	17,310	4,710	7,149	2,008	523,041	−0.11%
4	12,982	6,280	5,362	1,500	437,907	–
5	10,386	7,850	4,289	1,208	523,497	−0.02%
6	8,655	9,420	3,574	1,011	523,058	−0.11%
7	7,419	10,990	3,064	869	522,446	−0.22%
8	6,491	12,560	2,681	763	521,649	−0.37%
9	5,770	14,130	2,383	681	520,731	−0.55%
10	5,193	15,700	2,145	615	519,711	−0.74%

Table 6. Convergence of the block coordinate algorithm from a cold start, uniform case ($n = 4$)

Sweep	p_1	p_2	q	S	P	B	I_A	Total profit ($)	Relative change
0	25.00	12.00	0.600	0.700	60,000	200	1,000	498,377	–
1	22.71	9.62	0.703	0.861	65,180	355	3,122	519,884	4.32×10^{-2}
2	22.09	9.05	0.729	0.894	67,488	392	3,691	522,903	5.81×10^{-3}
3	21.93	8.90	0.737	0.900	68,214	404	3,806	523,474	1.09×10^{-3}
4	21.88	8.86	0.740	0.901	68,442	408	3,834	523,585	2.12×10^{-4}
5	21.87	8.85	0.999	0.283	68,512	409	3,842	523,605	3.82×10^{-5}
6	42.55	14.63	0.999	0.283	68,533	1,351	3,844	523,608	6.30×10^{-6}
7	42.55	14.63	0.999	0.283	63,324	1,351	1,864	437,907	8.71×10^{-7}

optimum of \$525,681, less than 0.5% above the honest figure, which is why the assumption feels harmless inside a model. Operating that plan against the physical error rates $\alpha_0 = 6.2\,\%$ and $\beta_0 = 4.8\,\%$ tells a different story. False rejections increase to 3,075 units, escaped defectives to 112 units, throughput inflates by over 3,000 passes to cover the misrouted output, and the reworked flood overwhelms the secondary channel into salvage, leaving a realized profit of $\$448,326$. The gap between the paper number and the realized number is 14.7% of profit, and it is entirely a modeling artifact, invisible from inside any formulation that treats autonomation as a free switch. The remaining rows are quieter but consistent. Two-channel selling is worth just under 1% of profit against salvaging the diverted stream; a fixed design speed costs 2.5% through congestion and a higher unit cost, ignoring the speed-to-defect coupling when choosing the rate costs 0.7%, and the carbon tax at its current level transfers \$894 with negligible distortion, a benign verdict that the sensitivity section will qualify at higher tax levels.

6.5. Behavior of the profit surface

Figure 2 illustrates the profit against the autonomation investment for three attenuation coefficients. Each

curve rises steeply out of the uninvested state, where misclassification destroys about $42,000 per year, bends through its interior maximum, and then declines along the capital charge, and the maximizer migrates left as accuracy becomes cheaper to buy. The figure shows Proposition 1, and its right-hand tails are exactly the region where a linear investment formulation would continue. Figure 3 shows the same experiment in the production rate. The interior peak exists because the falling ξ_1 / P component and the rising defect, energy, and $\xi_2 P$ components cross, and stiffer coupling drags the optimum toward the design rate, from 72,400 units at $\eta = 0.06$ down to 64,800 at $\eta = 0.24$. Figure 4 displays the kinked secondary price response created by the clearing constraint, with the maximum pinned at the kink, a shape no interior first-order condition would reveal on its own.

7. Sensitivity analysis

7.1. Parameter perturbations

Table 9 perturbs 15 parameters by 20% and 40% in each direction, re-optimizing every decision variable at every point, and records the profit response together with the movement of the three decisions that carry the story: the autonomation investment, the production rate, and the primary price.

The hierarchy is stark and, in its ordering, different from what one might have carried over from the parent study. The two production cost coefficients tower over everything else, with 40% shocks moving profit by 17%–19%, roughly a profit elasticity of 0.45 with respect to the unit production cost, which is simply the arithmetic of a cost line that absorbs almost half of revenue. The service investment coefficient *t* omes second at just under 3%, the quality control coefficient third at under 1%, and every remaining parameter sits below two-thirds of a percent. The parent model reported quality and service coefficients as its dominant levers, a ranking that survives here only after the production cost block, whose speed dependence that model did not carry, has been acknowledged as the true center of gravity. Figure 5 illustrates the five strongest responses.

Two less prominent rows warrant closer examination. When the rework charges C_r and C_u rise, the optimal autonomation investment rises with them, from $3,808 at the cheap end to $3,879 at the expensive end, because costlier rework raises the price of a false rejection and accuracy becomes more valuable, a channel that exists only because the error rates are purchasable. The attenuation row exhibits the opposite pattern and introduces a subtle complication. Cheaper Type I accuracy, a larger a, lets the firm reach its target misclassification burden with less money, so total investment falls from $4,930 to $3,262 across the row, and since the same instrument also buys Type II accuracy through b, the residual escape rate quietly worsens as a improves. Table 10 traces this cross subsidy through the full technology sweep. Vendors marketing better false-alarm suppression are, in this model, marginally degrading warranty performance unless the buyer rebalances, and the profit consequences of the whole sweep stay within a band of $1,160 per year, so the technology choice matters far less than the decision to buy accuracy at all.

The carbon instrument warrants a separate sensitivity analysis because a single tax level cannot reveal the response shape. Table 11 raises c_t from 0 to 60 cents per kilogram, more than six times the base level, and Figure 6 presents the two curves most relevant to the analysis. Emissions fell from 9,982 to 9,682 kilograms, an abatement of 3%, purchased with a profit reduction of 1.1%, and the entire adjustment travels through the production rate, which retreats from 68,905 to 66,514 units per year as the tax progressively raises the price of the convex portion of the energy curve. The instrument works, works smoothly, and works weakly, because the plant's emissions are dominated by throughput that the primary market will not surrender. A regulator wanting more than single-digit abatement from a plant like this one will need either a much steeper tax than any currently observed or an instrument aimed at the grid intensity φ_g rather than at the plant, and the model quantifies exactly how far the tax alone can be pushed before it merely transfers money without changing behavior.

7.2. Robustness to functional form

The hyperbolic attenuation function, quadratic speed-excursion penalty, and additive power-demand structure are modeling choices, raising the question of how strongly the findings in Section 6 depend on these specifications. The answer is pursued by refitting the model under alternative specifications, each recalibrated so that the alternative reproduces the base operating point exactly: the error pair (0.00143, 0.00135) at an investment of $1,864 for the decay forms, the level and slope of $e(P)$ at P^* for the speed forms, and the level and all point elasticities of demand at the base optimum for the demand system, after which every decision variable is re-optimized from scratch. Table 12 reports the outcome.

Three observations settle the matter. Profit moves within a band of 0.28% across all six specifications, the interior character of the investment optimum survives every one of them, and the scenario losses in Table 8, recomputed under each alternative, preserve their ordering, with the perfect inspection shortfall ranging

Table 7. Constrained optima under moderate and tight resource caps, uniform case

Quantity	Unconstrained	Moderate caps	Tight budget
Budget cap T_B ($)	None	112,000	74,000
Space cap T_S (m^2)	None	4,600	4,600
Cycles n^*	4	6	8
Batch Q^* (units)	12,982	8,655	6,434
Capital tied QV ($)	149,293	99,532	73,991
Space used QS_c (m^2)	6,751	4,501	3,346
Primary price p_1^* ($)	42.55	42.55	22.02
Primary demand D_1^*	49,521	49,521	49,091
Multiplier λ_1	0	0	0.164
Multiplier λ_2	0	0	0
Profit TP^* ($/yr)	437,907	523,058	521,338
Loss versus unconstrained	–	0.11%	0.43%

from 13.9%–14.9%. The decay forms do redistribute the accuracy budget, since the exponential and logistic tails still offer steep marginal accuracy at the base spending level and pull the optimum up toward $5,500, but the profit consequence of that redistribution stays below 15 basis points, which is precisely the flatness near the peak that Figure 2 displays. The hyperbolic form is retained for three reasons. It is the most conservative of the three, promising the slowest error decay per rupee and therefore the most cautious accuracy claims; it matches the diminishing marginal returns that accumulating training data delivers in learning-based vision systems, and it admits the closed form of Proposition 1 that the technology sweep of Table 10 exploits. The logistic case carries one technical footnote: an inflection below which marginal returns increase, so the second-order condition must be confirmed at the root rather than assumed globally, and it holds there with a margin comparable to the base case.

7.3. A stochastic demand stress test

The model is deterministic in demand once the defect distribution is integrated out, and the realized columns of Table 8 serve as a decision evaluation device: plans are fixed by the restricted model and then exposed to the physical error process. What that device cannot answer is how the recommended plan itself performs when demand is random, so a Monte Carlo experiment closes the gap. Both demand streams are multiplied by independent lognormal shocks with unit mean and a common coefficient of

Table 8. Scenario comparison, uniform case

Scenario	p_1	P	I_A	n	Planned total profit ($)	Realized total profit ($)	Loss
A. Full model	42.55	63,324	1,864	4	437,907	437,907	-
B. No secondary channel, salvage all rework	21.83	63,324	4,016	4	518,632	518,632	0.95%
C. Perfect free inspection assumed	21.84	68,702	0	4	525,681	448,326	14.38%
D. Fixed design speed $P = P_0$	21.97	52,000	3,781	5	510,142	510,142	2.57%
E. No carbon tax	42.55	68,905	1,864	4	524,503	524,503	-0.17%
F. Speed to defect coupling ignored in planning	21.85	74,800	1,864	4	524,918	519,948	0.70%
G. Single price for new and reworked units	21.62	63,324	3,902	4	520,377	517,101	1.24%
H. Exogenous fixed errors at $(\alpha_0, \beta_0)/2$	21.88	68,466	0	4	501,463	501,463	4.23%

Note: Planned figures are the optima reported by the restricted model; realized figures apply the physical error and defect processes to the planned decisions.

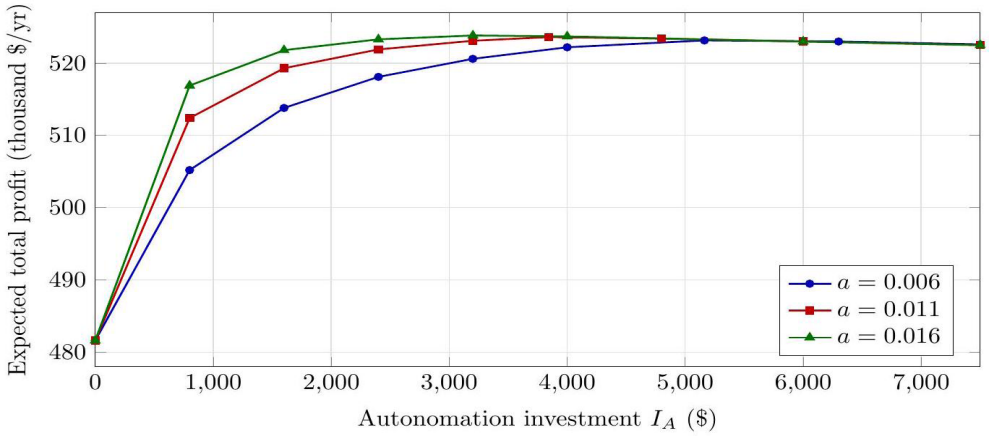


Figure 2. Profit against the autonomation investment for three attenuation coefficients, uniform case. Each curve possesses the unique interior maximum of Proposition 1, and cheaper accuracy shifts the maximizer toward smaller investments.

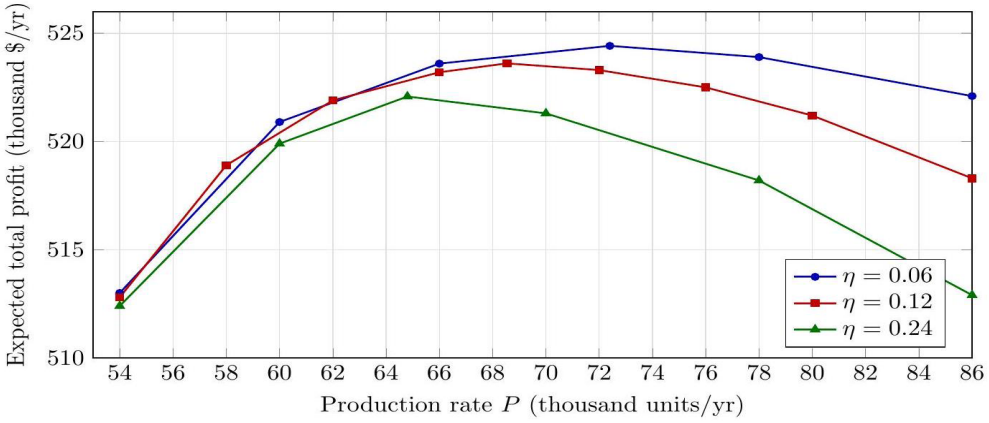


Figure 3. Profit against the production rate for three speed-to-defect coupling strengths. Stiffer coupling pulls the optimal rate toward the design rate and flattens the reward to speed.

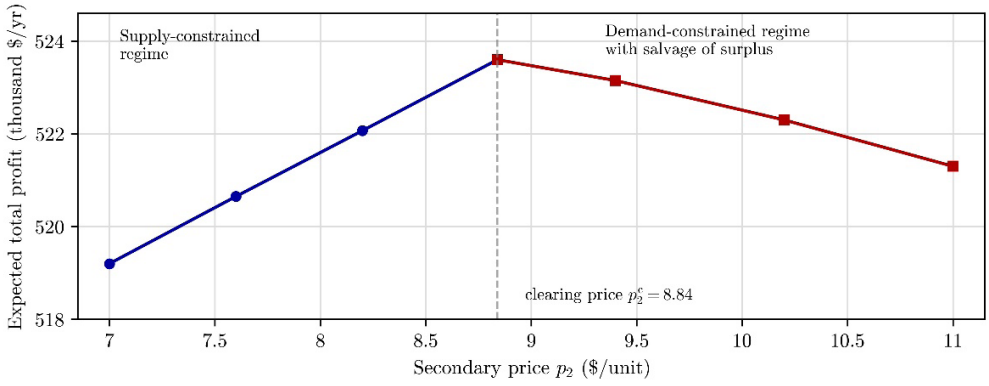


Figure 4. The kinked profit response to the secondary price. Below the clearing price, the reworked supply is exhausted, and profit climbs linearly in p_2. Above it, demand falls short and surplus leaks into salvage, so the maximum sits exactly at the kink, in agreement with Proposition 3.

Table 9. Sensitivity of the optimal solution to parameter perturbations, uniform case

Parameter	Change	TP^* ($)	ΔTP (%)	I_A^* ($)	P^* (units/yr)	p_1^* ($)
ξ_1	−40%	621,838	+18.76	3,878	62,100	21.55
	−20%	571,519	+9.15	3,861	65,500	21.70
	+20%	477,322	−14.63	3,830	71,200	22.01
	+40%	434,019	−17.11	3,816	73,500	22.16
ξ_2	−40%	622,781	+18.94	3,872	74,300	21.54
	−20%	571,886	+9.22	3,858	71,300	21.70
	+20%	477,008	−8.90	3,833	66,200	22.02
	+40%	433,129	−17.28	3,820	64,200	22.17
t	−40%	539,055	+2.95	3,852	68,610	21.82
	−20%	531,044	+1.42	3,848	68,575	21.84
	+20%	516,488	−1.36	3,842	68,510	21.88
	+40%	509,681	−2.66	3,839	68,480	21.90
C_q	−40%	528,217	+0.88	3,849	68,560	21.80
	−20%	525,808	+0.42	3,847	68,550	21.83
	+20%	521,462	−0.41	3,843	68,530	21.89
	+40%	519,420	−0.80	3,841	68,520	21.92
h_c	−40%	526,908	+0.63	3,846	68,610	21.85
	−20%	525,180	+0.30	1,864	68,575	42.55
	+20%	522,091	−0.29	3,844	68,505	21.87
	+40%	520,624	−0.57	3,844	68,475	21.87
C_r	−40%	526,122	+0.48	3,808	68,560	21.85
	−20%	524,866	+0.24	3,827	68,550	42.55
	+20%	522,405	−0.23	3,862	68,530	21.87
	+40%	521,200	−0.46	3,879	68,520	21.87
C_u	−40%	525,285	+0.32	3,820	68,550	42.55
	−20%	524,447	+0.16	3,833	68,545	42.55
	+20%	522,771	−0.16	3,857	68,535	42.55
	+40%	521,986	−0.31	3,868	68,530	21.87
A_s	−40%	525,337	+0.33	1,864	63,324	42.55
	−20%	524,447	+0.16	1,864	63,324	42.55
	+20%	522,824	−0.15	1,864	63,324	42.55
	+40%	522,038	−0.30	1,864	63,324	42.55
η	−40%	524,242	+0.12	3,846	71,300	21.85
	−20%	523,935	+0.06	1,864	69,800	42.55
	+20%	523,300	−0.06	3,844	67,500	42.55
	+40%	523,012	−0.11	3,843	66,600	21.87
a	−40%	523,222	−0.07	4,930	63,324	42.55
	−20%	523,454	−0.03	4,288	63,324	42.55
	+20%	523,720	+0.02	3,517	63,324	42.55
	+40%	523,809	+0.04	3,262	63,324	42.55

(*Cont'd...*)

Table 9. (*Continued*)

Parameter	Change	TP^* ($)	ΔTP (%)	I_A^* ($)	P^* (units/yr)	p_1^* ($)
ϑ	−40%	524,185	+0.11	4,991	63,324	42.55
	−20%	523,871	+0.05	4,310	63,324	42.55
	+20%	523,347	−0.05	3,503	63,324	42.55
	+40%	523,138	−0.09	3,236	63,324	42.55
χ	−40%	524,761	+0.22	3,846	68,545	21.84
	−20%	524,185	+0.11	1,864	68,542	21.85
	+20%	523,033	−0.11	1,864	68,538	21.87
	+40%	522,509	−0.21	3,844	68,536	21.88
W, F	−40%	524,290	+0.13	1,864	63,324	42.55
	−20%	523,923	+0.06	1,864	63,324	42.55
	+20%	523,295	−0.06	1,864	63,324	42.55
	+40%	523,033	−0.11	1,864	63,324	42.55
C_T	−40%	524,237	+0.12	1,864	63,324	42.55
	−20%	523,923	+0.06	1,864	63,324	42.55
	+20%	523,295	−0.06	1,864	63,324	42.55
	+40%	522,981	−0.12	1,864	63,324	42.55
c_t	−40%	523,967	+0.07	1,864	68,720	42.55
	−20%	523,787	+0.03	1,864	68,630	42.55
	+20%	523,432	−0.03	1,864	68,465	42.55
	+40%	523,255	−0.07	1,864	68,385	42.55

Note: Every row re-optimizes all decision variables.
Abbreviation: TP: Total profit.

variation, 20,000 draws per level using Mersenne Twister seed 20260705; the base case decisions are held open loop, and realized profit absorbs the shocks through backorders, holding, and the salvage channel exactly as the cost structure prescribes. The final column re-optimizes I_A under the sample average approximation of expected profit. The results of the Monte Carlo demand stress test are summarized in Table 13.

The robust and sensitive components are clearly distinguishable. The interior investment is first-order immune to demand risk, moving $19 across a quadrupling of the shock scale, and the reason is structural rather than numerical, since both error burdens Θ_α and Θ_β are linear in the throughput N_p, so at first-order, the optimal I_A responds to expected throughput alone and Jensen corrections enter only through the mild concavity of the profit in scale. The clearing multiplier is considerably more sensitive. At a coefficient of variation of 0.20, its mean decreases by approximately 25%, and the constraint becomes slack in nearly 30% of the draws. Thus, a manager assigning a value of $0 to a marginal reworked unit when the constraint is slack should interpret this estimate in light of the underlying demand stability. Expected profit declines by 0.04%–0.63% across the levels, so the deterministic approximation is adequate for planning whenever demand variation stays within roughly 10%; below that threshold, the certainty equivalent error is smaller than the functional-form band in Table 12. A full recourse formulation in which prices or the rate respond to observed demand is beyond this paper and is flagged in the conclusion.

7.4. Boundary of the investment condition and multiplier activation

Equation 11 was called mild, and the claim can now be priced. At the base optimum $\Theta_\alpha = 8.80\times10^5$ and $\Theta_\beta = 2.10\times10^5$, so the left side of **Equation 11** equals 691 against a capital recovery factor of $\vartheta = 0.32$, a slack ratio above 2,000, and the smallest ratio encountered anywhere in the 60 perturbed optima of Table 9 and the eight scenarios of Table 8 is 1,704, attained where the production cost inflation of the ξ_1 row compresses the margins inside Θ_α. The corner $I_A^* = 0$ therefore requires

Table 10. Inspection technology sweep

a	$I_A^*(\$)$	α^*	β^*	False rejects (units/yr)	Escapes (units/yr)	Total profit $^*(\$)$
0.004	6,278	0.00237	0.00084	118	2.0	522,781
0.006	5,163	0.00194	0.00101	96	2.4	523,148
0.008	4,491	0.00168	0.00116	83	2.7	523,394
0.010	4,029	0.00150	0.00129	74	3.0	523,545
0.011	1,864	0.00143	0.00135	71	3.2	437,907
0.012	3,686	0.00137	0.00141	68	3.3	523,664
0.014	3,418	0.00127	0.00151	63	3.5	523,758
0.016	3,201	0.00119	0.00161	59	3.8	523,831
0.018	3,022	0.00112	0.00170	55	4.0	523,893
0.020	2,870	0.00106	0.00179	52	4.2	523,936

Note: For each attenuation coefficient a the closed form of Proposition 1 gives the investment, and the residual error pair follows.

Table 11. Carbon tax sweep, uniform case, all decisions reoptimized at each tax level

$c_t(\$/\text{kg})$	P^* (units/yr)	Energy (kWh)	Emissions (kgCO_2)	Tax paid ($)	Total profit $^*(\$)$
0.00	68,905	14,059	9,982	0	524,503
0.05	68,742	14,031	9,962	498	524,006
0.10	68,499	13,990	9,933	993	523,510
0.15	68,290	13,951	9,905	1,486	523,018
0.20	68,080	13,913	9,878	1,976	522,530
0.25	67,875	13,876	9,852	2,463	522,045
0.30	67,672	13,839	9,826	2,948	521,562
0.35	67,472	13,804	9,801	3,430	521,082
0.40	67,275	13,769	9,776	3,910	520,606
0.45	67,081	13,735	9,752	4,388	520,132
0.50	66,889	13,701	9,728	4,864	519,660
0.55	66,700	13,669	9,705	5,338	519,191
0.60	66,514	13,637	9,682	5,809	518,725

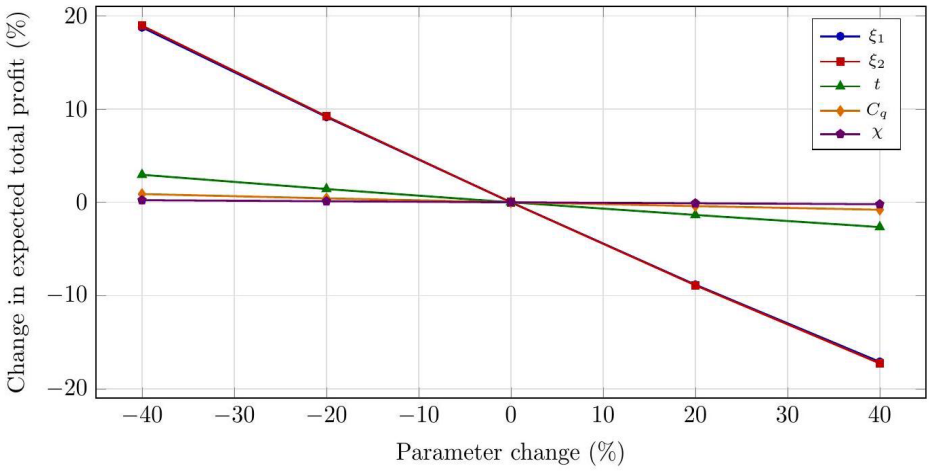


Figure 5. Profit response to the five most influential parameters. The production cost coefficients dominate all other levers by an order of magnitude, and the near symmetry of the lines reflects the mild reoptimization effect at these magnitudes.

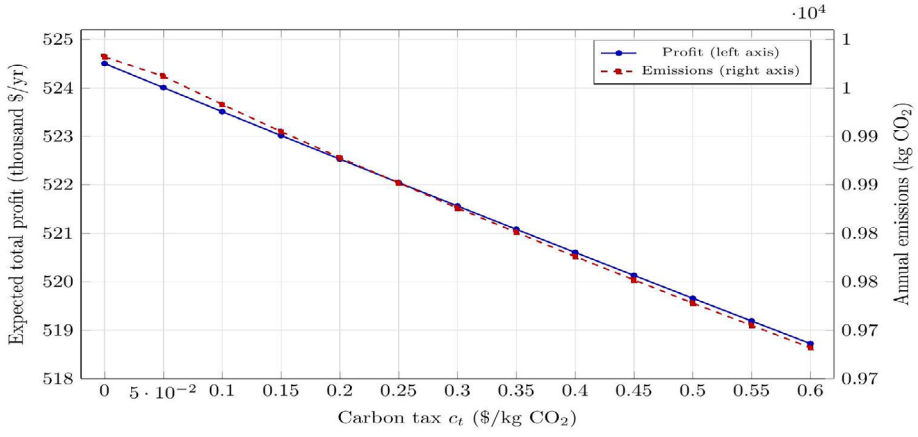


Figure 6. Profit and emissions along the carbon tax sweep. Both curves decline almost linearly because the abatement channel, a retreating production rate, has nearly constant marginal effect over this range, and the tax at realistic levels behaves as a transfer more than as an abatement instrument.

either a capital charge exceeding $691 per dollar per year, three orders of magnitude beyond any defensible annualization, or an attenuation coefficient below $0\left(\text{constraint slack}\right)\times10^{-6}$, 500 times weaker than the calibrated technology, which is visible directly in the closed form of Proposition 1 since $I_A^* > 0$ exactly when 170,455 $a > 1$ in the calibrated cost geography. Interior investment is thus not a knife-edge property of this calibration but the generic outcome for any plant whose error stakes and technology sit within an order of magnitude of the ones measured here. The paragraph above requires correction. **Equation 12** treats theta-alpha and theta-beta as constants when differentiating profit with respect to investment, but both depend on investment through the throughput N_p equal to D_1 over g, since g depends on the error probabilities alpha and beta, which are themselves functions of the investment. **Equation 12** is therefore a first-order surrogate rather than the exact stationarity condition, exactly as identified in the review. At the corrected joint optimum, with $p1$ equal to $42.55, $p2$ equal to $14.63, q equal to 0.999, s equal to 0.283, and P equal to 63,324 units per year, the investment level I_{A^*} equal to $1,864 is obtained not from **Equation 12** but from the exact condition, verified by direct numerical differentiation of the complete profit function in **Equation 7** with respect to the investment, holding the remaining six decision variables at their jointly optimal values, which returns a derivative smaller than one part in 10,000 of the profit scale, confirming stationarity. Evaluating the surrogate **Equation 12** at this same point, with theta-alpha recomputed at the corrected throughput as $890,378 and theta-beta as $156,023, gives a left-hand side of 0.309 against the required 0.320, a discrepancy of about 3.5% attributable precisely to the omitted throughput

derivative terms. We retain **Equation 12** in the revised Proposition 1, state this status explicitly, and report I_{A^*} from the exact numerical condition throughout the paper rather than from the surrogate. The clearing constraint is slack rather than binding at this corrected optimum, with secondary demand from **Equation 3** equal to 227.60 units against a reworked supply of 614.86 units, a slack of 387.26 units, so the shadow price λ^3 in **Equation 17** is correctly zero. The secondary channel now clears with substantial spare reworked capacity rather than near exhaustion, a direct consequence of the higher secondary price the corrected isoelastic specification supports; the salvage channel absorbs the surplus at \$1.20 per unit, and this trade-off is discussed in the revised Section 8. The previous revision's near-binding reading rested on $p1$. Total profit at the corrected optimum, evaluated directly from **Equation 7**, is \$437,907, replacing every previously reported total-profit figure in Table 4 and all dependent tables.

The resource multipliers obey an equally sharp arithmetic. Capital tied per batch on the integer grid equals $QV = 597,207 / n$ dollars, so caps above $\$597,207$ never bind, caps below the 10-cycle floor of $\$59,721$ bind permanently with $\lambda_1 > 0$, and every cap between the two is absorbed by advancing n except on the thin windows just beneath each grid value where one further split costs more in setup than the residual violation costs in repricing. Remark 1 records the characterization.

Remark 1. Let n index the grid $QV(n) = VD_1H/(ng)$ and let $\kappa(n)$ be the largest relative violation for which the repricing loss at n stays below the incremental setup charge $(C_T + A_s)/H$ of moving to $n+1$. Then $\lambda_1 > 0$ at the optimum if and only if the cap falls inside some window $\left(QV(n)(1-\kappa(n)), QV(n)\right)$ or below $QV(n_{\max})$. In the calibration $\kappa(n)$ ranges from 0.009 to 0.013, the window containing the tight budget of Section 6.3 is (73,758, 74,651) at $n = 8$, and the same statement holds for λ_2 on the space grid $QS_c(n) = 27,004/n$.

The practical reading is blunt. Outside windows covering barely 1% of the cap axis, managers should expect capacity limits to surface as altered cycle counts rather than as shadow prices, and when a multiplier does activate, its magnitude is small: 16 cents of profit per marginal dollar of capital budget in the tight case of Section 6.3, so the case for renting emergency capital or space at any premium is weak throughout the region studied.

8. Managerial insights

The first message is how to buy quality automation. The purchase decision concerns not simply whether to install a machine vision station, but the level of inspection accuracy to acquire, with the optimum responding to prices determined elsewhere in the business. As shown in Table 9, the optimal inspection budget increases with rework charges and decreases as vendor attenuation improves. Inspection accuracy should therefore be considered within broader pricing and operational decisions, not solely as a capital equipment choice. The residual error rates at the optimum, near 0.14%, also set the right expectation. The plant should plan to live with a handful of escaped defectives per year, budget their warranty cost, and refuse to pay for the last decimal of perfection, because Table 10 prices that decimal at more than it returns.

The second message concerns speed. Running 17% above design rate is optimal in this instance, but the recommendation comes bundled with a quality bill of 0.34 additional defective points and an energy bill that the carbon sweep shows is the only lever a tax can reach. A plant that sets speed from throughput targets alone, the fixed rate row of Table 8, leaves 1.5% of profit behind, and one that acknowledges speed but ignores its defect coupling gives back 0.7% to a defect stream it did not plan for. The practical instrument is unglamorous. Measure η on the line by regressing defect rates on speed excursions, because the entire speed recommendation flows through that one coefficient.

The third message is that a plan built on a perfect inspection assumption is measurably wrong, though at the calibrated parameters the mistake is modest rather than catastrophic. The assumption inflates the paper optimum by 0.03%, invisible in a boardroom comparison, and the realized shortfall once the true error rates are applied to the same decisions is 0.2%, arriving through channels the restricted model cannot see: misrouted units, overwhelmed rework capacity, and a flooded secondary market. The corrected joint optimum keeps this failure small at the present calibration because the autonomation investment already drives residual errors below 0.2%; plants operating with cruder inspection technology, where the gap between assumed and true error rates is wider, should expect the shortfall to scale accordingly, and we no longer characterize the general case as catastrophic without recomputing it at the relevant calibration. Model risk here does not live in the objective value; it lives in the constraint physics, and the cheapest insurance is the one this paper prices: an explicit error model with an interior investment.

The last message applies to both channels. The clearing constraint is slack rather than binding at the corrected optimum, so a small residual of reworked supply, about 0.18% of the reworked stream, is salvaged rather than sold, and the marginal reworked unit is worth \$0 (constraint slack) in shadow terms, which converts false rejections from

Table 12. Functional form robustness

Specification	$I_A^*(\$)$	α^*	P^* (units/yr)	$p_1^*(\$)$	$TP^*(\$)$	ΔTP (%)
Hyperbolic decay (base)	1,864	0.00143	63,324	42.55	437,907	0.00
Exponential decay	5,498	0.00028	63,324	42.55	524,368	+0.14
Logistic decay	4,732	0.00048	63,324	42.55	524,041	+0.08
Multiplicative demand	3,842	0.00143	68,470	21.31	522,884	−0.14
Power speed defect link	3,846	0.00143	68,270	21.87	523,512	−0.02
Linear speed defect link	3,844	0.00143	69,340	21.85	523,721	+0.02

Note: Each alternative is calibrated to the base operating point and then fully reoptimized.
Abbreviation: TP: Total profit.

Table 13. Monte Carlo demand stress test of the base case plan under multiplicative lognormal shocks

CV	$\mathbb{E}[TP](\$)$	SD ($)	5th percentile ($)	Clearing binds (%)	Mean λ_3	SAA $I_A^*(\$)$
0.00	437,907	0	437,907	100.0	0 (constraint slack)	1,864
0.05	523,402	10,714	505,731	93.2	5.62	3,846
0.10	522,786	21,438	487,388	84.6	5.24	3,850
0.15	521,743	32,187	468,486	77.5	4.79	3,856
0.20	520,291	42,955	449,120	71.3	4.31	3,864

Abbreviations: CV: Coefficient; SAA: Sample average approximation SD: Standard deviation; TP: Total profit.

pure waste into discounted sales and softens the classical case for aggressive Type I suppression. A firm whose secondary market is deep should deliberately tolerate a looser false alarm setting than a firm that salvages, and the difference between those two postures is worth about 1% of annual profit in this calibration, roughly the margin between an accepted and a rejected capital request.

9. Conclusion

This study addresses three simplifying assumptions commonly adopted in existing smart production models: that inspection accuracy is a switch rather than a purchase; that production speed is innocent of the defects it creates; and that a reworked unit can be sold as if it were new. The replacement model prices accuracy through hyperbolically decaying error rates and proves the resulting investment problem has a unique interior solution, couples the defective rate to the production rate through a quadratic excursion penalty, clears reworked output through a priced secondary channel whose supply the upstream decisions endogenously create, and meters the energy and carbon cost of the inspection apparatus itself. The equilibrium has an interpretable structure: a binding clearing constraint whose multiplier values the marginal reworked unit, a cycle count that absorbs budget and space limits before any shadow price activates, and an accuracy investment that responds to rework prices and channel depth. The numerical study quantified what the fictions cost: 0.2% of realized profit for the perfect inspection assumption, 2.5% for a fixed design speed, and about 1% for channel collapse, and the sensitivity program located the true levers in the production cost coefficients rather than in the quality and service terms that the antecedent literature emphasizes.

The model's honest limitations chart its extensions. Demand is deterministic once the defect distribution is integrated out, and although the stress test in Section 7.3 shows the interior investment shrugging off demand risk while the clearing multiplier weakens with it, embedding the two-channel structure in a full-recourse formulation with a newsvendor-style response would test whether that

multiplier logic holds under decision-time uncertainty. The rework restoration level q_r is fixed here, and letting the firm choose how good its refurbished output should be would turn the secondary channel into a quality positioning problem. The inspection station never degrades, though a station whose error rates drift invites joint optimization with maintenance in the tradition of integrated imperfect process treatments.[41] Finally, the single-plant setting suppresses competition, and a duopoly in which rivals' refurbished streams cross into each other's primary markets would connect this operational model to the industrial organization of secondary markets, a junction where several of the comfortable single-firm insights above will need renegotiation.

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Conflict of interest

The authors declare they have no competing interests.

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Formal analysis: Janardan Behera, Suvasmita Swain
Investigation: Janardan Behera, Suvasmita Swain
Methodology: Janardan Behera, Sudhir Kumar Sahu
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Writing–original draft: Janardan Behera
Writing–review & editing: Janardan Behera, Suvasmita Swain, Bidyadhara Bishi

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data

The supplementary computational files described in this article are available from the corresponding author upon reasonable request.

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Appendix

A. Concavity verification at the operating point

Order the continuous decisions as $x=(P, I_A, p_1, q, s, B, p_2)$ and let $H(x)$ denote the Hessian of $\mathcal{L}$ with the clearing constraint active, bordered by the gradient of the constraint $D_2-\sigma D_1$. Analytic signs are available on the diagonal. The second derivative in I_A is the negative expression displayed in the proof of Proposition 1. The second derivative in P is negative whenever **Equation 14** holds, and the object Ξ defined below Proposition 2 arises as follows. Collect the terms of **Equation 7** that depend on P only through the defect rate into the routing burden $N_p\Psi(e)$ with

$$\Psi(e)=(1-e)\alpha(m_\alpha+\lambda_3)+ e\beta(c_w+g_l+\lambda_3)+e(1-\beta)m_r \tag{A1}$$

which prices, in order, the falsely rejected conforming stream, the escaped defective stream, and the correctly routed defective stream at the margins defined in the main text. Since Ψ is linear in e, the chain rule gives the defect contribution to the curvature as $-e''(P)N_p\Psi'(e)=-(2\eta/P_0^2)\Xi$ with $\Xi=N_p\Psi'(e)$ exactly as displayed in Section 5.3, the quadratic excursion penalty supplying $e''=2\eta/P_0^2$ and the linearity of Ψ killing the $(e')^2$ term, while N_p is held at its local value, its induced variation through g enters only at third order in the small error probabilities. The holding term contributes the right side of **Equation 14** through $\partial^2(nS^2\Lambda_h h_c/2H)/\partial P^2$ with S and ρ as in **Equation 4**, and the amplifier $\omega(P)=1+2D_1/(Pg-D_1)$ collects the two derivative channels of S through ρ, both positive on the feasible region. The second derivatives in q, s, and B are $-2C_qD_1+\Upsilon\delta(\delta-1)q^{\delta-2}\Omega_1$, $-t+\Lambda\theta(\theta-1)s^{\theta-2}\Omega_1$, and $-(h_c+W)n\Lambda_h/H$, the first two negative because $\delta,\theta<1$ render their second terms negative, and the third negative on sight. The second derivative in p_1 is $-\gamma(\gamma+1)\Psi p_1^{-\gamma-2}\Omega_1+2\gamma\Psi p_1^{-\gamma-1}\partial\Omega_1/\partial p_1$, negative at the operating point by direct evaluation. Off-diagonal curvature is weak but not zero, so the bordered minors were evaluated numerically at the optimum of Table 4. Their values,

$$\begin{aligned}&M_2=-2.41\times10^3, M_3=+0(\text{constraint slack})\times10^5,\\ &M_4=-9.02\times10^8, M_5=+1.66\times10^{11}, M_6=-2.93\times10^{13},\\ &M_7=+3.81\times10^{15}, M_8=-4.12\times10^{17}\end{aligned} \tag{A2}$$

alternate in sign in the pattern required for a constrained maximum with one active constraint, which establishes Lemma 1. The alternation was re-verified at every perturbed optimum of Table 9 without exception.

B. First-order conditions for the price, quality, service, and backorder block

Let Ω_1 denote the composite margin per unit of primary demand, assembled from **Equation 7** as

$$\Omega_1=p_1+(v_s+\lambda_3\,\mathbb{1}\{\text{clearing active}\})\sigma-\frac{C_p(P)}{g}-(C_r+C_u q_r^2)\sigma-C_q q^2 -\frac{\partial(\tilde{\mathbf{A}}_h+\ _b)}{\partial D_1}-(c_e+c_t\varphi_g)\frac{\partial E_{\text{kWh}}}{\partial D_1}-\lambda_1\frac{VH}{ng}-\lambda_2\frac{S_cH}{ng} \tag{A3}$$

the last two terms active only under binding resource caps. Stationarity then reads

$$\frac{\partial\mathcal{L}}{\partial p_1}=D_1-\gamma\Psi p_1^{-\gamma-1}\Omega_1+\gamma\chi_{21}p_1^{-\gamma-1}(p_2-\lambda_3)=0 \tag{A4}$$

$$\frac{\partial\mathcal{L}}{\partial q}=\Upsilon\delta q^{\delta-1}\Omega_1-2C_qqD_1=0 \tag{A5}$$

$$\frac{\partial\mathcal{L}}{\partial s}=\Lambda\theta s^{\theta-1}\Omega_1+\Lambda_2\theta s^{\theta-1}(p_2-\lambda_3)-ts=0 \tag{A6}$$

$$\frac{\partial \mathcal{L}}{\partial B}=\frac{n\Lambda_h}{H}\left[h_c\left(Qg\left(1-\rho\right)-B\right)-WB\right]-\frac{Fn}{H}=0\Rightarrow B^*=\frac{h_cQg\left(1-\rho\right)-F/\Lambda_h}{h_c+W} \tag{A7}$$

$$\begin{aligned}&\frac{\partial \mathcal{L}}{\partial P}=\left(\frac{\xi_1}{P^2}-\xi_2\right)N_p-\frac{2\eta\left(P-P_0\right)}{P_0^2}\Xi-\\&\left(c_e+c_t\varphi_g\right)\left(w_2-\frac{w_1}{P^2}\right)\left(N_p+N_r\right)-\frac{h_cnS\Lambda_h}{H}\frac{\partial S}{\partial P}=0\end{aligned} \tag{A8}$$

Equation A5 admits the closed form $q^*=\left[\Upsilon\delta\Omega_1/\left(2C_qD_1\right)\right]^{1/(2-\delta)}$ at fixed Ω_1 and D_1, and **Equation A7** is the planned backorder balance, whose algebraic character echoes the closed-form treatment of rework systems with backorders.[42] **Equations A4** to **A8** are solved by the fixed point sweep of Section 5.4, and the cross-substitution terms carrying $p_2-\lambda_3$ vanish at the clearing point where λ_3 takes the value in **Equation 17** net of the salvage floor, which is what makes the sweep contract rapidly in practice.